AROUND ASSOCIATORS

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Abstract. This is a concise exposition of recent developments around the study of associators. It is based on the author's talk at the Mathematische Arbeitstagung in Bonn, June 2011 and at the Automorphic forms and Galois representations in Durham, July 2011. The first section is a review of Drinfeld's definition [Dr] of associators and the results [F10, F12] concerning the definition. The second section is to explain the four pro-unipotent algebraic groups related to associators; the motivic Galois group, the Grothendieck-Teichmüller group, the double shuffle group and the Kashiwara-Vergne group. Relationships, actually inclusions, between them are also discussed.

1. Associators

We recall the definition of associators [Dr] and explain our main results in [F10, F11] concerning the defining equations of associators.

The notion of associators was introduced by Drinfeld in [Dr]. They describe monodromies of the KZ (Knizhnik-Zamolodchikov) equations. They are essential for the construction of quasi-triangular quasi-Hopf quantized universal enveloping algebras (loc. cit), for the quantization of Lie-bialgebras (Etingof-Kazhdan quantization [EtK]), for the proof of formality chain operad of little discs by Tamarkin [Ta] (see also Ševera and Willwacher [SW]) and also for the combinatorial reconstruction of the universal Vassiliev knot invariant (the Kontsevich invariant [Kon, Ba95]) by Bar-Natan [Ba97], Cartier [C], Kassel and Turaev [KssT], Le and Murakami [LM96a] and Piunikhin [P].

Notation 1. Let k be a field of characteristic 0 and $\bar{k}$ be its algebraic closure. Denote by $U\mathfrak{F}_2 = k\langle\langle X_0, X_1\rangle\rangle$ the non-commutative formal power series ring defined as the universal enveloping algebra of the completed free Lie algebra $\mathfrak{F}_2$ with two variables X_0 and X_1. An element $\varphi = \varphi(X_0, X_1)$ of $U\mathfrak{F}_2$ is called *group-like* [1] if it satisfies

$$\Delta(\varphi) = \varphi \otimes \varphi \text{ and } \varphi(0, 0) = 1 \tag{1}$$

[1] It is equivalent to $\varphi \in \exp \mathfrak{F}_2$.

where $\Delta : U\mathfrak{F}_2 \to U\mathfrak{F}_2 \hat{\otimes} U\mathfrak{F}_2$ is given by $\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0$ and $\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1$. For any k-algebra homomorphism $\iota : U\mathfrak{F}_2 \to S$, the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(X_0), \iota(X_1))$.

Denote by $U\mathfrak{a}_3$ (resp. $U\mathfrak{a}_4$) the universal enveloping algebra of the *completed pure braid Lie algebra* $\mathfrak{a}_3$ (resp. $\mathfrak{a}_4$) over k with 3 (resp. 4) strings, which is generated by t_{ij} ($1 \leqslant i, j \leqslant 3$ (resp. 4)) with defining relations

$$t_{ii} = 0,\ t_{ij} = t_{ji},\ [t_{ij}, t_{ik} + t_{jk}] = 0 \ (i,j,k\text{: all distinct})$$
$$\text{and}\ \ [t_{ij}, t_{kl}] = 0 \ (i,j,k,l\text{: all distinct}).$$

Note that $X_0 \mapsto t_{12}$ and $X_1 \mapsto t_{23}$ give an isomorphism $U\mathfrak{F}_2 \simeq U\mathfrak{a}_3$.

Definition 2 ([Dr]). A pair (μ, φ) with a *non-zero* element μ in k and a group-like series $\varphi = \varphi(X_0, X_1) \in U\mathfrak{F}_2$ is called an *associator* if it satisfies *one pentagon equation*

(2)
$$\varphi(t_{12}, t_{23}+t_{24})\varphi(t_{13}+t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\varphi(t_{12}+t_{13}, t_{24}+t_{34})\varphi(t_{12}, t_{23})$$

in $U\mathfrak{a}_4$ and *two hexagon equations*

(3)
$$\exp\{\frac{\mu(t_{13}+t_{23})}{2}\} = \varphi(t_{13}, t_{12}) \exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{13}, t_{23})^{-1} \exp\{\frac{\mu t_{23}}{2}\}\varphi(t_{12}, t_{23}),$$

(4)
$$\exp\{\frac{\mu(t_{12}+t_{13})}{2}\} = \varphi(t_{23}, t_{13})^{-1} \exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{12}, t_{13}) \exp\{\frac{\mu t_{12}}{2}\}\varphi(t_{12}, t_{23})^{-1}$$

in $U\mathfrak{a}_3$.

Remark 3. (i). Drinfeld [Dr] proved that such a pair always exists for any filed k of characteristic 0.

(ii). The equations (2)$\sim$(4) reflect the three axioms of braided monoidal categories [JS]. We note that for any k-linear *infinitesimal* tensor category $\mathcal{C}$, each associator gives a structure of a braided monoidal category on $\mathcal{C}[[h]]$ (cf. [C, Dr, KssT]). Here $\mathcal{C}[[h]]$ denotes the category whose set of objects is equal to that of $\mathcal{C}$ and whose set of morphisms $Mor_{\mathcal{C}[[h]]}(X, Y)$ is $Mor_{\mathcal{C}}(X, Y) \otimes k[[h]]$ (h: a formal parameter).

Actually, the two hexagon equations are a consequence of the one pentagon equation:

Theorem 4 ([F10]). *Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies the pentagon equation* (2). *Then there always exists $\mu \in \bar{k}$ (unique up to signature) such that the pair (μ, φ) satisfies two hexagon equations* (3) *and* (4).

Recently several different proofs of the above theorem were obtained (see [AlT, BaD, Wi]).

One of the nicest examples of associators is the Drinfeld associator:

Example 5. The *Drinfeld associator* $\Phi_{KZ}=\Phi_{KZ}(X_0,X_1)\in\mathbf{C}\langle\langle X_0,X_1\rangle\rangle$ is defined to be the quotient $\Phi_{KZ}=G_1(z)^{-1}G_0(z)$ where G_0 and G_1 are the solutions of the *formal KZ-equation*, which is the following differential equation for multi-valued functions $G(z):\mathbf{C}\backslash\{0,1\}\to\mathbf{C}\langle\langle X_0,X_1\rangle\rangle$

$$\frac{d}{dz}G(z)=\Big(\frac{X_0}{z}+\frac{X_1}{z-1}\Big)G(z),$$

such that $G_0(z)\approx z^{X_0}$ when $z\to 0$ and $G_1(z)\approx(1-z)^{X_1}$ when $z\to 1$ (cf.[Dr]). It is shown in [Dr] (see also [Wo]) that the pair $(2\pi\sqrt{-1},\Phi_{KZ})$ forms an associator for $k=\mathbf{C}$. Namely Φ_{KZ} satisfies (1)$\sim$(4) with $\mu=2\pi\sqrt{-1}$.

Remark 6. (i). The Drinfeld associator is expressed as follows:

$$\Phi_{KZ}(X_0,X_1)=1+\sum_{\substack{m,k_1,\dots,k_m\in\mathbf{N}\\ k_m>1}}(-1)^m\zeta(k_1,\cdots,k_m)X_0^{k_m-1}X_1\cdots X_0^{k_1-1}X_1$$
$$+\ (\text{regularized terms}).$$

Here $\zeta(k_1,\cdots,k_m)$ is the *multiple zeta value* (MZV in short), the real number defined by the following power series

$$\zeta(k_1,\cdots,k_m):=\sum_{0<n_1<\cdots<n_m}\frac{1}{n_1^{k_1}\cdots n_m^{k_m}} \tag{5}$$

for m, $k_1,\dots$, $k_m\in\mathbf{N}(=\mathbf{Z}_{>0})$ with $k_m>1$ (its convergent condition). All of the coefficients of Φ_{KZ} (including its regularized terms) are explicitly calculated in terms of MZV's in [F03] Proposition 3.2.3 by Le-Murakami's method in [LM96b].

(ii). Since all of the coefficients of Φ_{KZ} are described by MZV's, the equations (1)$\sim$(4) for $(\mu,\varphi)=(2\pi\sqrt{-1},\Phi_{KZ})$ yield algebraic relations among them, which are called *associator relations*. It is expected that the associator relations might produce all algebraic relations among MZV's.

The above MZV's were introduced by Euler in [Eu] and have recently undergone a huge revival of interest due to their appearance in various different branches of mathematics and physics. In connection with motive theory, linear and algebraic relations among MZV's are particularly important. The regularized double shuffle relations which were initially introduced by Ecalle and Zagier in the early 90's might be one of the most fascinating ones. To state them let us fix notations again:

Notation 7. Let $\pi_Y : k\langle\langle X_0, X_1\rangle\rangle \to k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ be the k-linear map between non-commutative formal power series rings that sends all the words ending in X_0 to zero and the word $X_0^{n_m-1}X_1 \cdots X_0^{n_1-1}X_1$ $(n_1, \dots, n_m \in \mathbf{N})$ to $(-1)^m Y_{n_m} \cdots Y_{n_1}$. Define the coproduct Δ_* on $k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ by

$$\Delta_*(Y_n) = \sum_{i=0}^{n} Y_i \otimes Y_{n-i}$$

for all $n \geqslant 0$ with $Y_0 := 1$. For $\varphi = \sum_{W:\text{word}} c_W(\varphi) W \in U\mathfrak{F}_2 = k\langle\langle X_0, X_1\rangle\rangle$ with $c_W(\varphi) \in k$ (a 'word' is a monic monomial element or 1 in $U\mathfrak{F}_2$), put

$$\varphi_* = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{X_0^{n-1}X_1}(\varphi) Y_1^n\right) \cdot \pi_Y(\varphi).$$

The *regularized double shuffle relations* for a group-like series $\varphi \in U\mathfrak{F}_2$ is a relation of the form

$$\Delta_*(\varphi_*) = \varphi_* \otimes \varphi_*. \tag{6}$$

Remark 8. The *regularized double shuffle relations* for MZV's are the algebraic relations among them obtained from (1) and (6) for $\varphi = \Phi_{KZ}$ (cf. [IkKZ, R]). It is also expected that the relations produce all algebraic relations among MZV's.

The following is the simplest example of the relations.

Example 9. For $a, b > 1$,

$$\zeta(a)\zeta(b) = \sum_{i=0}^{a-1} \binom{b-1+i}{i} \zeta(a-i, b+i) + \sum_{j=0}^{b-1} \binom{a-1+j}{j} \zeta(b-j, a+j),$$

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(a+b) + \zeta(b,a).$$

The former follows from (1) and the latter follows from (6).

The regularised double shuffle relations are also a consequence of the pentagon equation:

Theorem 10 ([F11])**.** *Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies the pentagon equation* (2)*. Then it also satisfies the regularised double shuffle relations* (6)*.*

This result attains the final goal of the project posed by Deligne-Terasoma [Te]. Their idea is to use some convolutions of perverse sheaves, whereas our proof is to use Chen's bar construction calculus. It would be our next project to complete their idea and to get another proof of Theorem 10.

Remark 11. Our Theorem 10 was extended cyclotomically in [F12].

The following Zagier's relation which is essential for Brown's proof of Theorem 17 might be also one of the most fascinating ones. The author does not know if it also follows from our pentagon equation (2).

Theorem 12 ([Z]). *For $a, b \geqslant 0$*

$$\zeta(2^{\{a\}}, 3, 2^{\{b\}}) = 2 \sum_{r=1}^{a+b+1} (-1)^r (A^r_{a,b} - B^r_{a,b}) \zeta(2r+1) \zeta(2^{\{a+b+1-r\}})$$

with $A^r_{a,b} = \binom{2r}{2a+2}$ *and* $B^r_{a,b} = (1 - 2^{-2r})\binom{2r}{2b+1}$.

2. Four Groups

We explain recent developments on the four pro-unipotent algebraic groups related to associators; the motivic Galois group, the Grothendieck-Teichmüller group, the double shuffle group and the Kashiwara-Vergne group, all of which are regarded as subgroups of $Aut \exp \mathfrak{F}_2$. In the end of this section we discuss natural inclusions betweeen them.

2.1. Motivic Galois group.

We review the formulations of the motivic Galois groups (consult also [An] as a nice exposition).

Notation 13. We work in the triangulated category $DM(\mathbf{Q})_{\mathbf{Q}}$ of *mixed motives* over $\mathbf{Q}$ (a part of idea of mixed motives is explained in [De] §1) constructed by Hanamura, Levine and Voevodsky. *Tate motives* $\mathbf{Q}(n)$ ($n \in \mathbf{Z}$) are (Tate) objects of the category. Let $DMT(\mathbf{Q})_{\mathbf{Q}}$ be the triangulated sub-category of $DM(\mathbf{Q})_{\mathbf{Q}}$ generated by Tate motives $\mathbf{Q}(n)$ ($n \in \mathbf{Z}$). By the work of Levine a neutral tannakian $\mathbf{Q}$-category $MT(\mathbf{Q}) = MT(\mathbf{Q})_{\mathbf{Q}}$ of *mixed Tate motives over* $\mathbf{Q}$ is extracted by taking the heart with respect to a t-structure of $DMT(\mathbf{Q})_{\mathbf{Q}}$. Deligne and Goncharov [DeG] introduced the full subcategory $MT(\mathbf{Z}) = MT(\mathbf{Z})_{\mathbf{Q}}$ of *unramified mixed Tate motives* inside of $MT(\mathbf{Q})_{\mathbf{Q}}$, All objects there are mixed Tate motives M (i.e. an object of $MT(\mathbf{Q})$) such that for each subquotient E of M which is an extension of $\mathbf{Q}(n)$ by $\mathbf{Q}(n+1)$ for $n \in \mathbf{Z}$, the extension class of E in

$$Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(n), \mathbf{Q}(n+1)) = Ext^1_{MT(\mathbf{Q})}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Q}^\times \otimes \mathbf{Q}$$

is equal to in $\mathbf{Z}^\times \otimes \mathbf{Q} = \{0\}$.

In the category $MT(\mathbf{Z})$ of unramified mixed Tate motives, the following holds:

$$\dim_{\mathbf{Q}} Ext^1_{MT(\mathbf{Z})}(\mathbf{Q}(0),\mathbf{Q}(m)) = \begin{cases} 1 \ (m=3,5,7,\dots), \\ 0 \ (m : \text{others}), \end{cases} \tag{7}$$

$$\dim_{\mathbf{Q}} Ext^2_{MT(\mathbf{Z})}(\mathbf{Q}(0),\mathbf{Q}(m)) = 0. \tag{8}$$

The category $MT(\mathbf{Z})$ forms a neutral tannakian $\mathbf{Q}$-category (consult [DeM]) with the fiber functor

$$\omega_{\mathrm{can}} : MT(\mathbf{Z}) \to Vect_{\mathbf{Q}}$$

($Vect_{\mathbf{Q}}$: the category of $\mathbf{Q}$-vector spaces) sending each motive M to $\oplus_n Hom(\mathbf{Q}(n), Gr^W_{-2n}M)$.

Definition 14. The *motivic Galois group* here is defined to be the Galois group of $MT(\mathbf{Z})$, which is the pro-$\mathbf{Q}$-algebraic group defined by $Gal^{\mathcal{M}}(\mathbf{Z}) := \underline{Aut}^{\otimes}(MT(\mathbf{Z}) : \omega_{\mathrm{can}})$.

By the fundamental theorem of tannakian category theory, ω_{can} induces an equivalence of categories

$$MT(\mathbf{Z}) \simeq Rep\ Gal^{\mathcal{M}}(\mathbf{Z}) \tag{9}$$

where the right hand side of the isomorphism denotes the category of finite dimensional $\mathbf{Q}$-vector spaces with $Gal^{\mathcal{M}}(\mathbf{Z})$-action.

Remark 15. The action of $Gal^{\mathcal{M}}(\mathbf{Z})$ on $\omega_{\mathrm{can}}(\mathbf{Q}(1)) = \mathbf{Q}$ defines a surjection $Gal^{\mathcal{M}}(\mathbf{Z}) \to \mathbf{G}_m$ and its kernel $Gal^{\mathcal{M}}(\mathbf{Z})_1$ is the unipotent radical of $Gal^{\mathcal{M}}(\mathbf{Z})$. There is a canonical splitting $\tau : \mathbf{G}_m \to Gal^{\mathcal{M}}(\mathbf{Z})$ which gives a negative grading on its associated Lie algebra $LieGal^{\mathcal{M}}(\mathbf{Z})_1$. From (7) and (8) it follows that the Lie algebra is the graded *free* Lie algebra generated by one element in each degree $-3,-5,-7,\dots$. (consult [De] §8 for the full story).

The *motivic fundamental group* $\pi_1^{\mathcal{M}}(\mathbf{P}^1\backslash\{0,1,\infty\} : \overrightarrow{01})$ constructed in [DeG] §4 is a (pro-)object of $MT(\mathbf{Z})$. The Drinfeld associator (cf. Example 5) is essential in describing the Hodge realization of the motive (cf. [An, DeG, F07]). By our tannakian equivalence (9), it gives a (pro-)object of the right hand side of (9), which induces a (graded) action

$$\Psi : Gal^{\mathcal{M}}(\mathbf{Z})_1 \to Aut \exp \mathfrak{F}_2. \tag{10}$$

Remark 16. For each $\sigma \in Gal^{\mathcal{M}}(\mathbf{Z})_1(k)$, its action on $\exp \mathfrak{F}_2$ is described by $e^{X_0} \mapsto e^{X_0}$ and $e^{X_1} \mapsto \varphi_\sigma^{-1} e^{X_1} \varphi_\sigma$ for some $\varphi_\sigma \in \exp \mathfrak{F}_2$.

The following has been conjectured (Deligne-Ihara conjecture) for a long time and finally proved by Brown by using Zagier's relation (Theorem 12).

Theorem 17 ([Br])**.** *The map* Ψ *is injective.*

It is a pro-unipotent analogue of the so-called Belyĭ's theorem [Bel] in the pro-finite group setting. The theorem says that all unramified mixed Tate motives are associated with MZV's.

2.2. **Grothendieck-Teichmüller group.** The Grothendieck-Teichmüller group was introduced by Drinfeld [Dr] in his study of deformations of quasi-triangular quasi-Hopf quantized universal enveloping algebras. It was defined to be the set of 'degenerated' associators. The construction of the group was also stimulated by the previous idea of Grothendieck, *un jeu de Teichmüller-Lego*, which was posed in his article *Esquisse d'un programme* [G].

Definition 18 ([Dr])**.** The *Grothendieck-Teichmüller group* GRT_1 is defined to be the pro-algebraic variety whose set of k-valued points consists of *degenerated associators*, which are group-like series $\varphi \in U\mathfrak{F}_2$ satisfying the defining equations (2)∼(4) of associators with $\mu = 0$.

Remark 19. (i). By Theorem 4, GRT_1 is reformulated to be the set of group-like series satisfying (2) without quadratic terms.

(ii). It forms a group [Dr] by the multiplication below

$$\varphi_2 \circ \varphi_1 := \varphi_1(\varphi_2 X_0 \varphi_2^{-1}, X_1) \cdot \varphi_2 = \varphi_2 \cdot \varphi_1(X_0, \varphi_2^{-1} X_1 \varphi_2). \tag{11}$$

By the map $X_0 \mapsto X_0$ and $X_1 \mapsto \varphi^{-1} X_1 \varphi$, the group GRT_1 is regarded as a subgroup of $Aut \exp \mathfrak{F}_2$.

(iii). Ihara came to the Lie algebra of GRT_1 independently of Drinfeld's work in his arithmetic study of Galois action on fundamental groups (cf. [Iy90]).

(iv). The cyclotomic analogues of associators and that of the Grothendieck-Teichmüller group were introduced by Enriquez [Eu]. Some elimination results on their defining equations in special case were obtained in [EnF].

Geometric interpretation (cf. [Dr, Iy90, Iy94]) of equations (2)∼ (4) implies the following (for a proof, see also [An, F07])

Theorem 20. $Im\Psi \subset GRT_1$.

Related to the questions posed in [De, Dr, Iy90], it is expected that they are isomorphic.

Remark 21. (i). The Drinfeld associator Φ_{KZ} is an associator (cf. example 5) but is not a degenerated associator, i.e. $\Phi_{KZ} \notin GRT_1(\mathbf{C})$.

(ii). The p-adic Drinfeld associator Φ^p_{KZ} introduced in [F04] is not an associator but a degenerated associator, i.e. $\Phi^p_{KZ} \in GRT_1(\mathbf{Q}_p)$ (cf. [F07]).

2.3. **Double shuffle group.** The double shuffle group was introduced by Racinet as the set of solutions of the regularized double shuffle relations with 'degeneration' condition (no quadratic terms condition).

Definition 22 ([R])**.** The *double shuffle group* DMR_0 is the pro-algebraic variety whose set of k-valued points consists of the group-like series $\varphi \in U\mathfrak{F}_2$ satisfying the regularized double shuffle relations (6) without linear terms and quadratic terms.

Remark 23. (i). We note that DMR stands for *double mélange regularisé* ([R]).

(ii). It was shown in [R] that it forms a group by the operation (11).

(iii). By the same way to remark 19 (ii), the group DMR_0 is regarded as a subgroup of $Aut \exp \mathfrak{F}_2$.

It is also shown that $\mathrm{Im}\Psi$ is contained in DMR_0 (cf.[F07])). Actually it is expected that they are isomorphic. Theorem 10 follows the inclusion between GRT_1 and DMR_0:

Theorem 24 ([F11])**.** *$GRT_1 \subset DMR_0$.*

It is also expected that they are isomorphic.

Remark 25. (i). The Drinfeld associator Φ_{KZ} satisfies the regularized double shuffle relations (cf. Remark 8) but it is not an element of the double shuffle group, i.e. $\Phi_{KZ} \notin DMR_0(\mathbf{C})$, because its quadratic term is non-zero, actually is equal to $\zeta(2)X_1X_0 - \zeta(2)X_0X_1$.

(ii). The p-adic Drinfeld associator Φ^p_{KZ} satisfies the regularized double shuffle relations (cf. [BeF, FJ]) and it is an element of the double shuffle group, i.e. $\Phi^p_{KZ} \in DMR_0(\mathbf{Q}_p)$, which also follows from remark 21.(ii) and Theorem 24.

2.4. **Kashiwara-Vergne group.** In [KswV] Kashiwara and Vergne proposed a conjecture related to the Campbell-Baker-Hausdorff series which generalizes Duflo's theorem (Duflo isomorphism) to some extent. The conjecture was settled generally by Alekseev and Meinrenken [AlM]. The Kashiwara-Vergne group was introduced as a 'degeneration' of the set of solution of the conjecture by Alekseev and Torossian in [AlT], where they gave another proof of the conjecture by using Drinfeld's [Dr] theory of associators.

The following is one of the formulations of the conjecture stated in [AlET].

Generalized Kashiwara-Vergne problem: Find a group automorphism $P : \exp\mathfrak{F}_2 \to \exp\mathfrak{F}_2$ such that P belongs to $TAut\exp\mathfrak{F}_2$ (that is, $P \in Aut\exp\mathfrak{F}_2$ such that

$$P(e^{X_0}) = p_1 e^{X_0} p_1^{-1} \text{ and } P(e^{X_1}) = p_2 e^{X_1} p_2^{-1}$$

for some $p_1,\ p_2 \in \exp\mathfrak{F}_2$) and P satisfies

$$P(e^{X_0}e^{X_1}) = e^{(X_0+X_1)}$$

and the coboundary Jacobian condition

$$\delta \circ J(P) = 0.$$

Here J stands for the Jacobian cocycle $J : TAut\exp\mathfrak{F}_2 \to \mathfrak{tr}_2$ and δ denotes the differential map $\delta : \mathfrak{tr}_n \to \mathfrak{tr}_{n+1}$ for $n = 1, 2, \dots$ (for their precise definitions see [AlT]). We note that P is uniquely determined by the pair (p_1, p_2).

The following is essential for the proof of the conjecture.

Theorem 26 ([AlT, AlET])**.** *Let (μ, φ) be an associator. Then the pair*

$$(p_1, p_2) = \Big(\varphi(X_0/\mu, X_\infty/\mu),\ e^{X_\infty/2}\varphi(X_1/\mu, X_\infty/\mu)\Big)$$

with $X_\infty = -X_0 - X_1$ gives a solution to the above problem.

The Kashiwara-Vergne group is defined to be the set of solutions of the problem with 'degeneration condition' ('the condition $\mu = 0$'):

Definition 27 ([AlT, AlET])**.** The *Kashiwara-Vergne group KRV* is defined to be the set of $P \in Aut\exp\mathfrak{F}_2$ which satisfies $P \in TAut\exp\mathfrak{F}_2$,

$$P(e^{(X_0+X_1)}) = e^{(X_0+X_1)}$$

and the coboundary Jacobian condition $\delta \circ J(P) = 0$.

The above KRV forms a subgroup of $Aut\exp\mathfrak{F}_2$. We denote by KRV_0 the subgroup of KRV consisting of P without linear terms in both p_1 and p_2. Theorem 26 yields the inclusion below.

Theorem 28 ([AlT, AlET])**.** *$GRT_1 \subset KRV_0$.*

Actually it is expected that they are isomorphic (cf. [AlT]). Recent result of Schneps in [S] also leads to

Theorem 29 ([S])**.** *$DMR_0 \subset KRV_0$.*

2.5. **Comparison.** By Theorem 17, 20, 24, 28 and 29, we obtain

Theorem 30. $Gal^{\mathcal{M}}(\mathbf{Z})_1 \subseteq GRT_1 \subseteq DMR_0 \subseteq KRV_0$.

We finish our exposition by posing the following question:

Question 31. Are they all equal? Namely,

$$Gal^{\mathcal{M}}(\mathbf{Z})_1 = GRT_1 = DMR_0 = KRV_0 \quad ?$$

These four groups were constructed independently and there are no philosophical reasonings why we expect that they are all equal. Though it might be not so good mathematically to believe such equalities without any strong conceptual support, the author believes that it might be good at least spiritually to dream a hidden theory to relate them.

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