BLOCKS FOR MOD p REPRESENTATIONS OF $\mathrm{GL}_2(\mathbb{Q}_p)$

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ABSTRACT. Let π_1 and π_2 be absolutely irreducible smooth representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with a central character, defined over a finite extension of $\mathbb{F}_p$. We show that if there exists a non-split extension between π_1 and π_2 then they both appear as subquotients of the reduction modulo p of a unit ball in a crystalline Banach space representation of G. The results of Berger-Breuil describe such reductions and allow us to organize the irreducible representation into blocks. The result is new for $p = 2$, the proof, which works for all p, is new.

1. INTRODUCTION

Let L be a finite extension of $\mathbb{Q}_p$, with the ring of integers $\mathcal{O}$, a uniformizer ϖ, and residue field k, and let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and let B be the subgroup of upper-triangular matrices in G.

Theorem 1.1. *Let π_1, π_2 be smooth, absolutely irreducible k-representations of G with a central character. Suppose that $\mathrm{Ext}^1_G(\pi_2, \pi_1) \neq 0$ then after replacing L by a finite extension, we may find integers $(l, k) \in \mathbb{Z} \times \mathbb{N}$ and unramified characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to L^\times$ with $\chi_2 \neq \chi_1 |\centerdot|$, such that π_1 and π_2 are subquotients of $\overline{\Pi}^{ss}$, where $\overline{\Pi}^{ss}$ is the semi-simplification of the reduction modulo ϖ of an open bounded G-invariant lattice in Π, where Π is the universal unitary completion of*

$$(\mathrm{Ind}_B^G \chi_1 \otimes \chi_2 |\centerdot|^{-1})_{\mathrm{sm}} \otimes \det{}^l \otimes \mathrm{Sym}^{k-1} L^2.$$

The results of Berger-Breuil [3], Berger [2], Breuil-Emerton [6] and [22] describe explicitly the possibilities for $\overline{\Pi}^{ss}$, see Proposition 3.11. These results and the Theorem imply that $\mathrm{Ext}^1_G(\pi_2, \pi_1)$ vanishes in many cases. Let us make this more precise.

Let $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ be the category of smooth G-representation on $\mathcal{O}$-torsion modules. It contains $\mathrm{Mod}^{\mathrm{sm}}_G(k)$, the category of smooth G-representations on k-vector spaces, as a full subcategory. Every irreducible object π of $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ is killed by ϖ, and hence is an object of $\mathrm{Mod}^{\mathrm{sm}}_G(k)$. Barthel-Livné [1] and Breuil [4] have classified the absolutely irreducible smooth representations π admitting a central character. They fall into four disjoint classes:

(i) characters $\delta \circ \det$;
(ii) special series $\mathrm{Sp} \otimes \delta \circ \det$;
(iii) principal series $(\mathrm{Ind}_B^G \delta_1 \otimes \delta_2)_{\mathrm{sm}}$, $\delta_1 \neq \delta_2$;
(iv) supersingular representations,

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where Sp is the Steinberg representation, that is the locally constant functions from $\mathbb{P}^1(\mathbb{Q}_p)$ to k modulo the constant functions; $\delta, \delta_1, \delta_2 : \mathbb{Q}_p^\times \to k^\times$ are smooth characters and we consider $\delta_1 \otimes \delta_2$ as a character of B, which sends $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to $\delta_1(a)\delta_2(d)$. Using their results and some easy arguments, see [25, §5.3], one may show that for an irreducible smooth representations π the following are equivalent: 1) π is admissible, which means that π^H is finite dimensional for all open subgroups H of G; 2) $\mathrm{End}_G(\pi)$ is finite dimensional over k; 3) there exists a finite extension k' of k, such that $\pi \otimes_k k'$ is isomorphic to a finite direct sum of distinct absolutely irreducible k'-representations with a central character.

Let $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$, consisting of representations, which are equal to the union of their admissible subrepresentations. The categories $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ and $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ are abelian, see [15, Prop.2.2.18]. We define $\mathrm{Mod}_G^{\mathrm{l.adm}}(k)$ in exactly the same way with $\mathcal{O}$ replaced by k. Let $\mathrm{Irr}_G^{\mathrm{adm}}$ be the set of irreducible representation in $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$, then $\mathrm{Irr}_G^{\mathrm{adm}}$ is the set of irreducible representations in $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ satisfying the equivalent conditions described above. We define an equivalence relation $\sim$ on $\mathrm{Irr}_G^{\mathrm{adm}}$: $\pi \sim \tau$, if there exists a sequence of irreducible admissible representations $\pi = \pi_1, \pi_2, \ldots, \pi_n = \tau$, such that for each i one of the following holds: 1) $\pi_i \cong \pi_{i+1}$; 2) $\mathrm{Ext}_G^1(\pi_i, \pi_{i+1}) \neq 0$; 3) $\mathrm{Ext}_G^1(\pi_{i+1}, \pi_i) \neq 0$. We note that it does not matter for the definition of $\sim$, whether we compute Ext_G^1 in $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$, $\mathrm{Mod}_G^{\mathrm{sm}}(k)$, $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ or $\mathrm{Mod}_G^{\mathrm{l.adm}}(k)$, since we only care about vanishing or non-vanishing of $\mathrm{Ext}_G^1(\pi_i, \pi_{i+1})$ for distinct irreducible representations. A block is an equivalence class of $\sim$.

Corollary 1.2. *The blocks containing an absolutely irreducible representation are given by the following:*

(i) $\mathfrak{B} = \{\pi\}$ *with* π *supersingular;*
(ii) $\mathfrak{B} = \{(\mathrm{Ind}_B^G \delta_1 \otimes \delta_2\omega^{-1})_{\mathrm{sm}}, (\mathrm{Ind}_B^G \delta_2 \otimes \delta_1\omega^{-1})_{\mathrm{sm}}\}$ *with* $\delta_2\delta_1^{-1} \neq \omega^{\pm 1}, \mathbf{1}$*;*
(iii) $p > 2$ *and* $\mathfrak{B} = \{(\mathrm{Ind}_B^G \delta \otimes \delta\omega^{-1})_{\mathrm{sm}}\}$*;*
(iv) $p = 2$ *and* $\mathfrak{B} = \{\mathbf{1}, \mathrm{Sp}\} \otimes \delta \circ \det$*;*
(v) $p \geq 5$ *and* $\mathfrak{B} = \{\mathbf{1}, \mathrm{Sp}, (\mathrm{Ind}_B^G \omega \otimes \omega^{-1})_{\mathrm{sm}}\} \otimes \delta \circ \det$*;*
(vi) $p = 3$ *and* $\mathfrak{B} = \{\mathbf{1}, \mathrm{Sp}, \omega \circ \det, \mathrm{Sp} \otimes \omega \circ \det\} \otimes \delta \circ \det$*;*

where $\delta, \delta_1, \delta_2 : \mathbb{Q}_p^\times \to k^\times$ *are smooth characters and where* $\omega : \mathbb{Q}_p^\times \to k^\times$ *is the character* $\omega(x) = x|x| \pmod{\varpi}$.

One may view the cases (iii) to (vi) as degenerations of case (ii). A finitely generated smooth admissible representation of G is of finite length, [15, Thm.2.3.8]. This makes $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ into a locally finite category. It follows from [17] that every locally finite category decomposes into blocks. In our situation we obtain:

$$\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O}) \cong \prod_{\mathfrak{B} \in \mathrm{Irr}_G^{\mathrm{adm}}/\sim} \mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})[\mathfrak{B}], \tag{1}$$

where $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})[\mathfrak{B}]$ is the full subcategory of $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ consisting of representations, with all irreducible subquotients in $\mathfrak{B}$. One can deduce a similar result for the category of admissible unitary L-Banach space representations of G, see [25, Prop.5.32].

The result has been previously known for $p > 2$. Breuil and the author [7, §8], Colmez [8, §VII], Emerton [16, §4] and the author [23] have computed $\mathrm{Ext}_G^1(\pi_2, \pi_1)$ by different characteristic p methods, which do not work in the exceptional cases, when $p = 2$. In this paper, we go via characteristic 0 and make use of a deep Theorem of Berger-Breuil. The proof is less involved, but it does not give any information about the extensions between irreducible representations lying in the same block.

The motivation for these calculations comes from the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Colmez in [8] to a 2-dimensional absolutely irreducible L-representation of the absolute Galois group of $\mathbb{Q}_p$ has associated an admissible unitary absolutely irreducible non-ordinary L-Banach space representation of G. He showed that his construction induces an injection on the isomorphism classes and asked whether it is a bijection, see [8, §0.13]. This has been answered affirmatively in [25] for $p \geq 5$, where the knowledge of blocks has been used in an essential way. The results of this paper should be useful in dealing with the remaining cases.

Let us give a rough sketch of the argument. Let $0 \to \pi_1 \to \pi \to \pi_2 \to 0$ be a non-split extension. The method of [7] allows us to embed π into Ω, such that $\Omega|_K$ is admissible and an injective object in $\mathrm{Mod}^{\mathrm{sm}}_K(k)$, where $K = \mathrm{GL}_2(\mathbb{Z}_p)$. Using the results of [24] we may lift Ω to an admissible unitary L-Banach space representation E of G, in the sense that we may find a G-invariant unit ball E^0 in E, such that $E^0/\varpi E^0 \cong \Omega$. Moreover, $E|_K$ is isomorphic to a direct summand of $\mathcal{C}(K, L)^{\oplus r}$, where $\mathcal{C}(K, L)$ is the space of continuous function with the supremum norm. This implies, using an argument of Emerton, that the K-algebraic vectors are dense in E. As a consequence we find a closed G-invariant subspace Π of E, such that the reduction of $\Pi \cap E^0$ modulo ϖ contains π as a subrepresentation, and Π contains $\oplus_{i=1}^m \frac{\text{c-Ind}_{KZ}^G \tilde{\mathbf{1}}_i}{(T-a_i)^{n_i}} \otimes \det^{l_i} \otimes \mathrm{Sym}^{k_i-1} L^2$ as a dense subrepresentation, where Z is the centre of G, $\tilde{\mathbf{1}}_i : KZ \to L^\times$ is a character, trivial on K, $a_i \in L$, and T is a certain Hecke operator in $\mathrm{End}_G(\text{c-Ind}_{KZ}^G \tilde{\mathbf{1}}_i)$, such that $\frac{\text{c-Ind}_{KZ}^G \tilde{\mathbf{1}}_i}{(T-a_i)}$ is an unramified principal series representation. Once we have this we are in a good shape to prove Theorem 1.1.

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2. Notation

Let L be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k. We normalize the valuation val on L so that $\mathrm{val}(p) = 1$, and the norm $|\centerdot|$, so that $|x| = p^{-\mathrm{val}(x)}$, for all $x \in L$. Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$; Z the centre of G; B the subgroup of upper triangular matrices; $K = \mathrm{GL}_2(\mathbb{Z}_p)$; $I = \{g \in K : g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod p\}$; $I_1 = \{g \in K : g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod p\}$; let $\mathfrak{K}$ be the G-normalizer of I; let $H = \{\left(\begin{smallmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{smallmatrix}\right) : \lambda, \mu \in \mathbb{F}_p^\times\}$, where $[\lambda]$ is the Teichmüller lift of λ; let $\mathcal{G}$ be the subgroup of G generated by matrices $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ and H. Let $G^+ = \{g \in G : \mathrm{val}(\det(g)) \equiv 0 \pmod 2\}$. Since we are working with representations of locally pro-p groups in characteristic p, these representations will not be semi-simple in general; socle is the maximal semi-simple subobject. So for example, $\mathrm{soc}_G\, \tau$ means the maximal semi-simple G-subrepresentation of τ. Let $\mathrm{Ban}^{\mathrm{adm}}_G(L)$ be the category of admissible unitary L-Banach space representations of G, studied in [26]. This category is abelian. Let Π be an admissible unitary L-Banach space representation of G, and let Θ be an open bounded G-invariant lattice in Π, then $\Theta/\varpi\Theta$ is a smooth admissible k-representation of G. If $\Theta/\varpi\Theta$

is of finite length as a G-representation, then we let $\overline{\Pi}^{ss}$ be the semi-simplification of $\Theta/\varpi\Theta$. Since any two such Θ's are commensurable, $\overline{\Pi}^{ss}$ is independent of the choice of Θ. Universal unitary completions are discussed in [11, §1].

3. Main

Let π_1, π_2 be distinct smooth absolutely irreducible k-representation of G with a central character. It follows from [1] and [4] that π_1 and π_2 are admissible. We suppose that there exists a non-split extension in $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$:

$$0 \to \pi_1 \to \pi \to \pi_2 \to 0. \tag{2}$$

Since π_1 and π_2 are distinct and irreducible, by examining the long exact sequence induced by multiplication with ϖ, we deduce that π is killed by ϖ. A similar argument shows that the existence of a non-split extension implies that the central character of π_1 is equal to the central character of π_2. Moreover, π also has a central character, which is then equal to the central character of π_1, see [23, Prop.8.1]. We denote this central character by $\zeta : Z \to k^{\times}$. After replacing L by a quadratic extension and twisting by a character we may assume that $\zeta(\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)) = 1$.

Lemma 3.1. *If $\pi_1^{I_1} \neq \pi^{I_1}$ then Theorem 1.1 holds for π_1 and π_2.*

Proof. Since ζ is continuous, it is trivial on the pro-p group $Z \cap I_1$. We thus may extend ζ to ZI_1, by letting $\zeta(zu) = \zeta(z)$ for all $z \in Z$, $u \in I_1$. If τ is a smooth k-representation of G with a central character ζ then $\tau^{I_1} \cong \mathrm{Hom}_{I_1 Z}(\zeta, \tau) \cong \mathrm{Hom}_G(\text{c-}\mathrm{Ind}^G_{KZ}\, \zeta, \tau)$. Thus τ^{I_1} is naturally an $\mathcal{H} := \mathrm{End}_G(\text{c-}\mathrm{Ind}^G_{I_1 Z}\, \zeta)$ module. Taking I_1-invariants of (2) we get an exact sequence of $\mathcal{H}$-modules:

$$0 \to \pi_1^{I_1} \to \pi^{I_1} \to \pi_2^{I_1}. \tag{3}$$

Since π_2 is irreducible, $\pi_2^{I_1}$ is an irreducible $\mathcal{H}$-module by [27]. Hence, if $\pi_1^{I_1} \neq \pi^{I_1}$, then the last arrow is surjective. It is shown in [20], that if τ is a smooth k-representation of G, with a central character ζ, generated as a G-representation by its I_1-invariants, then the natural map $\tau^{I_1} \otimes_{\mathcal{H}} \text{c-}\mathrm{Ind}^G_{KZ}\, \zeta \to \tau$ is an isomorphism. This implies that the sequence $0 \to \pi_1^{I_1} \to \pi^{I_1} \to \pi_2^{I_1} \to 0$ is non-split, and hence defines a non-zero element of $\mathrm{Ext}^1_{\mathcal{H}}(\pi_2^{I_1}, \pi_1^{I_1})$. Since $\pi_i \cong \pi_i^{I_1} \otimes_{\mathcal{H}} \text{c-}\mathrm{Ind}^G_{KZ}\, \zeta$ for $i = 1, 2$, the $\mathcal{H}$-modules $\pi_1^{I_1}$ and $\pi_2^{I_1}$ are non-isomorphic. Non-vanishing of $\mathrm{Ext}^1_{\mathcal{H}}(\pi_2^{I_1}, \pi_1^{I_1})$ implies that there exists a smooth character $\eta : G \to k^{\times}$ such that either ($\pi_1 \cong \eta$ and $\pi_2 \cong \mathrm{Sp} \otimes \eta$) or ($\pi_2 \cong \eta$ and $\pi_1 \cong \mathrm{Sp} \otimes \eta$), [25, Lem.5.24], where Sp is the Steinberg representation. In both cases the universal unitary completion of $(\mathrm{Ind}^G_B |\centerdot| \otimes |\centerdot|^{-1})_{\mathrm{sm}} \otimes \tilde{\eta}$, where $\tilde{\eta} : G \to \mathcal{O}^{\times}$ is any smooth character lifting η, will satisfy the conditions of Theorem 1.1 by [13, 5.3.18]. $\square$

Lemma 3.1 allows to assume that $\pi^{I_1} = \pi_1^{I_1}$. We note that this implies that $\mathrm{soc}_K\, \pi_1 \cong \mathrm{soc}_K\, \pi$, and, since I_1 is contained in G^+, the restriction of (2) to G^+ is a non-split extension of G^+-representations.

Now we perform a renaming trick, the purpose of which is to get around some technical issues, when $p = 2$. If either $p > 2$ or $p = 2$ and π_1 is neither a special series nor a character then we let $\tau_1 = \pi_1$, $\tau = \pi$ and $\tau_2 = \pi_2$. If $p = 2$ and π_1 is either a special series representation or a character, then we let $0 \to \tau_1 \to \tau \to \tau_2 \to 0$ be the exact sequence obtained by tensoring (2) with $\mathrm{Ind}^G_{G^+} \mathbf{1}$. In particular, $\tau \cong \pi \otimes \mathrm{Ind}^G_{G^+} \mathbf{1}$, which implies that $\tau|_{G^+} \cong \pi|_{G^+} \oplus \pi|_{G^+}$ and $\tau_1|_{G^+} \cong \pi_1|_{G^+} \oplus \pi_1|_{G^+}$.

Hence, $\tau^{I_1}=\tau_1^{I_1}$ and $\mathrm{soc}_K\,\tau\cong\mathrm{soc}_K\,\tau_1\cong\mathrm{soc}_K\,\pi_1\oplus\mathrm{soc}_K\,\pi_1$. This implies that $\mathrm{soc}_G\,\tau\cong\mathrm{soc}_G\,\tau_1$.

Lemma 3.2. $\mathrm{soc}_G\,\tau\cong\mathrm{soc}_G\,\tau_1\cong\pi_1$.

Proof. We already know that $\mathrm{soc}_G\,\tau\cong\mathrm{soc}_G\,\tau_1$ and we only need to consider the case $p=2$ and π_1 is either special series or a character. The assumption on π_1 implies that $\pi_1^{I_1}$ is one dimensional. Let $\mathfrak{K}$ be the normalizer of I_1 in G, then I_1Z is a subgroup of $\mathfrak{K}$ of index 2. We note that $I=I_1$ as $p=2$. Thus $\mathfrak{K}$ acts on $\pi_1^{I_1}$ by a character χ, such that the restriction of χ to I_1Z is equal to ζ. Since $p=2$, we have an exact non-split sequence of G-representations $0\to\mathbf{1}\to\mathrm{Ind}_{G^+}^{G}\mathbf{1}\to\mathbf{1}\to 0$. We note that G^+ and hence ZI_1 act trivially on all the terms in this sequence. By tensoring with π_1 we obtain an exact sequence $0\to\pi_1\to\tau_1\to\pi_1\to 0$ of G-representations. Taking I_1-invariants, gives us an isomorphism of $\mathfrak{K}$-representations $\tau_1^{I_1}\cong\pi_1^{I_1}\otimes\mathrm{Ind}_{ZI_1}^{\mathfrak{K}}\mathbf{1}$. This representation is a non-split extension of χ by itself. Thus τ_1 is a non-split extension of π_1 by itself. Hence, $\mathrm{soc}_G\,\tau_1\cong\pi_1$. ☐

If $p=2$ then $\tau_1^{I_1}$ is 2-dimensional and has a basis of the form $\{v,\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)v\}$: if π_1 is either a character or special series, this follows from the isomorphism $\tau_1^{I_1}\cong\pi_1^{I_1}\otimes\mathrm{Ind}_{ZI_1}^{\mathfrak{K}}\mathbf{1}$, otherwise $\tau_1=\pi_1$ and the assertion follows from [7, Cor. 6.4 (i)] noting that the work of Bartel-Livné [1] and Breuil [4] on classification of irreducible representations of G implies that π^{I_1} is isomorphic as a module of the pro-p Iwahori Hecke algebra to $M(r,\lambda,\eta)$ defined in [7, Def. 6.2]. Since $\tau^{I_1}=\tau_1^{I_1}$, [7, Prop.9.2] implies that the inclusion $\tau^{I_1}\hookrightarrow\tau$ has a $\mathcal{G}$-equivariant section.

Proposition 3.3. *There exists a G-equivariant injection $\tau\hookrightarrow\Omega$, where Ω is a smooth k-representation of G, such that $\Omega|_K$ is an injective envelope of $\mathrm{soc}_K\,\tau$ in $\mathrm{Mod}_K^{\mathrm{sm}}(k)$, $\left(\begin{smallmatrix}p&0\\0&p\end{smallmatrix}\right)$ acts trivially on Ω and $\Omega|_{\mathfrak{K}}\cong\mathrm{Ind}_{\mathcal{G}}^{\mathfrak{K}}\,\Omega^{I_1}$.*

Proof. The existence of Ω satisfying the first two conditions follows from [7, Cor.9.11]. The last condition is satisfied as a byproduct of the construction of the action of $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$ in [7, Lem. 9.6]. ☐

Corollary 3.4. *Let Ω be as above then $\mathrm{soc}_K\,\Omega\cong\mathrm{soc}_K\,\tau_1$ and $\mathrm{soc}_G\,\Omega\cong\pi_1$.*

Proof. Since τ is a subrepresentation of Ω, $\mathrm{soc}_K\,\tau$ is contained in $\mathrm{soc}_K\,\Omega$. Since $\Omega|_K$ is an injective envelope of $\mathrm{soc}_K\,\tau$, every non-zero K-invariant subspace of Ω intersects $\mathrm{soc}_K\,\tau$ non-trivially. This implies that $\mathrm{soc}_K\,\tau\cong\mathrm{soc}_K\,\Omega$. This implies the first assertion, as $\mathrm{soc}_K\,\tau\cong\mathrm{soc}_K\,\tau_1$. Moreover, every G-invariant non-zero subspace of Ω intersects τ non-trivially, since those are also K-invariant. This implies $\mathrm{soc}_G\,\Omega\cong\mathrm{soc}_G\,\tau\cong\pi_1$, where the last isomorphism follows from Lemma 3.2. ☐

Lemma 3.5. *Let κ be a finite dimensional k-representation of $\mathcal{G}$ on which $\left(\begin{smallmatrix}p&0\\0&p\end{smallmatrix}\right)$ acts trivially. There exists an admissible unitary L-Banach space representation $(E,\|\centerdot\|)$ of $\mathfrak{K}$, such that $\|E\|\subset|L|$, $\left(\begin{smallmatrix}p&0\\0&p\end{smallmatrix}\right)$ acts trivially on E, and the reduction modulo ϖ of the unit ball in E is isomorphic to $(\mathrm{Ind}_{\mathcal{G}}^{\mathfrak{K}}\,\kappa)_{\mathrm{sm}}$ as a $\mathfrak{K}$-representation.*

Proof. It is enough to prove the statement, when κ is indecomposable, which we now assume. Let $p^{\mathbb{Z}}$ be the subgroup of G generated by $\left(\begin{smallmatrix}p&0\\0&p\end{smallmatrix}\right)$. Since the order of H is prime to p, and H has index 2 in $\mathcal{G}/p^{\mathbb{Z}}$, κ is either a character or an induction of a character from H to $\mathcal{G}/p^{\mathbb{Z}}$. In both cases we may lift κ to a representation $\tilde{\kappa}^0$

of $\mathcal{G}/p^{\mathbb{Z}}$ on a free $\mathcal{O}$-module of rank 1 or rank 2 respectively. Let $\tilde{\kappa} = \tilde{\kappa}^0 \otimes_{\mathcal{O}} L$ and let $\| \centerdot \|$ be the gauge of $\tilde{\kappa}^0$. Then $\| \centerdot \|$ is $\mathcal{G}$-invariant and $\tilde{\kappa}^0$ is the unit ball with respect to $\| \centerdot \|$. Then $(\mathrm{Ind}_{\mathcal{G}/p^{\mathbb{Z}}}^{\mathfrak{K}/p^{\mathbb{Z}}} \tilde{\kappa})_{\mathrm{cont}}$ with the norm $\|f\|_1 := \sup_{g \in \mathfrak{K}/p^{\mathbb{Z}}} \|f(g)\|$ is a lift of $(\mathrm{Ind}_{\mathcal{G}/p^{\mathbb{Z}}}^{\mathfrak{K}/p^{\mathbb{Z}}} \kappa)_{\mathrm{sm}}$, where the subscript cont indicates continuous induction: the space of continuous functions with the right transformation property. $\square$

Theorem 3.6. *Let Ω be any representation given by Proposition 3.3. Then there exists an admissible unitary L-Banach space representation $(E, \| \centerdot \|)$ of G, such that $\|E\| \subset |L|$, $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ acts trivially on E, and the reduction modulo ϖ of the unit ball in E is isomorphic to Ω as a G-representation.*

Proof. If $p \neq 2$ this is shown in [24, Thm.6.1]. We will observe that the renaming trick allows us to carry out essentially the same proof when $p = 2$. We make no assumption on p.

We first lift $\Omega|_K$ to characteristic 0. Let σ be the K-socle of Ω. Pontryagin duality induces an anti-equivalence of categories between $\mathrm{Mod}_K^{\mathrm{sm}}(k)$ and the category of pseudocompact $k[\![K]\!]$-modules, which we denote by $\mathrm{Mod}_K^{\mathrm{pro.aug}}(k)$. Since Ω is an injective envelope of σ in $\mathrm{Mod}_K^{\mathrm{sm}}(k)$, its Pontryagin dual $\Omega^\vee$ is a projective envelope of $\sigma^\vee$ in $\mathrm{Mod}_K^{\mathrm{pro.aug}}(k)$. Let $\widetilde{P}_{\sigma^\vee}$ be a projective envelope of $\sigma^\vee$ in the category of pseudocompact $\mathcal{O}[\![K]\!]$-modules. Then $\widetilde{P}_{\sigma^\vee}/\varpi \widetilde{P}_{\sigma^\vee}$ is a projective envelope of $\sigma^\vee$ in $\mathrm{Mod}_K^{\mathrm{pro.aug}}(k)$. Since projective envelopes are unique up to isomorphism, we obtain $\Omega^\vee \cong \widetilde{P}_{\sigma^\vee}/\varpi \widetilde{P}_{\sigma^\vee}$. Since τ_1 is admissible and $\sigma \cong \mathrm{soc}_K \tau_1$ by Corollary 3.4, σ is a finite dimensional k-vector space. In particular, $\sigma^\vee$ is a finitely generated $\mathcal{O}[\![K]\!]$-module, and so there exists a surjection of $\mathcal{O}[\![K]\!]$-modules $\mathcal{O}[\![K]\!]^{\oplus r} \twoheadrightarrow \sigma^\vee$. Since $\mathcal{O}[\![K]\!]^{\oplus r}$ is projective, and $\widetilde{P}_{\sigma^\vee} \twoheadrightarrow \sigma^\vee$ is essential, the surjection factors through $\mathcal{O}[\![K]\!]^{\oplus r} \twoheadrightarrow \widetilde{P}_{\sigma^\vee}$, and so $\widetilde{P}_{\sigma^\vee}$ is a finitely generated $\mathcal{O}[\![K]\!]$-module. Since $\widetilde{P}_{\sigma^\vee}$ is projective, we deduce that it is a direct summand of $\mathcal{O}[\![K]\!]^{\oplus r}$, and hence it is $\mathcal{O}$-torsion free.

Thus $\widetilde{P}_{\sigma^\vee}$ is an $\mathcal{O}$-torsion free, finitely generated $\mathcal{O}[\![K]\!]$-module, and its reduction modulo ϖ is isomorphic to $\Omega^\vee$ in $\mathrm{Mod}_K^{\mathrm{pro.aug}}(k)$. Let $E_0 = \mathrm{Hom}_{\mathcal{O}}^{cont}(\widetilde{P}_{\sigma^\vee}, L)$, and let $\| \centerdot \|_0$ be the supremum norm. It follows from [26] that E_0 is an admissible unitary L-Banach space representation of K. Moreover, the unit ball E_0^0 in E_0 is $\mathrm{Hom}_{\mathcal{O}}^{cont}(\widetilde{P}_{\sigma^\vee}, \mathcal{O})$ and

$$\mathrm{Hom}_{\mathcal{O}}^{cont}(\widetilde{P}_{\sigma^\vee}, \mathcal{O}) \otimes_{\mathcal{O}} k \cong \mathrm{Hom}_{\mathcal{O}}^{cont}(\widetilde{P}_{\sigma^\vee}, k) \cong \mathrm{Hom}_k^{cont}(P_{\sigma^\vee}, k) \cong (\Omega^\vee)^\vee \cong \Omega,$$

see [24, §5] for details. We extend the action of K on E_0 to the action of KZ by letting $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ act trivially.

Since σ is finite dimensional, it follows from [21, Lem.6.2.4] that Ω^{I_1} is a finite dimensional k-vector space. Since $\Omega|_{\mathfrak{K}} \cong (\mathrm{Ind}_{\mathcal{G}}^{\mathfrak{K}} \Omega^{I_1})_{\mathrm{sm}}$ by Proposition 3.3, Lemma 3.5 implies that there exists a unitary L-Banach space representation $(E_1, \| \centerdot \|_1)$ of $\mathfrak{K}$, such that $\|E_1\| \subseteq |L|$, $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ acts trivially on E_1 and the reduction of the unit ball E_1^0 in E_1 modulo ϖ is isomorphic to $\Omega|_{\mathfrak{K}}$. We claim that there exists an isometric, IZ-equivariant isomorphism $\varphi : E_1 \to E_0$ such that the following diagram of IZ-representations:

$$\begin{tikzcd} E_1^0/\varpi E_1^0 \arrow[r, "\varphi", "\text{mod } \varpi"'] \arrow[d, "\cong"] & E_0^0/\varpi E_0^0 \arrow[d, "\cong"] \\ \Omega \arrow[r, "\mathrm{id}"] & \Omega \end{tikzcd} \tag{4}$$

commutes, where the left vertical arrow is the given $\mathfrak{K}$-equivariant isomorphism $E_1^0/\varpi E_1^0 \cong \Omega|_{\mathfrak{K}}$ and the right vertical arrow is the given KZ-equivariant isomorphism $E_0^0/\varpi E_0^0 \cong \Omega|_{KZ}$. Granting the claim, we may transport the action of $\mathfrak{K}$ on E_0 by using φ to obtain a unitary action of KZ and $\mathfrak{K}$ on E_0, such that the two actions agree on $KZ \cap \mathfrak{K}$, which is equal to IZ. The resulting action glues to the unitary action of G on E_0, see [21, Cor.5.5.5], which is stated for smooth representations, but the proof of which works for any representation. The commutativity of the above diagram implies that $E_0^0 \otimes_{\mathcal{O}} k \cong \Omega$ as a G-representation.

We will prove the claim now. Let $M = \mathrm{Hom}_{\mathcal{O}}^{cont}(E_1^0, \mathcal{O})$ equipped with the topology of pointwise convergence. Then M is an object of $\mathrm{Mod}_I^{\mathrm{pro.aug}}(\mathcal{O})$, and $M \otimes_{\mathcal{O}} k \cong \Omega^\vee$ in $\mathrm{Mod}_I^{\mathrm{pro.aug}}(k)$, see [24, Lem.5.4]. Since $\Omega|_K$ is injective in $\mathrm{Mod}_K^{\mathrm{sm}}(k)$, $\Omega|_I$ is injective in $\mathrm{Mod}_I^{\mathrm{sm}}(k)$. Since I_1 is a pro-p group, every non-zero I-invariant subspace of Ω intersects Ω^{I_1} non-trivially. Thus $\Omega|_I$ is an injective envelope of Ω^{I_1} in $\mathrm{Mod}_I^{\mathrm{sm}}(k)$. Hence, $\Omega^\vee$ is a projective envelope of $(\Omega^{I_1})^\vee$ in $\mathrm{Mod}_I^{\mathrm{pro.aug}}(k)$. Since M is $\mathcal{O}$-torsion free, and $M \otimes_{\mathcal{O}} k$ is a projective envelope of $(\Omega^{I_1})^\vee$ in $\mathrm{Mod}_I^{\mathrm{pro.aug}}(k)$, [24, Prop.4.6] implies that M is a projective envelope of $(\Omega^{I_1})^\vee$ in $\mathrm{Mod}_I^{\mathrm{pro.aug}}(\mathcal{O})$. The same holds for $\widetilde{P}_{\sigma^\vee}$. Since projective envelopes are unique up to isomorphism, there exists an isomorphism $\psi : \widetilde{P}_{\sigma^\vee} \xrightarrow{\cong} M$ in $\mathrm{Mod}_I^{\mathrm{pro.aug}}(\mathcal{O})$. It follows from [24, Cor.4.7] that the natural map $\mathrm{Aut}_{\mathcal{O}[\![I]\!]}(\widetilde{P}_{\sigma^\vee}) \to \mathrm{Aut}_{k[\![I]\!]}(\widetilde{P}_{\sigma^\vee}/\varpi \widetilde{P}_{\sigma^\vee})$ is surjective. Using this we may choose ψ so that the following diagram in $\mathrm{Mod}_K^{\mathrm{pro.aug}}(k)$:

$$\begin{tikzcd} \widetilde{P}_{\sigma^\vee}/\varpi \widetilde{P}_{\sigma^\vee} \arrow[r, "\psi", "\text{mod } \varpi"'] \arrow[d, "\cong"] & M/\varpi M \arrow[d, "\cong"] \\ \Omega^\vee \arrow[r, "\mathrm{id}"] & \Omega^\vee \end{tikzcd} \tag{5}$$

commutes. Dually we obtain an isometric I-equivariant isomorphism of unitary L-Banach space representations of I, $\psi^d : \mathrm{Hom}_{\mathcal{O}}^{cont}(M, L) \to \mathrm{Hom}_{\mathcal{O}}^{cont}(\widetilde{P}_{\sigma^\vee}, L)$. It follows from [26, Thm.1.2] that $(E_1, \|\centerdot\|_1)$ is naturally and isometrically isomorphic to $\mathrm{Hom}_{\mathcal{O}}^{cont}(M, L)$ with the supremum norm. This gives our φ. The commutativity of (5) implies the commutativity of (4). $\square$

Corollary 3.7. *The Banach space representation $(E, \|\centerdot\|)$ constructed in Theorem 3.6 is isometrically, K-equivariantly isomorphic to a direct summand of $\mathcal{C}(K, L)^{\oplus r}$, where $\mathcal{C}(K, L)$ is the space of continuous functions from K to L, equipped with the supremum norm, and r is a positive integer.*

Proof. It follows from the construction of E, that $(E, \|\centerdot\|)$ is isometrically, K-equivariantly isomorphic to $\mathrm{Hom}_{\mathcal{O}}^{cont}(\widetilde{P}_{\sigma^\vee}, L)$ with the supremum norm. Moreover, it follows from the proof of Theorem 3.6 that $\widetilde{P}_{\sigma^\vee}$ is a direct summand of $\mathcal{O}[\![K]\!]^{\oplus r}$. It is shown in [26, Lem.2.1, Cor.2.2] that the natural map $K \to \mathcal{O}[\![K]\!]$, $g \mapsto g$ induces an isometrical, K-equivariant isomorphism between $\mathcal{C}(K, L)$ and $\mathrm{Hom}_{\mathcal{O}}^{cont}(\mathcal{O}[\![K]\!], L)$. $\square$

If F is a finite extension of $\mathbb{Q}_p$ then exactly the same proof works. We note that [7, Thm.9.8] is proved for $\mathrm{GL}_2(F)$. We record this as a corollary below. Let $\mathcal{O}_F$ be the ring of integers of F, ϖ_F a uniformizer, k_F the residue field, let $\mathcal{G}_F$ be the subgroup of $\mathrm{GL}_2(F)$ generated by the matrices $\left(\begin{smallmatrix} \varpi_F & 0 \\ 0 & \varpi_F \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ \varpi_F & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{smallmatrix}\right)$, for $\lambda, \mu \in k_F^{\times}$, where $[\lambda]$ is the Teichmüller lift of λ. Let I_1 be the standard pro-p Iwahori subgroup of G.

Corollary 3.8. *Let τ be an admissible smooth k-representation of $\mathrm{GL}_2(F)$, such that $\left(\begin{smallmatrix} \varpi_F & 0 \\ 0 & \varpi_F \end{smallmatrix}\right)$ acts trivially on τ and if $p = 2$ assume that the inclusion $\tau^{I_1} \hookrightarrow \tau$ has a $\mathcal{G}_F$-equivariant section. Then there exists a $\mathrm{GL}_2(F)$-equivariant embedding $\tau \hookrightarrow \Omega$, such that $\Omega|_{\mathrm{GL}_2(\mathcal{O}_F)}$ is an injective envelope of $\mathrm{GL}_2(\mathcal{O}_F)$-socle of τ in the category of smooth k-representations of $\mathrm{GL}_2(\mathcal{O}_F)$ and $\left(\begin{smallmatrix} \varpi_F & 0 \\ 0 & \varpi_F \end{smallmatrix}\right)$ acts trivially on Ω. Moreover, we may lift Ω to an admissible unitary L-Banach space representation of $\mathrm{GL}_2(F)$.*

Remark 3.9. *We also note that one could work with a fixed central character throughout.*

Let $V_{l,k} = \det^l \otimes \mathrm{Sym}^{k-1} L^2$, for $k \in \mathbb{N}$ and $l \in \mathbb{Z}$. Rather unfortunately k also denotes the residue field of L, we hope that this will not cause any confusion.

Proposition 3.10. *Let $(E, \|\centerdot\|)$ be a unitary L-Banach space representation of K isomorphic in the category of unitary admissible L-Banach space representations of K to a direct summand of $\mathcal{C}(K, L)^{\oplus r}$. The evaluation map*

$$\bigoplus_{(l,k)\in\mathbb{Z}\times\mathbb{N}} \mathrm{Hom}_K(V_{l,k}, E) \otimes V_{l,k} \to E \tag{6}$$

is injective and the image is a dense subspace of E. Moreover, the subspaces $\mathrm{Hom}_K(V_{l,k}, E)$ are finite dimensional.

Proof. The argument is the same as given in the proof of [14, Prop.5.4.1]. We have provided some details in the Appendix at the request of the referee. It is enough to prove the statement for $\mathcal{C}(K, L)$, since then it is true for $\mathcal{C}(K, L)^{\oplus r}$ and by applying the idempotent, which cuts out E, we may deduce the same statement for E. In the case $E = \mathcal{C}(K, L)$, the assertion follows from Proposition A.3 applied to $G = \mathrm{GL}_2$. We note that every rational irreducible representation of GL_2 / L is isomorphic to $V_{l,k}$ for a unique pair $(l, k) \in \mathbb{Z} \times \mathbb{N}$. The last assertion follows from (13) below. $\square$

Proposition 3.11. *Let $\rho = (\mathrm{Ind}_B^G \chi_1 \otimes \chi_2 |\centerdot|^{-1})_{\mathrm{sm}}$ be a smooth principal series representation of G, where $\chi_1, \chi_2 : \mathbb{Q}_p^{\times} \to L^{\times}$ smooth characters with $\chi_1 |\centerdot| \neq \chi_2$. Let Π be the universal unitary completion of $\rho \otimes V_{l,k}$. Then Π is an admissible, finite length L-Banach space representation of G. Moreover, if Π is non-zero and we let $\overline{\Pi}^{ss}$ be the semi-simplification of the reduction modulo ϖ of an open bounded G-invariant lattice in Π, then either $\overline{\Pi}^{ss}$ is irreducible supersingular, or*

$$\overline{\Pi}^{ss} \subseteq (\mathrm{Ind}_B^G \delta_1 \otimes \delta_2 \omega^{-1})^{ss}_{\mathrm{sm}} \oplus (\mathrm{Ind}_B^G \delta_2 \otimes \delta_1 \omega^{-1})^{ss}_{\mathrm{sm}}, \tag{7}$$

for some smooth characters $\delta_1, \delta_2 : \mathbb{Q}_p^{\times} \to k^{\times}$, where the superscript ss indicates the semi-simplification.

Proof. If $\Pi \neq 0$ then $-(k + l) \leq \mathrm{val}(\chi_1(p)) \leq -l$, $-(k + l) \leq \mathrm{val}(\chi_2(p)) \leq -l$ and $\mathrm{val}(\chi_1(p)) + \mathrm{val}(\chi_2(p)) = -(k+2l)$, [24, Lem.7.9], [11, Lem.2.1]. If both inequalities are strict and $\chi_1 \neq \chi_2$ then it is shown in [3, §5.3] that Π is non-zero, admissible and absolutely irreducible. The assertion about $\overline{\Pi}^{ss}$ then follows from [2].

If both inequalities are strict, $\chi_1 = \chi_2$ and Π is non-zero it is shown in [22, Prop.4.2] that there exist $\mathcal{O}$-lattices M in $\rho \otimes V_{l,k}$ and M' in $\rho' \otimes V_{l,k}$, where $\rho' = (\mathrm{Ind}_B^G \chi_1' \otimes \chi_2' |\centerdot|^{-1})_{\mathrm{sm}}$ for some distinct smooth characters, $\chi_1', \chi_2' : \mathbb{Q}_p^\times \to L^\times$ congruent to χ_1, χ_2 modulo $1 + (\varpi)$, such that both lattices are finitely generated $\mathcal{O}[G]$-modules and their reductions modulo ϖ are isomorphic. Since M is $\mathcal{O}$-torsion free, the completion of $\rho \otimes V_{l,k}$ with respect to the gauge of M is non-zero, and since M is a finitely generated $\mathcal{O}[G]$-module, the completion is the universal unitary completion, [11, Prop.1.17], thus is isomorphic to Π. Let Π^0 be the unit ball in Π with respect to the gauge of M. Then $\Pi^0/\varpi\Pi^0 \cong M/\varpi M \cong M'/\varpi M'$. Now by the same argument the completion of $\rho' \otimes V_{l,k}$ with respect to the gauge of M' is the universal unitary completion of of $\rho' \otimes V_{l,k}$. Since $\chi_1' \neq \chi_2'$ we may apply the results of Berger-Breuil [3] to conclude that the semi-simplification of $M'/\varpi M'$ has the desired form.

Suppose that either $\mathrm{val}(\chi_1(p)) = -l$ or $\mathrm{val}(\chi_2(p)) = -l$. If $\chi_1 = \chi_2|\centerdot|$ then this forces $k = 1$, so that $V_{l,k}$ is a character and $\rho \otimes V_{l,k} \cong (\mathrm{Ind}_B^G |\centerdot| \otimes |\centerdot|^{-1})_{\mathrm{sm}} \otimes \eta$, where $\eta : G \to L^\times$ is a unitary character. It follows from [13, Lem.5.3.18] that the universal unitary completion of $\rho \otimes V_{l,k}$ is admissible and of length 2. Moreover, $\overline{\Pi}^{ss} \cong \overline{\eta} \oplus \mathrm{Sp} \otimes \overline{\eta} \cong (\mathrm{Ind}_B^G \overline{\eta} \otimes \overline{\eta})^{ss}_{\mathrm{sm}}$. If $\chi_1 \neq \chi_2|\centerdot|$ then it follows from [6, Lem.2.2.1] that the universal unitary completion of $\rho \otimes V_{l,k}$ is isomorphic to a continuous induction of a unitary character. Hence $\overline{\Pi}^{ss}$ is isomorphic to the semi-simplification of a principal series representation. □

Proof of Theorem 1.1. Let $(E, \|\centerdot\|)$ be the unitary L-Banach space representation of G constructed in the proof of Theorem 3.6. Let E^0 be the unit ball in E, then by construction we have $E^0/\varpi E^0 \cong \Omega$, where Ω is a smooth k-representation of G, satisfying the conditions of Proposition 3.3. Let $V = \oplus \mathrm{Hom}_K(V_{l,k}, E) \otimes V_{l,k}$, where the sum is taken over all $(l,k) \in \mathbb{Z} \times \mathbb{N}$. It follows from Corollary 3.7 and Proposition 3.10 that the natural map $V \to E$ is injective and the image is dense. Let $\{V^i\}_{i \geq 0}$ be any increasing, exhaustive filtration of V by finite dimensional K-invariant subspaces. Then $V^i \cap E^0$ is a K-invariant $\mathcal{O}$-lattice in V^i, and we denote by $\overline{V}^i$ its reduction modulo ϖ. It follows from [24, Lem.5.5] that the reduction modulo ϖ induces a K-equivariant injection $\overline{V}^i \hookrightarrow \Omega$. The density of V in E implies that $\{\overline{V}^i\}_{i\geq 0}$ is an increasing, exhaustive filtration of Ω by finite dimensional, K-invariant subspaces. Recall that Ω contains τ as a subrepresentation, see Proposition 3.3. Now τ is finitely generated as a G-representation, since it is of finite length. Thus we may conclude, that there exists a finite dimensional K-invariant subspace W of V, such that τ is contained in the G-subrepresentation of Ω generated by $\overline{W}$.

Let $\varphi : V_{l,k} \to E$ be a non-zero K-equivariant, L-linear homomorphism. Let $R(\varphi)$ be the G-subrepresentation of E in the category of (abstract) G-representations on L-vector spaces, generated by the image of φ. Frobenius reciprocity gives us a surjection $\text{c-Ind}_{KZ}^G \tilde{\mathbf{1}} \otimes V_{l,k} \twoheadrightarrow R(\varphi)$, where $\tilde{\mathbf{1}} : KZ \to L^\times$ is an unramified character, such that $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ acts trivially on $V_{l,k} \otimes \tilde{\mathbf{1}}$. Now $\mathrm{End}_G(\text{c-Ind}_{KZ}^G \tilde{\mathbf{1}})$ is isomorphic to the ring of polynomials over L in one variable T. It follows from the proof of [24, Cor.7.4] that the surjection factors through $\frac{\text{c-Ind}_{KZ}^{G}\,\tilde{\mathbf{1}}}{P(T)}\otimes V_{l,k}\twoheadrightarrow R(\varphi)$, for some non-zero $P(T)\in L[T]$.

Let R be the (abstract) G-subrepresentation of E generated by W, and let Π be the closure of R in E. Since W is isomorphic to a finite direct sum of $V_{l,k}$'s, we deduce that if we replace L by a finite extension there exists a surjection:

$$\bigoplus_{i=1}^{m}\frac{\text{c-Ind}_{KZ}^{G}\,\tilde{\mathbf{1}}_i}{(T-a_i)^{n_i}}\otimes V_{l_i,k_i}\twoheadrightarrow R, \tag{8}$$

for some $a_i\in L$, $n_i\in\mathbb{N}$ and $(l_i,k_i)\in\mathbb{Z}\times\mathbb{N}$. Let $\rho_i=\frac{\text{c-Ind}_{KZ}^{G}\,\tilde{\mathbf{1}}_i}{T-a_i}$, then using (8) we may construct a finite, increasing, exhaustive filtration $\{R^j\}_{j\geq 0}$ of R by G-invariant subspaces, such that for each j there exists a surjection $\rho_i\otimes V_{l_i,k_i}\twoheadrightarrow R^j/R^{j-1}$, for some $1\leq i\leq m$. Moreover, by choosing n_i and m in (8) to be minimal, we may assume that $\mathrm{Hom}_G(\rho_i\otimes V_{l_i,k_i},R)$ is non-zero for all $1\leq i\leq m$. Let Π^j be the closure of R^j in E. We note that since E is admissible, Π^j is an admissible unitary L-Banach space representation of G, moreover the category $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ is abelian. Since R^j is dense in Π^j, its image is dense in Π^j/Π^{j-1}. Hence, for each j there exists a G-equivariant map $\varphi_j:\rho_i\otimes V_{l_i,k_i}\to\Pi^j/\Pi^{j-1}$ with a dense image. Let Π_i be the universal unitary completion of $\rho_i\otimes V_{l_i,k_i}$. Since the target of φ_j is unitary, we can extend it to a continuous G-equivariant map $\tilde{\varphi}_j:\Pi_i\to\Pi^j/\Pi^{j-1}$. Moreover, since the target of φ_j is admissible and the image is dense, $\tilde{\varphi}_j$ is surjective.

For each closed subspace U of E, we let $\overline{U}$ be the reduction of $(U\cap E^0)$ modulo ϖ. It follows from [24, Lem.5.5] that the reduction modulo ϖ induces an injection $\overline{U}\hookrightarrow\Omega$. Since Π contains W, $\overline{\Pi}$ will contain $\overline{W}$. Since $\overline{\Pi}$ is G-invariant, it will contain τ. Now $\{\overline{\Pi}^j\}_{j\geq 0}$ defines a finite, increasing, exhaustive filtration of $\overline{\Pi}$ by G-invariant subspaces. Since π_2 is an irreducible subquotient of τ, there exists j, such that π_2 is an irreducible subquotient of $\overline{\Pi}^j/\overline{\Pi}^{j-1}$.

Each representation ρ_i is an unramified principal series representation, considered in Proposition 3.11, see [5, Prop.3.2.1]. Hence, Π_i is an admissible, finite length L-Banach space representation of G, moreover $\overline{\Pi}_i^{ss}$ is of finite length as described in Proposition 3.11. The surjection $\tilde{\varphi}_j:\Pi_i\twoheadrightarrow\Pi^j/\Pi^{j-1}$ induces a surjection $\overline{\Pi}_i^{ss}\twoheadrightarrow\overline{(\Pi^j/\Pi^{j-1})}^{ss}$. It follows from [24, Lem.5.5] that the semi-simplification of $\overline{\Pi}^j/\overline{\Pi}^{j-1}$ is isomorphic to $\overline{(\Pi^j/\Pi^{j-1})}^{ss}$. Thus π_2 is a subquotient of $\overline{\Pi}_i^{ss}$.

Since $\mathrm{Hom}_G(\rho_i\otimes V_{l_i,k_i},\Pi)$ is non-zero, there exists a non-zero continuous G-invariant homomorphism $\varphi:\Pi_i\to\Pi$. Let Σ be the image of φ. Since Π_i and Π are admissible, we have a surjection $\Pi_i\twoheadrightarrow\Sigma$ and an injection $\Sigma\hookrightarrow\Pi$ in the abelian category $\mathrm{Ban}_G^{\mathrm{adm}}(L)$. The surjection induces a surjection $\overline{\Pi}_i^{ss}\twoheadrightarrow\overline{\Sigma}^{ss}$. The injection induces an injection $\overline{\Sigma}\hookrightarrow\overline{\Pi}\hookrightarrow\Omega$. Since $\mathrm{soc}_G\,\Omega\cong\pi_1$ by Corollary 3.4 and $\overline{\Sigma}$ is non-zero, we deduce that $\pi_1\cong\mathrm{soc}_G\,\overline{\Sigma}$. Hence, π_1 is a subquotient of $\overline{\Pi}_i^{ss}$. □

Lemma 3.12. *Let κ and λ be smooth k-representations of G and let l be a finite extension of k. Then $\mathrm{Ext}^i_G(\kappa,\lambda)\otimes_k l\cong\mathrm{Ext}^i_G(\kappa\otimes_k l,\lambda\otimes_k l)$, for all $i\geq 0$, where the* Ext *groups are computed in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$ and $\mathrm{Mod}_G^{\mathrm{sm}}(l)$, respectively.*

Proof. The assertion for $i=0$ follows from [25, Lem.5.1]. Hence, it is enough to find an injective resolution of λ in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$, which remains injective after tensoring with l. Such resolution may be obtained by considering $(\mathrm{Ind}_{\{1\}}^G V)_{\mathrm{sm}}$, where $\{1\}$ is the trivial subgroup of G and V is a k-vector space. We note that $(\mathrm{Ind}_{\{1\}}^G V)_{\mathrm{sm}}\otimes_k l\cong(\mathrm{Ind}_{\{1\}}^G V\otimes_k l)_{\mathrm{sm}}$, since l is finite over k. □

Proof of Corollary 1.2. Lemma 3.12 implies that replacing L by a finite extension does not change the blocks. It follows from Proposition 3.11 and Theorem 1.1 that an irreducible supersingular representation is in a block on its own. Let $\pi\{\delta_1, \delta_2\}$ be the semi-simple representation defined by (7), where $\delta_1, \delta_2 : \mathbb{Q}_p^\times \to k^\times$ are smooth characters. We have to show that all irreducible subquotients of $\pi\{\delta_1, \delta_2\}$ lie in the same block. We adopt an argument used in [8]. It follows from [5, 5.3.3.1, 5.3.3.2, 5.3.4.1] that there exists an irreducible unitary L-Banach space representation Π of G, such that $\overline{\Pi}^{ss} \cong \pi\{\delta_1, \delta_2\}$, then [8, Prop.VII.4.5(i)] asserts that we may choose an open bounded G-invariant lattice Θ in Π such that $\Theta/\varpi\Theta$ is indecomposable. It follows from (1) that all the irreducible subquotients of $\Theta/\varpi\Theta$ lie in the same block.

We will list explicitly the irreducible subquotients of $\pi\{\delta_1, \delta_2\}$. It is shown in [1] that if $\delta_2\delta_1^{-1} \neq \omega$ then $(\mathrm{Ind}_B^G \delta_1 \otimes \delta_2\omega^{-1})_{\mathrm{sm}}$ is absolutely irreducible, and there exists a non-split exact sequence

$$0 \to \delta_1 \circ \det \to (\mathrm{Ind}_B^G \delta_1 \otimes \delta_2\omega^{-1})_{\mathrm{sm}} \to \mathrm{Sp} \otimes \delta_1 \circ \det \to 0 \tag{9}$$

if $\delta_2\delta_1^{-1} = \omega$. Taking this into account there are the following possibilities for decomposing $\pi\{\delta_1, \delta_2\}$ into irreducible direct summands depending on δ_1, δ_2 and p:

(i) If $\delta_2\delta_1^{-1} \neq \omega^{\pm 1}, \mathbf{1}$ then

$$\pi\{\delta_1, \delta_2\} \cong (\mathrm{Ind}_B^G \delta_1 \otimes \delta_2\omega^{-1})_{\mathrm{sm}} \oplus (\mathrm{Ind}_B^G \delta_2 \otimes \delta_1\omega^{-1})_{\mathrm{sm}};$$

(ii) if $\delta_2 = \delta_1 (= \delta)$ then
  (a) if $p > 2$ then $\pi\{\delta, \delta\} \cong (\mathrm{Ind}_B^G \delta \otimes \delta\omega^{-1})_{\mathrm{sm}}^{\oplus 2}$;
  (b) if $p = 2$ then $\pi\{\delta, \delta\} \cong (\mathrm{Sp}^{\oplus 2} \oplus \mathbf{1}^{\oplus 2}) \otimes \delta \circ \det$.

(iii) if $\delta_2\delta_1^{-1} = \omega^{\pm 1}$ then
  (a) if $p \geq 5$ then $\pi\{\delta_1, \delta_2\} \cong (\mathbf{1} \oplus \mathrm{Sp} \oplus (\mathrm{Ind}_B^G \omega \otimes \omega^{-1})_{\mathrm{sm}}) \otimes \delta \circ \det$;
  (b) if $p = 3$ then $\pi\{\delta_1, \delta_2\} \cong (\mathbf{1} \oplus \mathrm{Sp} \oplus \omega \circ \det \oplus \mathrm{Sp} \otimes \omega \circ \det) \otimes \delta \circ \det$;
  (c) if $p = 2$ then we are in the case (ii)(b),
  where δ is either δ_1 or δ_2.

Finally, we note that in the case (ii)(b) instead of using [5, 5.3.3.2], which is stated without proof, we could have observed that since (9) is non-split, $\mathrm{Sp} \otimes \delta_1 \circ \det$ and $\delta_1 \circ \det$ lie in the same block. □

Appendix A. Density of algebraic vectors

Let X be an affine scheme of finite type over $\mathbb{Z}_p$ and let $A = \Gamma(X, \mathcal{O}_X)$. By choosing an isomorphism $A \cong \mathbb{Z}_p[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ we may identify the $X(\mathbb{Z}_p)$ with a closed subset of $\mathbb{Z}_p^n$. The induced topology on $X(\mathbb{Z}_p)$ is independent of a choice of the isomorphism, see [9, Prop.2.1]. Let $\mathcal{C}(X(\mathbb{Z}_p), L)$ be the space of continuous functions from $X(\mathbb{Z}_p)$ to L. Since $X(\mathbb{Z}_p)$ is compact, $\mathcal{C}(X(\mathbb{Z}_p), L)$ equipped with the supremum norm is an L-Banach space. Recall that $X(\mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}_p-alg}(A, \mathbb{Z}_p)$. We denote by $\mathcal{C}^{\mathrm{alg}}(X(\mathbb{Z}_p), L)$ the functions $f : X(\mathbb{Z}_p) \to L$, which are obtained by evaluating elements of $A \otimes_{\mathbb{Z}_p} L$ at $\mathbb{Z}_p$-valued points of X.

Lemma A.1. *$\mathcal{C}^{\mathrm{alg}}(X(\mathbb{Z}_p), L)$ is a dense subspace of $\mathcal{C}(X(\mathbb{Z}_p), L)$.*

Proof. We first look at the special case, when $X = \mathbb{A}^n$, so that $A = \mathbb{Z}_p[x_1, \dots, x_n]$ and $X(\mathbb{Z}_p) = \mathbb{Z}_p^n$. Since addition and multiplication in $\mathbb{Z}_p$ are continuous functions, we deduce that $\mathcal{C}^{\mathrm{alg}}(\mathbb{A}(\mathbb{Z}_p), L)$ is a subspace of $\mathcal{C}(\mathbb{A}(\mathbb{Z}_p), L)$. The density follows from the theory of Mahler expansions, see for example [19, III.1.2.4]. In the general case, we choose an isomorphism $A \cong \mathbb{Z}_p[x_1, \dots, x_n]/(f_1, \dots, f_m)$ and identify $X(\mathbb{Z}_p)$ with a closed subset of $\mathbb{A}^n(\mathbb{Z}_p) = \mathbb{Z}_p^n$. The restriction of functions to $X(\mathbb{Z}_p)$ induces a surjective map $r : \mathcal{C}(\mathbb{A}^n(\mathbb{Z}_p), L) \to \mathcal{C}(X(\mathbb{Z}_p), L)$, see for example [10, Thm.3.1(1)]. Since $\mathcal{C}^{\mathrm{alg}}(\mathbb{A}^n(\mathbb{Z}_p), L)$ is dense in $\mathcal{C}(\mathbb{A}^n(\mathbb{Z}_p), L)$ and $\sup_{x \in X(\mathbb{Z}_p)} |r(f)(x)| \le \sup_{x \in \mathbb{A}^n(\mathbb{Z}_p)} |f(x)|$ for all $f \in \mathcal{C}(\mathbb{A}^n(\mathbb{Z}_p), L)$ we deduce that $r(\mathcal{C}^{\mathrm{alg}}(\mathbb{A}^n(\mathbb{Z}_p), L))$ is a dense subspace. Since it is equal to $\mathcal{C}^{\mathrm{alg}}(X(\mathbb{Z}_p), L)$ we are done. $\square$

Remark A.2. *If X is an affine scheme of finite type over $\mathcal{O}_F$, where $\mathcal{O}_F$ is a ring of integers in a finite field extension F over $\mathbb{Q}_p$, then there are two ways to topologize $X(\mathcal{O}_F)$: as $\mathcal{O}_F$ points of X and as $\mathbb{Z}_p$-points of the Weil restriction of X to $\mathbb{Z}_p$. However, they coincide, see* [9, Ex.2.4].

Proposition A.3. *Let G be an affine group scheme of finite type over $\mathbb{Z}_p$ such that G_L is a split connected reductive group over L. Then the evaluation map*

$$\bigoplus_{[V]} \mathrm{Hom}_{G(\mathbb{Z}_p)}(V, \mathcal{C}(G(\mathbb{Z}_p), L)) \otimes V \to \mathcal{C}(G(\mathbb{Z}_p), L), \tag{10}$$

where the sum is taken over all the isomorphism classes of irreducible rational representations of G_L, is injective and the image is equal to $\mathcal{C}^{\mathrm{alg}}(G(\mathbb{Z}_p), L)$. In particular, the image of (10) *is a dense subspace of $\mathcal{C}(G(\mathbb{Z}_p), L)$.*

Proof. The category of rational representations of G_L is semi-simple as L is of characteristic 0, see [18, II.5.6 (6)]. Hence, the regular representation $\mathcal{O}(G_L)$ decomposes into a direct sum of irreducible representations. Since we have assumed that G_L is split, every irreducible rational representation V of G_L is absolutely irreducible [18, II.2.9]. This implies that $\mathrm{End}_{G_L}(V) = L$ for every irreducible representation V. It follows from Frobenius reciprocity [18, I.3.7 (3)] and the semi-simplicity of $\mathcal{O}(G_L)$ that we have an isomorphism of G_L-representations:

$$\mathcal{O}(G_L) \cong \bigoplus_{[V]} V^* \otimes V \tag{11}$$

where the G_L-action on V^* is trivial. The isomorphism (11) is $G(L)$-equivariant, and hence $G(\mathbb{Z}_p)$-equivariant, which gives us an isomorphism of $G(\mathbb{Z}_p)$-representations:

$$\mathcal{C}^{\mathrm{alg}}(G(\mathbb{Z}_p), L) \cong \bigoplus_{[V]} V^* \otimes V. \tag{12}$$

The map $\varphi \mapsto [v \mapsto \varphi(v)(1)]$ induces an isomorphism

$$\mathrm{Hom}_{G(\mathbb{Z}_p)}(V, \mathcal{C}(G(\mathbb{Z}_p), L)) \cong V^*, \tag{13}$$

with the inverse map given by $\ell \mapsto [v \mapsto [g \mapsto \ell(gv)]]$. Since every V is a finite dimensional L-vector space, we conclude from (12), (13) that the injection

$$\mathrm{Hom}_{G(\mathbb{Z}_p)}(V, \mathcal{C}^{\mathrm{alg}}(G(\mathbb{Z}_p), L)) \hookrightarrow \mathrm{Hom}_{G(\mathbb{Z}_p)}(V, \mathcal{C}(G(\mathbb{Z}_p), L)) \tag{14}$$

is an isomorphism. Moreover, as a byproduct we obtain that $\mathrm{End}_{G(\mathbb{Z}_p)}(V) = L$ and $\mathrm{Hom}_{G(\mathbb{Z}_p)}(V, W) = 0$, if V and W are non-isomorphic irreducible representations of G_L. We conclude that the evaluation map is injective, and the image is equal to $\mathcal{C}^{\mathrm{alg}}(G(\mathbb{Z}_p), L)$. Lemma A.1 implies the last assertion. $\square$

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