From étale P_+-representations to G-equivariant sheaves on G/P

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Abstract

Let $K/\mathbb{Q}_p$ be a finite extension with ring of integers o, let G be a connected reductive split $\mathbb{Q}_p$-group of Borel subgroup $P = TN$ and let α be a simple root of T in N. We associate to a finitely generated module D over the Fontaine ring over o endowed with a semilinear étale action of the monoid T_+ (acting on the Fontaine ring via α), a $G(\mathbb{Q}_p)$-equivariant sheaf of o-modules on the compact space $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$. Our construction generalizes the representation $D \boxtimes \mathbb{P}^1$ of $GL(2, \mathbb{Q}_p)$ associated by Colmez to a (φ, Γ)-module D endowed with a character of $\mathbb{Q}_p^*$.

Contents

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1 Introduction

1.1 Notations

We fix a finite extension $K/\mathbb{Q}_p$ of ring of integers o and an algebraic closure $\overline{\mathbb{Q}}_p$ of K. We denote by $\mathcal{G}_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ the absolute Galois group of $\mathbb{Q}_p$, by $\Lambda(\mathbb{Z}_p) = o[[\mathbb{Z}_p]]$ the Iwasawa o-algebra of maximal ideal $\mathcal{M}(\mathbb{Z}_p)$, and by $\mathcal{O}_{\mathcal{E}}$ the Fontaine ring which is the p-adic completion of the localisation of $\Lambda(\mathbb{Z}_p)$ with respect to the elements not in $p\Lambda(\mathbb{Z}_p)$. We put on $\mathcal{O}_{\mathcal{E}}$ the weak topology inducing the $\mathcal{M}(\mathbb{Z}_p)$-adic topology on $\Lambda(\mathbb{Z}_p)$, a fundamental system of neighborhoods of 0 being $(p^n\mathcal{O}_{\mathcal{E}} + \mathcal{M}(\mathbb{Z}_p)^n)_{n\in\mathbb{N}}$. The action of $\mathbb{Z}_p - \{0\}$ by multiplication on $\mathbb{Z}_p$ extends to an action on $\mathcal{O}_{\mathcal{E}}$.

We fix an arbitrary split reductive connected $\mathbb{Q}_p$-group G and a Borel $\mathbb{Q}_p$-subgroup $P = TN$ with maximal $\mathbb{Q}_p$-subtorus T and unipotent radical N. We denote by w_0 the longest element of the Weyl group of T in G, by Φ_+ the set of roots of T in N, and by $u_\alpha : \mathbb{G}_a \to N_\alpha$, for $\alpha \in \Phi_+$, a $\mathbb{Q}_p$-homomorphism onto the root subgroup N_α of N such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $x \in \mathbb{Q}_p$ and $t \in T(\mathbb{Q}_p)$, and $N_0 = \prod_{\alpha\in\Phi_+} u_\alpha(\mathbb{Z}_p)$ is a subgroup of $N(\mathbb{Q}_p)$. We denote by T_+ the monoid of dominant elements t in $T(\mathbb{Q}_p)$ such that $\mathrm{val}_p(\alpha(t)) \geq 0$ for all $\alpha \in \Phi_+$, by $T_0 \subset T_+$ the maximal subgroup, by T_{++} the subset of strictly dominant elements, i.e. $\mathrm{val}_p(\alpha(t)) > 0$ for all $\alpha \in \Phi_+$, and we put $P_+ = N_0T_+, P_0 = N_0T_0$. The natural action of T_+ on N_0 extends to an action on the Iwasawa o-algebra $\Lambda(N_0) = o[[N_0]]$. The compact set $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ contains the open dense subset $\mathcal{C} = N(\mathbb{Q}_p)w_0P(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ homeomorphic to $N(\mathbb{Q}_p)$ and the compact subset $\mathcal{C}_0 = N_0w_0P(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ homeomorphic to N_0. We put $\overline{P}(\mathbb{Q}_p) = w_0P(\mathbb{Q}_p)w_0^{-1}$.

Each simple root α gives a $\mathbb{Q}_p$-homomorphism $x_\alpha : N \to \mathbb{G}_a$ with section u_α. We denote by $\ell_\alpha : N_0 \to \mathbb{Z}_p$, resp. $\iota_\alpha : \mathbb{Z}_p \to N_0$, the restriction of x_α, resp. u_α, to N_0, resp. $\mathbb{Z}_p$.

For example, $G = GL(n)$, P is the subgroup of upper triangular matrices, N consists of the strictly upper triangular matrices (1 on the diagonal), T is the diagonal subgroup, $N_0 = N(\mathbb{Z}_p)$, the simple roots are $\alpha_1, \ldots, \alpha_{n-1}$ where $\alpha_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_it_{i+1}^{-1}$, x_{α_i} sends a matrix to its $(i, i+1)$-coefficient, $u_{\alpha_i}(.)$ is the strictly upper triangular matrix, with $(i, i+1)$-coefficient . and 0 everywhere else.

We denote by $C^\infty(X, o)$ the o-module of locally constant functions on a locally profinite space X with values in o, and by $C_c^\infty(X, o)$ the subspace of compactly supported functions.

1.2 General overview

Colmez established a correspondence $V \mapsto \Pi(V)$ from the absolutely irreducible K-representations V of dimension 2 of the Galois group $\mathcal{G}_p$ to the unitary admissible absolutely irreducible K-representations Π of $GL(2, \mathbb{Q}_p)$ admitting a central character [6]. This correspondence relies on the construction of a representation $D(V)\boxtimes\mathbb{P}^1$ of $GL(2, \mathbb{Q}_p)$ for any representation V (not necessarily of dimension 2) of $\mathcal{G}_p$ and any unitary character $\delta : \mathbb{Q}_p^* \to o^*$. When the dimension of V is 2 and when $\delta = (x|x|)^{-1}\delta_V$, where δ_V is the character of $\mathbb{Q}_p^*$ corresponding to the representation $\det V$ by local class field theory, then $D(V)\boxtimes\mathbb{P}^1$ is an extension of $\Pi(V)$ by its dual twisted by $\delta \circ \det$. It is a general belief that the correspondence $V \to \Pi(V)$ should extend to a correspondence from representations V of dimension d to representations Π of $GL(d, \mathbb{Q}_p)$.

We generalize here Colmez's construction of the representation $D \boxtimes \mathbb{P}^1$ of $GL(2, \mathbb{Q}_p)$, replacing $GL(2)$ by the arbitrary split reductive connected $\mathbb{Q}_p$-group G. More precisely, we denote by $O_{\mathcal{E},\alpha}$ the ring $\mathcal{O}_\mathcal{E}$ with the action of T_+ via a simple root $\alpha \in \Delta$ (if the rank of G is 1, α is unique and we omit α). For any finitely generated $\mathcal{O}_{\mathcal{E},\alpha}$-module D with an étale semilinear action of T_+, we construct a representation of $G(\mathbb{Q}_p)$. It is realized as the space of global sections of a $G(\mathbb{Q}_p)$-equivariant sheaf on the compact quotient $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$. When the rank of G is 1, the compact space $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ is isomorphic to $\mathbb{P}^1(\mathbb{Q}_p)$ and when $G = GL(2)$ we recover Colmez's sheaf.

We review briefly the main steps of our construction.

1. We show that the category of étale T_+-modules finitely generated over $\mathcal{O}_{\mathcal{E},\alpha}$ is equivalent to the category of étale T_+-modules finitely generated over $\Lambda_{\ell_\alpha}(N_0)$, for a topological ring $\Lambda_{\ell_\alpha}(N_0)$ generalizing the Fontaine ring $\mathcal{O}_\mathcal{E}$, which is better adapted to the group G, and depends on the simple root α.

2. We show that the sections over $\mathcal{C}_0 \simeq N_0$ of a $P(\mathbb{Q}_p)$-equivariant sheaf $\mathcal{S}$ of o-modules over $\mathcal{C} \simeq N$ is an étale $o[P_+]$-module $\mathcal{S}(\mathcal{C}_0)$ and that the functor $\mathcal{S} \mapsto \mathcal{S}(\mathcal{C}_0)$ is an equivalence of categories.

3. When $\mathcal{S}(\mathcal{C}_0)$ is an étale T_+-module finitely generated over $\Lambda_{\ell_\alpha}(N_0)$, and the root system of G is irreducible, we show that the $P(\mathbb{Q}_p)$-equivariant sheaf $\mathcal{S}$ on $\mathcal{C}$ extends to a $G(\mathbb{Q}_p)$-equivariant sheaf over $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ if and only if the rank of G is 1.

4. For any strictly dominant element $s \in T_{++}$, we associate functorially to an étale T_+-module M finitely generated over $\Lambda_{\ell_\alpha}(N_0)$, a $G(\mathbb{Q}_p)$-equivariant sheaf $\mathfrak{Y}_s$ of o-modules

over $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ with sections over $\mathcal{C}_0$ a dense étale $\Lambda(N_0)[T_+]$-submodule M_s^{bd} of M. When the rank of G is 1, the sheaf $\mathfrak{Y}_s$ does not depend on the choice of $s \in T_{++}$, and $M_s^{bd} = M$; when $G = GL(2)$ we recover the construction of Colmez. For a general G, the sheaf $\mathfrak{Y}_s$ depends on the choice of $s \in T_{++}$, the system $(\mathfrak{Y}_s)_{s \in T_{++}}$ of sheaves is compatible, and we associate functorially to M the $G(\mathbb{Q}_p)$-equivariant sheaves $\mathfrak{Y}_\cup$ and $\mathfrak{Y}_\cap$ of o-modules over $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ with sections over $\mathcal{C}_0$ equal to $\cup_{s \in T_{++}} M_s^{bd}$ and $\cap_{s \in T_{++}} M_s^{bd}$, respectively.

1.3 The rings $\Lambda_{\ell_\alpha}(N_0)$ and $\mathcal{O}_{\mathcal{E},\alpha}$

Fixing a simple root $\alpha \in \Delta$, the topological local ring $\Lambda_{\ell_\alpha}(N_0)$, generalizing the Fontaine ring $\mathcal{O}_{\mathcal{E}}$, is defined as in [11] with the surjective homomorphism $\ell_\alpha : N_0 \to \mathbb{Z}_p$.

We denote by $\mathcal{M}(N_{\ell_\alpha})$ the maximal ideal of the Iwasawa o-algebra $\Lambda(N_{\ell_\alpha}) = o[[N_{\ell_\alpha}]]$ of the kernel N_{ℓ_α} of ℓ_α. The ring $\Lambda_{\ell_\alpha}(N_0)$ is the $\mathcal{M}(N_{\ell_\alpha})$-adic completion of the localisation of $\Lambda(N_0)$ with respect to the Ore subset of elements which are not in $\mathcal{M}(N_{\ell_\alpha})\Lambda(N_0)$. This is a noetherian local ring with maximal ideal $\mathcal{M}_{\ell_\alpha}(N_0)$ generated by $\mathcal{M}(N_{\ell_\alpha})$. We put on $\Lambda_{\ell_\alpha}(N_0)$ the weak topology with fundamental system of neighborhoods of 0 equal to $(\mathcal{M}_{\ell_\alpha}(N_0)^n + \mathcal{M}(N_0)^n)_{n \in \mathbb{N}}$. The action of T_+ on N_0 extends to an action on $\Lambda_{\ell_\alpha}(N_0)$. We denote by $\mathcal{O}_{\mathcal{E},\alpha}$ the ring $\mathcal{O}_{\mathcal{E}}$ with the action of T_+ induced by $(t,x) \mapsto \alpha(t)x : T_+ \times \mathbb{Z}_p \to \mathbb{Z}_p$. The homomorphism ℓ_α and its section ι_α induce T_+-equivariant ring homomorphisms

$$\ell_\alpha : \Lambda_{\ell_\alpha}(N_0) \to \mathcal{O}_{\mathcal{E},\alpha} \ , \ \iota_\alpha : \mathcal{O}_{\mathcal{E},\alpha} \to \Lambda_{\ell_\alpha}(N_0) \ , \text{ such that } \ell_\alpha \circ \iota_\alpha = \mathrm{id} \, .$$

1.4 Equivalence of categories

An étale T_+-module over $\Lambda_{\ell_\alpha}(N_0)$ is a finitely generated $\Lambda_{\ell_\alpha}(N_0)$-module M with a semilinear action $T_+ \times M \to M$ of T_+ which is étale, i.e. the action φ_t on M of each $t \in T_+$ is injective and

$$M = \oplus_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(M) \ ,$$

if $J(N_0/tN_0t^{-1}) \subset N_0$ is a system of representatives of the cosets N_0/tN_0t^{-1}; in particular, the action of each element of the maximal subgroup T_0 of T_+ is invertible. We denote by ψ_t the left inverse of φ_t vanishing on $u\varphi_t(M)$ for $u \in N_0$ not in tN_0t^{-1}. These modules form an abelian category $\mathcal{M}^{et}_{\Lambda_{\ell_\alpha}(N_0)}(T_+)$.

We define analogously the abelian category $\mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E},\alpha}}(T_+)$ of finitely generated $\mathcal{O}_{\mathcal{E},\alpha}$-modules with an étale semilinear action of T_+. The action φ_t of each element $t \in T_+$ such that $\alpha(t) \in \mathbb{Z}_p^*$ is invertible. We show that the action $T_+ \times D \to D$ of T_+ on $D \in \mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E},\alpha}}(T_+)$ is continuous for the weak topology on D; the canonical action of the inverse T_- of T is also continuous.

Theorem 1.1. *The base change functors $\mathcal{O}_{\mathcal{E}} \otimes_{\ell_\alpha} -$ and $\Lambda_{\ell_\alpha}(N_0) \otimes_{\iota_\alpha} -$ induce quasi-inverse isomorphisms*

$$\mathbb{D} : \mathcal{M}^{et}_{\Lambda_{\ell_\alpha}(N_0)}(T_+) \to \mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E},\alpha}}(T_+) \ , \ \mathbb{M} : \mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E},\alpha}}(T_+) \to \mathcal{M}^{et}_{\Lambda_{\ell_\alpha}(N_0)}(T_+) \ .$$

Using this theorem, we show that the action of T_+ and of the inverse monoid T_- (given by the operators ψ) on an étale T_+-module over $\Lambda_{\ell_\alpha}(N_0)$ is continuous for the weak topology.

1.5 P-equivariant sheaves on $\mathcal{C}$

The o-algebra $C^\infty(N_0, o)$ is naturally an étale $o[P_+]-$module, and the monoid P_+ acts on the o-algebra $\mathrm{End}_o M$ by $(b, F) \mapsto \varphi_b \circ F \circ \psi_b$. We show that there exists a unique $o[P_+]$-linear map

$$\mathrm{res} : C^\infty(N_0, o) \to \mathrm{End}_o M$$

sending the characteristic function 1_{N_0} of N_0 to the identity id_M; moreover res is an algebra homomorphism which sends $1_{b.N_0}$ to $\varphi_b \circ \psi_b$ for all $b \in P_+$ acting on $x \in N_0$ by $(b,x) \mapsto b.x$.

For the sake of simplicity, we denote now by the same letter a group defined over $\mathbb{Q}_p$ and the group of its $\mathbb{Q}_p$-rational points.

Let M^P be the $o[P]$-module induced by the canonical action of the inverse monoid P_- of P_+ on M; as a representation of N, it is isomorphic to the representation induced by the action of N_0 on M. The value at 1, denoted by $\mathrm{ev}_0 : M^P \to M$, is P_--equivariant, and admits a P_+-equivariant splitting $\sigma_0 : M \to M^P$ sending $m \in M$ to the function equal to $n \mapsto nm$ on N_0 and vanishing on $N - N_0$. The $o[P]$-submodule M_c^P of M^P generated by $\sigma_0(M)$ is naturally isomorphic to $A[P] \otimes_{A[P_+]} M$. When $M = C^\infty(N_0, o)$ then $M_c^P = C_c^\infty(N, o)$ and $M^P = C^\infty(N, o)$ with the natural $o[P]-$module structure. We have the natural o-algebra embedding

$$F \mapsto \sigma_0 \circ F \circ \mathrm{ev}_0 : \mathrm{End}_o M \to \mathrm{End}_o M^P \ .$$

sending id_M to the idempotent $R_0 = \sigma_0 \circ ev_0$ in $\mathrm{End}_o M^P$.

Proposition 1.2. *There exists a unique $o[P]$-linear map*

$$\mathrm{Res} : C_c^\infty(N, o) \to \mathrm{End}_o M^P$$

sending 1_{N_0} to R_0; moreover Res *is an algebra homomorphism.*

The topology of N is totally disconnected and by a general argument, the functor of compact global sections is an equivalence of categories from the P-equivariant sheaves on $N \simeq \mathcal{C}$ to the non-degenerate modules on the skew group ring

$$C_c^\infty(N, o)\#P \ = \oplus_{b \in P} b C_c^\infty(N, o) \ .$$

in which the multiplication is determined by the rule $(b_1 f_1)(b_2 f_2) = b_1 b_2 f_1^{b_2} f_2$ for $b_i \in P, f_i \in C_c^\infty(N, o)$ and $f_1^{b_2}(.) = f_1(b_2.)$.

Theorem 1.3. *The functor of sections over $N_0 \simeq \mathcal{C}_0$ from the P-equivariant sheaves on $N \simeq \mathcal{C}$ to the étale $o[P_+]$-modules is an equivalence of categories.*

The space of global sections of a P-equivariant sheaf $\mathcal{S}$ on $\mathcal{C}$ is $\mathcal{S}(C) = \mathcal{S}(C_0)^P$.

1.6 Generalities on G-equivariant sheaves on G/P

The functor of global sections from the G-equivariant sheaves on G/P to the modules on the skew group ring $\mathcal{A}_{G/P} = C^\infty(G/P, o)\#G$ is an equivalence of categories. We have the intermediate ring $\mathcal{A}$

$$\mathcal{A}_\mathcal{C} = C_c^\infty(\mathcal{C}, o)\#P \ \subset \ \mathcal{A} = \oplus_{g \in G} g C_c^\infty(g^{-1}\mathcal{C} \cap \mathcal{C}, o) \ \subset \ \mathcal{A}_{G/P},$$

and the o-module

$$\mathcal{Z} = \oplus_{g \in G} g C_c^\infty(\mathcal{C}, o)$$

which is a left ideal of $\mathcal{A}_{G/P}$ and a right $\mathcal{A}$-submodule.

Proposition 1.4. *The functor*

$$Z \mapsto Y(Z) = \mathcal{Z} \otimes_\mathcal{A} Z$$

from the non-degenerate $\mathcal{A}$-modules to the $\mathcal{A}_{G/P}$-modules is an equivalence of categories; moreover the G-sheaf on G/P corresponding to $Y(Z)$ extends the P-equivariant sheaf on $\mathcal{C}$ corresponding to $Z|_{\mathcal{A}_\mathcal{C}}$.

Given an étale $o[P_+]$-module M, we consider the problem of extending to $\mathcal{A}$ the o-algebra homomorphism

$$\mathrm{Res} : A_{\mathcal{C}} \to \mathrm{End}_o(M^P_c) \quad , \quad \sum_{b\in P} b f_b \mapsto b \circ \mathrm{Res}(f_b) \ .$$

We introduce the subrings

$$\mathcal{A}_0 = 1_{\mathcal{C}_0}\mathcal{A}1_{\mathcal{C}_0} = \oplus_{g\in G} g C^\infty(g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, o) \ \subset \ \mathcal{A} \ ,$$
$$\mathcal{A}_{\mathcal{C}0} = 1_{\mathcal{C}_0}\mathcal{A}_{\mathcal{C}}1_{\mathcal{C}_0} = \oplus_{b\in P} b C^\infty(b^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, o) \ \subset \ \mathcal{A}_{\mathcal{C}} \ .$$

The skew monoid ring $\mathcal{A}_{\mathcal{C}_0} = C^\infty(\mathcal{C}_0, o)\#P_+ = \oplus_{b\in P_+} b C^\infty(\mathcal{C}_0, o)$ is contained in $\mathcal{A}_{\mathcal{C}0}$. The intersection $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ is not 0 if and only if $g \in N_0\overline{P}N_0$. The subring $\mathrm{Res}(\mathcal{A}_{\mathcal{C}0})$ of $\mathrm{End}_o(M^P)$ necessarily lies in the image of $\mathrm{End}_o(M)$.

The group P acts on $\mathcal{A}$ by $(b, y) \mapsto (b1_{G/P})y(b1_{G/P})^{-1}$ for $b \in P$, and the map $b \otimes y \mapsto (b1_{G/P})y(b1_{G/P})^{-1}$ gives $o[P]$ isomorphisms

$$o[P] \otimes_{o[P_+]} \mathcal{A}_0 \to \mathcal{A} \quad \text{and} \quad o[P] \otimes_{o[P_+]} \mathcal{A}_{\mathcal{C}0} \to \mathcal{A}_{\mathcal{C}} \ .$$

Proposition 1.5. *Let M be an étale $o[P_+]$-module. We suppose given, for any $g \in N_0\overline{P}N_0$, an element $\mathcal{H}_g \in \mathrm{End}_o(M)$. The map*

$$\mathcal{R}_0 : \mathcal{A}_0 \to \mathrm{End}_o(M) \quad , \quad \sum_{g\in N_0\overline{P}N_0} g f_g \mapsto \sum_{g\in N_0\overline{P}N_0} \mathcal{H}_g \circ \mathrm{res}(f_g)$$

is a P_+-equivariant o-algebra homomorphism which extends $\mathrm{Res}\,|_{\mathcal{A}_{\mathcal{C}0}}$ if and only if, for all $g, h \in N_0\overline{P}N_0$, $b \in P \cap N_0\overline{P}N_0$, and all compact open subsets $\mathcal{V} \subset \mathcal{C}_0$, the relations

- *H1.* $\mathrm{res}(1_{\mathcal{V}}) \circ \mathcal{H}_g = \mathcal{H}_g \circ \mathrm{res}(1_{g^{-1}\mathcal{V}\cap\mathcal{C}_0})$,
- *H2.* $\mathcal{H}_g \circ \mathcal{H}_h = \mathcal{H}_{gh} \circ \mathrm{res}(1_{h^{-1}\mathcal{C}_0\cap\mathcal{C}_0})$,
- *H3.* $\mathcal{H}_b = b \circ \mathrm{res}(1_{b^{-1}\mathcal{C}_0\cap\mathcal{C}_0})$.

hold true. In this case, the unique $o[P]$-equivariant map $\mathcal{R} : \mathcal{A} \to \mathrm{End}_A(M^P_c)$ extending $\mathcal{R}_0$ is multiplicative.

When these conditions are satisfied, we obtain a G-equivariant sheaf on G/P with sections on $\mathcal{C}_0$ equal to M.

1.7 $(s, \mathrm{res}, \mathfrak{C})$-integrals $\mathcal{H}_g$

Let M be an étale T_+-module M over $\Lambda_{\ell_\alpha}(N_0)$ with the weak topology. We denote by $\mathrm{End}^{cont}_o(M)$ the o-module of continuous o-linear endomorphisms of M, and for g in $N_0\overline{P}N_0$, by $U_g \subseteq N_0$ the compact open subset such that

$$U_g w_0 P/P = g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 \ .$$

For $u \in U_g$, we have a unique element $\alpha(g, u) \in N_0T$ such that $guw_0N = \alpha(g, u)uw_0N$. We consider the map

$$\alpha_{g,0} : N_0 \to \mathrm{End}^{cont}_o(M)$$
$$\alpha_{g,0}(u) = \mathrm{Res}(1_{\mathcal{C}_0}) \circ \alpha(g, u) \circ \mathrm{Res}(1_{\mathcal{C}_0}) \text{ for } u \in U_g \text{ and } \alpha_{g,0}(u) = 0 \text{ otherwise.}$$

The module M is Hausdorff complete but not compact, also we introduce a notion of integrability with respect to a special family $\mathfrak{C}$ of compact subsets $C \subset M$, i.e. satisfying:

$\mathfrak{C}(1)$ Any compact subset of a compact set in $\mathfrak{C}$ also lies in $\mathfrak{C}$.

$\mathfrak{C}(2)$ If $C_1, C_2, \ldots, C_n \in \mathfrak{C}$ then $\bigcup_{i=1}^n C_i$ is in $\mathfrak{C}$, as well.

$\mathfrak{C}(3)$ For all $C \in \mathfrak{C}$ we have $N_0 C \in \mathfrak{C}$.

$\mathfrak{C}(4)$ $M(\mathfrak{C}) := \bigcup_{C \in \mathfrak{C}} C$ is an étale $o[P_+]$-submodule of M.

A map from $M(\mathfrak{C})$ to M is called $\mathfrak{C}$-continuous if its restriction to any $C \in \mathfrak{C}$ is continuous. The o-module $\mathrm{Hom}_o^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ of $\mathfrak{C}$-continuous o-linear homomorphisms from $M(\mathfrak{C})$ to M with the $\mathfrak{C}$-open topology, is a topological complete o-module.

For $s \in T_{++}$, the open compact subgroups $N_k = s^k N_0 s^{-k} \subset N$ for $k \in \mathbb{Z}$, form a decreasing sequence of union N and intersection $\{1\}$. A map $F\colon N_0 \to \mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ is called $(s, \mathrm{res}, \mathfrak{C})$-integrable if the limit

$$\int_{N_0} F d\,\mathrm{res} := \lim_{k\to\infty} \sum_{u \in J(N_0/N_k)} F(u) \circ \mathrm{res}(1_{uN_k}) \ ,$$

where $J(N_0/N_k) \subseteq N_0$, for any $k \in \mathbb{N}$, is a set of representatives for the cosets in N_0/N_k, exists in $\mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ and is independent of the choice of the sets $J(N_0/N_k)$. We denote by $\mathcal{H}_{g, J(N_0/N_k)}$ the sum in the right hand side when $F = \alpha_{g,0}(.)|_{M(\mathfrak{C})}$.

Proposition 1.6. *For all $g \in N_0 \overline{P} N_0$, the map $\alpha_{g,0}(.)|_{M(\mathfrak{C})}\colon N_0 \to \mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ is (s,* res*, $\mathfrak{C}$)-integrable when*

$\mathfrak{C}(5)$ *For any $C \in \mathfrak{C}$ the compact subset $\psi_s(C) \subseteq M$ also lies in $\mathfrak{C}$.*

$\mathfrak{T}(1)$ *For any $C \in \mathfrak{C}$ such that $C = N_0 C$, any open $A[N_0]$-submodule $\mathcal{M}$ of M, and any compact subset $C_+ \subseteq L_+$ there exists a compact open subgroup $P_1 = P_1(C, \mathcal{M}, C_+) \subseteq P_0$ and an integer $k(C, \mathcal{M}, C_+) \geq 0$ such that*

$$s^k (1 - P_1) C_+ \psi_s^k \subseteq E(C, \mathcal{M}) \qquad \textit{for any } k \geq k(C, \mathcal{M}, C_+) \ .$$

The integrals $\mathcal{H}_g$ of $\alpha_{g,0}(.)|_{M(\mathfrak{C})}$ satisfy the relations H1, H2, H3, when they belong to $\mathrm{End}_A(M(\mathfrak{C}))$*, and when*

$\mathfrak{C}(6)$ *For any $C \in \mathfrak{C}$ the compact subset $\varphi_s(C) \subseteq M$ also lies in $\mathfrak{C}$.*

$\mathfrak{T}(2)$ *Given a set $J(N_0/N_k) \subset N_0$ of representatives for cosets in N_0/N_k, for $k \geq 1$, for any $x \in M(\mathfrak{C})$ and $g \in N_0 \overline{P} N_0$ there exists a compact A-submodule $C_{x,g} \in \mathfrak{C}$ and a positive integer $k_{x,g}$ such that $\mathcal{H}_{g, J(N_0/N_k)}(x) \subseteq C_{x,g}$ for any $k \geq k_{x,g}$.*

When $\mathfrak{C}$ satisfies $\mathfrak{C}(1), \ldots, \mathfrak{C}(6)$ and the technical properties $\mathfrak{T}(1), \mathfrak{T}(2)$ are true, we obtain a G-equivariant sheaf on G/P with sections on $\mathcal{C}_0$ equal to $M(\mathfrak{C})$.

1.8 Main theorem

Let M be an étale T_+-module M over $\Lambda_{\ell_\alpha}(N_0)$ with the weak topology and let $s \in T_{++}$. We have the natural T_+-equivariant quotient map

$$\ell_M : M \to D = \mathcal{O}_{\mathcal{E},\alpha} \otimes_{\ell_\alpha} M \quad , \quad m \mapsto 1 \otimes m$$

from M to $D = \mathbb{D}(M) \in \mathcal{M}_{\mathcal{O}_{\mathcal{E},\alpha}}(T_+)$, of T_+-equivariant section

$$\iota_D : D \to M = \Lambda_{\ell_\alpha}(N_0) \otimes_{\iota_\alpha} D \quad , \quad d \mapsto 1 \otimes d \ .$$

We note that $o[N_0]\iota_D(D)$ is dense in M. A lattice D_0 in D is a $\Lambda(\mathbb{Z}_p)$-submodule generated by a finite set of generators of D over $\mathcal{O}_{\mathcal{E}}$. When D is killed by a power of p, the o-module

$$M_s^{bd}(D_0) := \{m \in M \mid \ell_M(\psi_s^k(u^{-1} m)) \in D_0 \text{ for all } u \in N_0 \text{ and } k \in \mathbb{N}\}$$

of M is compact and is a $\Lambda(N_0)$-module. Let $\mathfrak{C}_s$ be the family of compact subsets of M contained in $M_s^{bd}(D_0)$ for some lattice D_0 of D, and let $M_s^{bd} = \cup_{D_0} M_s^{bd}(D_0)$ the union being taken over all lattices D_0 in D. In general, M is p-adically complete, M/p^nM is an étale T_+-module over $\Lambda_{\ell_\alpha}(N_0)$, and $D/p^nD = \mathbb{D}(M/p^nM)$. We denote by $p_n : M \to M/p^nM$ the reduction modulo p^n, and by $\mathfrak{C}_{s,n}$ the family of compact subsets constructed above for M/p^nM. We define the family $\mathfrak{C}_s$ of compact subsets $C \subset M$ such that $p_n(C) \in \mathfrak{C}_{s,n}$ for all $n \geq 1$, and the o-module M_s^{bd} of $m \in M$ such that the set of $\ell_M(\psi_s^k(u^{-1}m))$ for $k \in \mathbb{N}, u \in N_0$ is bounded in D for the weak topology.

By reduction to the easier case where M is killed by a power of p, we show that $\mathfrak{C}_s$ satisfies $\mathfrak{C}(1), \ldots, \mathfrak{C}(6)$ and that the technical properties $\mathfrak{T}(1), \mathfrak{T}(2)$ are true.

Proposition 1.7. *Let M be an étale T_+-module M over $\Lambda_{\ell_\alpha}(N_0)$ and let $s \in T_{++}$.*

(i) M_s^{bd} is a dense $\Lambda(N_0)[T_+]$-étale submodule of M containing $\iota_D(D)$.

(ii) For $g \in N_0\overline{P}N_0$, the $(s, \mathrm{res}, \mathfrak{C}_s)$-integrals $\mathcal{H}_{g,s}$ of $\alpha_{g,0}|_{M_s^{bd}}$ exist, lie in $\mathrm{End}_o(M_s^{bd})$, and satisfy the relations H1, H2, H3.

(iii) For $s_1, s_2 \in T_{++}$, there exists $s_3 \in T_{++}$ such that $M_{s_3}^{bd}$ contains $M_{s_1}^{bd} \cup M_{s_2}^{bd}$ and $\mathcal{H}_{g,s_1} = \mathcal{H}_{g,s_2}$ on $M_{s_1}^{bd} \cap M_{s_2}^{bd}$.

The endomorphisms $\mathcal{H}_{g,s} \in \mathrm{End}_o(M_s^{bd})$ induce endomorphisms of $\cap_{s\in T_{++}} M_s^{bd}$ and of $\cup_{s\in T_{++}} M_s^{bd} = \sum_{s\in T_{++}} M_s^{bd}$ satisfying the relations H1, H2, H3. Moreover $\cup_{s\in T_{++}} M_s^{bd}$ and $\cap_{s\in T_{++}} M_s^{bd}$ are $\Lambda(N_0)[T_+]$-étale submodules of M containing $\iota_D(D)$. Our main theorem is the following:

Theorem 1.8. *There are faithful functors*

$$\mathbb{Y}_\cap,\ (\mathbb{Y}_s)_{s\in T_{++}},\ \mathbb{Y}_\cup : \mathcal{M}^{et}_{\mathcal{O}_{\mathcal{E},\alpha}}(T_+) \longrightarrow \text{ G-equivariant sheaves on } G/P\ ,$$

sending $D = \mathbb{D}(M)$ to a sheaf with sections on $\mathcal{C}_0$ equal to the dense $\Lambda(N_0)[T_+]$-submodules of M

$$\bigcap_{s\in T_{++}} M_s^{bd}, \quad (M_s^{bd})_{s\in T_{++}}, \text{ and } \bigcup_{s\in T_{++}} M_s^{bd}\ ,$$

respectively.

When $G = GL(2, \mathbb{Q}_p)$, the sheaves $\mathbb{Y}_s(D)$ are all equal to the G-equivariant sheaf on $G/P \simeq \mathbb{P}^1(\mathbb{Q}_p)$ of global sections $D \boxtimes \mathbb{P}^1$ constructed by Colmez. When the root system of G is irreducible of rank > 1, we check that $\cup_{s\in T_{++}} M_s^{bd}$ is never equal to M.

1.9 Structure of the paper

In section 2, we consider a general commutative (unital) ring A and A-modules M with two endomorphisms ψ, φ such that $\psi \circ \varphi = \mathrm{id}$. We show that the induction functor $\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}} = \varprojlim_\psi$ is exact and that the module $A[\mathbb{Z}] \otimes_{\mathbb{N},\varphi} M$ is isomorphic to the subrepresentation of $\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M) = \varprojlim_\psi M$ generated by the elements of the form $(\varphi^k(m))_{k\in\mathbb{N}}$.

In section 3, we consider a general monoid $P_+ = N_0 \rtimes L_+$ contained in a group P with the property that N_0 is a group such that $tN_0t^{-1} \subset N_0$ has a finite index for all $t \in L_+$ and we study the étale $A[P_+]$-modules M. We show that the inverse monoid $P_- = L_-N_0 \subset P$ acts on M, the inverse of $t \in L_+$ acting by the left inverse ψ_t of the action φ_t of t with kernel $\sum u\varphi_t(M)$ for $u \in N_0$ not in tN_0t^{-1}. We add the hypothesis that L_+ contains a central element s such that the sequence $(s^kN_0s^{-k})_{k\in\mathbb{Z}}$ is decreasing of trivial intersection, of union a group N, and that $P = N \rtimes L$ is the semi-direct product of N and of $L = \cup_{k\in\mathbb{N}} L_-s^k$. An $A[P_+]$-submodule of M is étale if and only if it is stable by ψ_s. The representation M^P of P induced by $M|_{P_-}$, restricted to N is the representation induced from $M|_{N_0}$, and restricted to $s^{\mathbb{Z}}$ is the representation $\varprojlim_{\psi_s} M$ induced from $M|_{s^{-\mathbb{N}}}$. The natural $A[P_+]$-embedding $M \to M^P$ generates a subrepresentation M_c^P of M^P isomorphic to $A[P] \otimes_{A[P_+]} M$. When N is a locally profinite group and N_0 an open compact subgroup, we show the existence and the uniqueness of a unit-preserving $A[P_+]$-map res : $C^\infty(N_0,A)\to \mathrm{End}_A(M)$, we extend it uniquely to an $A[P]$-map Res : $C^\infty(N,A)\to \mathrm{End}_A(M^P)$, and we prove our first theorem: the equivalence between the P-equivariant sheaves of A-modules on N and the étale $A[P_+]$-modules on N_0.

In section 4, we suppose that A is a linearly topological commutative ring, that P is a locally profinite group and that M is a complete linearly topological A-module with a continuous étale action of P_+ such that the action of P_- is also continuous, or equivalently ψ_s is continuous (we say that M is a topologically étale module). Then M^P is complete for the compact-open topology and Res is a measure on N with values in the algebra E^{cont} of continuous endomorphisms of M^P. We show that E^{cont} is a complete topological ring for the topology defined by the ideals $E^{cont}_{\mathcal{L}}$ of endomorphisms with image in an open A-submodule $\mathcal{L}\subset M^P$, and that any continuous map $N\to E^{cont}$ with compact support can be integrated with respect to Res.

In section 5, we introduce a locally profinite group G containing P as a closed subgroup with compact quotient set G/P, such that the double cosets $P\backslash G/P$ admit a finite system W of representatives normalizing L, of image in $N_G(L)/L$ equal to a group, and the image $\mathcal{C}=Pw_0P/P$ in G/P of a double coset (with $w_0\in W$) is open dense and homeomorphic to N by the map $n\mapsto nw_0P/P$. We show that any compact open subset of G/P is a finite disjoint union of $g^{-1}Uw_0P/P$ for $g\in G$ and $U\subset N$ a compact open subgroup. We prove the basic result that the G-equivariant sheaves of A-modules on G/P identify with modules over the skew group ring $C^\infty(G/P,A)\#G$, or with non-degenerate modules over a (non unital) subring $\mathcal{A}$, and that an étale $A[P_+]$-module M endowed with endomorphisms $\mathcal{H}_g\in\mathrm{End}_A(M)$, for $g\in N_0\overline{P}N_0$, satisfying certain relations H1, H2, H3, gives rise to a non-degenerate $\mathcal{A}$-module. For $g\in G$ we denote $N_g\subset N$ such that $N_gw_0P/P=g^{-1}\mathcal{C}\cap\mathcal{C}$. We study the map α from the set of (g,u) with $g\in G$ and $u\in N_g$ to P defined by $guw_0N=\alpha(g,u)uw_0N$. In particular, we show the cocycle relation $\alpha(gh,u)=\alpha(g,h.u)\alpha(h,u)$ when each term makes sense. When M is compact, then M^P is compact and the action of P on M^P induces a continuous map $P\to E^{cont}$. We show that the A-linear map $\mathcal{A}\to E^{cont}$ given by the integrals of $\alpha(g,.)f(.)$ with respect to Res, for $f\in C^\infty_c(N_g,A)$, is multiplicative. As explained above, we obtain a G-equivariant sheaf of A-modules on G/P with sections M on $\mathcal{C}_0$.

In section 6, we do not suppose that M is compact and we introduce the notion of $(s,\mathrm{res},\mathfrak{C})$-integrability for a special family $\mathfrak{C}$ of compact subsets of M. We give an $(s,\mathrm{res},\mathfrak{C})$-integrability criterion for the function $\alpha_{g,0}(u)=\mathrm{Res}(1_{N_0})\alpha(gh,u)\,\mathrm{Res}(1_{N_0})$ on the open subset $U_g\subset N_0$ such that $U_gw_0P/P=g^{-1}\mathcal{C}_0\cap\mathcal{C}_0$, for $g\in N_0w_0Pw_0N_0$, a criterion which ensures that the integrals $\mathcal{H}_g$ of $\alpha_{g,0}$ satisfy the relations H1, H2, H3, as well as a method of reduction to the case where M is killed by a power of p. When these criterions are satisfied, as explained in section 5, one gets a G-equivariant sheaf of A-modules on G/P with sections M on $\mathcal{C}_0$.

The section 7 concerns classical (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E}}$, seen as étale $o[P_+^{(2)}]$-module D, where the upper exponent indicates that $P_+^{(2)}$ is the upper triangular monoid P_+ of $GL(2,\mathbb{Q}_p)$. Using the properties of treillis we apply the method explained in section 6 to this case and we obtain the sheaf constructed by Colmez.

In section 8 we consider the case where N_0 is a compact p-adic Lie group endowed with a continuous non-trivial homomorphism $\ell:N_0\to N_0^{(2)}$ with a section ι, that $L_*\subset L$ is a monoid acting by conjugation on N_0 and $\iota(N_0^{(2)})$, that ℓ extends to a continuous homomorphism $\ell:P_*=N_0\rtimes L_*\to P_+^{(2)}$ sending L_* to $L_+^{(2)}$ and that ι is L_* equivariant. We consider the abelian categories of étale L_*-modules finitely generated over the microlocalized ring $\Lambda_\ell(N_0)$ resp. over $\mathcal{O}_{\mathcal{E}}$ (with the action of L_* induced by ℓ). Between these categories we have the base change functors given by the natural L_*-equivariant algebra homomorphisms $\ell:\Lambda_\ell(N_0)\to\mathcal{O}_{\mathcal{E}}$ and $\iota:\mathcal{O}_{\mathcal{E}}\to\Lambda_\ell(N_0)$. We show our second theorem: the base change functors are quasi-inverse equivalences of categories. When L_* contains an open topologically finitely generated pro-p-subgroup, we show that an étale

L_*-module over $\mathcal{O}_{\mathcal{E}}$ is automatically topologically étale for the weak topology; the result extends to étale L_*-modules over $\Lambda_\ell(N_0)$, with the help of this last theorem.

In the section 9, we suppose that $\ell : P \to P^{(2)}(\mathbb{Q}_p)$ is a continuous homomorphism with $\ell(L) \subset L^{(2)}(\mathbb{Q}_p)$, and that $\iota : N^{(2)}(\mathbb{Q}_p) \to N$ is a L-equivariant section of $\ell|_N$ (as L acts on $N^{(2)}(\mathbb{Q}_p)$ via ℓ) sending $\ell(N_0)$ in N_0. The assumptions of section 8 are satisfied for $L_* = L_+$. Given an étale L_+-module M over $\Lambda_\ell(N_0)$, we exhibit a special family $\mathfrak{C}_s$ of compact subsets in M which satisfies the criterions of section 6 with $M(\mathfrak{C}_s)$ equal to a dense $\Lambda(N_0)[L_+]$-submodule $M_s^{bd} \subset M$. We obtain our third theorem: there exists a faithful functor from the étale L_+-modules over $\Lambda_\ell(N_0)$ to the G-equivariant sheaves on G/P sending M to the sheaf with sections M_s^{bd} on $\mathcal{C}_0$.

In section 10, we check that our theory applies to the group $G(\mathbb{Q}_p)$ of rational points of a split reductive group of $\mathbb{Q}_p$, to a Borel subgroup $P(\mathbb{Q}_p)$ of maximal split torus $T(\mathbb{Q}_p) = L$ and to a natural homomorphism $\ell_\alpha : P(\mathbb{Q}_p) \to P^{(2)}(\mathbb{Q}_p)$ associated to a simple root α. We obtain our main theorem: there are compatible faithful functors from the étale $T(\mathbb{Q}_p)_+$-modules D over $\mathcal{O}_{\mathcal{E}}$ (where $T(\mathbb{Q}_p)_+$ acts via α) to the $G(\mathbb{Q}_p)$-equivariant sheaves on $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ sheaves with sections $\mathbb{M}(D)_s^{bd}$ on $\mathcal{C}_0$, for all strictly dominant $s \in T(\mathbb{Q}_p)$. When the root system of G is irreducible of rank > 1, we show that $\cup_s M_s^{bd} \neq M = \mathbb{M}(D)$.

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2 Induction Ind_H^G for monoids $H \subset G$

A monoid is supposed to have a unit.

2.1 Definition and remarks

Let A be a commutative ring, let G be a monoid and let H be a submonoid of G. We denote by $A[G]$ the monoid A-algebra of G and by $\mathfrak{M}_A(G)$ the category of left $A[G]$-modules, which has no reason to be equivalent to the category of right $A[G]$-modules. One can construct $A[G]$-modules starting from $A[H]$-modules in two natural ways, by taking the two adjoints of the restriction functor $\mathrm{Res}_H^G : \mathfrak{M}_A(G) \to \mathfrak{M}_A(H)$ from G to H. For $M \in \mathfrak{M}_A(H)$ and $V \in \mathfrak{M}_A(G)$ we have the isomorphism

$$\mathrm{Hom}_{A[G]}(A[G] \otimes_{A[H]} M, V) \xrightarrow{\simeq} \mathrm{Hom}_{A[H]}(M, V)$$

and the isomorphism

$$\mathrm{Hom}_{A[G]}(V, \mathrm{Hom}_{A[H]}(A[G], M)) \xrightarrow{\simeq} \mathrm{Hom}_{A[H]}(V, M) \ . \tag{1}$$

For monoid algebras, restriction of homomorphisms induces the identification

$$\mathrm{Hom}_{A[H]}(A[G], M) = \mathrm{Ind}_H^G(M)$$

where $\mathrm{Ind}_H^G(M)$ is formed by the functions

$$f : G \to M \text{ such that } f(hg) = hf(g) \text{ for any } \ h \in H, g \in G \ ;$$

the group G acts by right translations, $gf(x) = f(xg)$ for $g, x \in G$. The isomorphism (1) pairs ϕ of the left side and Φ of the right side satisfying ([14] I.5.7)

$$\phi(v)(g) = \Phi(gv) \quad \text{for } (v, g) \in V \times G.$$

It is well known that the left and right adjoint functors of Res_H^G are transitive (for monoids $H \subset K \subset G$), the left adjoint is right exact, the right adjoint is left exact.

We observe important differences between monoids and groups:

1) The binary relation $g \sim g'$ if $g \in Hg'$ is not symmetric, there is no "quotient space" $H \backslash G$, no notion of function with finite support modulo H in $\mathrm{Ind}_H^G(M)$.

2) When $hM = 0$ for some $h \in H$ such that $hG = G$, then $\mathrm{Ind}_H^G(M) = 0$. Indeed $f(hg) = hf(g)$ implies $f(hg) = 0$ for any $g \in G$.

3) When G is a group generated, as a monoid, by H and the inverse monoid $H^{-1} := \{h \in G \mid h^{-1} \in H\}$, and when M in an $A[H]$-module such that the action of any element $h \in H$ on M is invertible, then $f(g) = gf(1)$ for all $g \in G$ and $f \in \mathrm{Ind}_H^G(M)$. This can be seen by induction on the minimal number $m \in \mathbb{N}$ such that $g = g_1 \ldots g_m$ with $g_i \in H \cup H^{-1}$. Then $g_1 \in H$ implies $f(g) = g_1 f(g_2 \ldots g_m)$, and $g_1 \in H^{-1}$ implies $f(g_2 \ldots g_m) = f(g_1^{-1} g_1 g_2 \ldots g_m) = g_1^{-1} f(g)$. The representation $\mathrm{Ind}_H^G(M)$ is isomorphic by $f \mapsto f(1)$ to the natural representation of G on M.

2.2 From $\mathbb{N}$ to $\mathbb{Z}$

An A-module with an endomorphism φ is equivalent to an $A[\mathbb{N}]$-module, φ being the action of $1 \in \mathbb{N}$, and an A-module with an automorphism φ is equivalent to an $A[\mathbb{Z}]$-module. When φ is bijective, $A[\mathbb{Z}] \otimes_{A[\mathbb{N}]} M$ and $\mathrm{Ind}_{\mathbb{N}}^{\mathbb{Z}}(M)$ are isomorphic to M.

In general, $A[\mathbb{Z}] \otimes_{A[\mathbb{N}]} M$ is the limit of an inductive system and $\mathrm{Ind}_{\mathbb{N}}^{\mathbb{Z}}(M)$ is the limit of a projective system. The first one is interesting when φ is injective, the second one when φ is surjective.

For $r \in \mathbb{N}$ let $M_r = M$. The general element of M_r is written x_r with $x \in M$. Let $\varinjlim\,(M, \varphi)$ be the quotient of $\sqcup_{r \in \mathbb{N}} M_r$ by the equivalence relation generated by $\varphi(x)_{r+1} \equiv x_r$, with the isomorphism induced by the maps $x_r \to \varphi(x)_r \ : \ M_r \to M_r$ of inverse induced by the maps $x_r \to x_{r+1} \ : \ M_r \to M_{r+1}$. Let $x \mapsto [x] : \mathbb{Z} \to A[\mathbb{Z}]$ be the canonical map. The maps $x_r \to [-r] \otimes x \ : \ M_r \to A[\mathbb{Z}] \otimes_{A[\mathbb{N}]} M$ for $r \in \mathbb{N}$ induce an isomorphism of $A[\mathbb{Z}]$-modules

$$\varinjlim\, M \quad \to \quad A[\mathbb{Z}] \otimes_{A[\mathbb{N}]} M \quad .$$

Let

$$(2) \qquad \varprojlim\, M := \{x = (x_m)_{m \in \mathbb{N}} \in \prod_{m \in \mathbb{N}} M : \varphi(x_{m+1}) = x_m \text{ for any } \ m \in \mathbb{N}\}$$

seen as an $A[\mathbb{Z}]$-module via the automorphism

$$x \mapsto (\varphi(x_0), x_0, x_1, \ldots) = (\varphi(x_0), \varphi(x_1), \varphi(x_2) \ldots)$$

of inverse $x \mapsto (x_1, x_2, \ldots)$. The map $f \mapsto (f(-m))_{m \in \mathbb{N}}$ is an isomorphism of $A[\mathbb{Z}]$-modules

$$\mathrm{Ind}_{\mathbb{N}}^{\mathbb{Z}}(M) \quad \to \quad \varprojlim\, M \quad .$$

The submodules of M

$$M^{\varphi^\infty = 0} \ := \ \cup_{k \in \mathbb{N}} M^{\varphi^k = 0} \quad , \quad \varphi^\infty(M) \ := \ \cap_{n \in \mathbb{N}} \varphi^n(M)$$

are stable by φ. The inductive limit sees only the quotient $M/M^{\varphi^\infty = 0}$ and the projective limit sees only the submodule $\varphi^\infty(M)$,

$$\varinjlim\, M \ = \ \varinjlim\,(M/M^{\varphi^\infty = 0}) \quad , \quad \varprojlim\, M \ = \ \varprojlim\,(\varphi^\infty(M)) \quad .$$

Lemma 2.1. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of A-modules with an endomorphism φ.*

a) The sequence

$$0 \to \varinjlim M_1 \to \varinjlim M_2 \to \varinjlim M_3 \to 0$$

is exact.

b) When φ is surjective on M_1, the sequence

$$0 \to \varprojlim M_1 \to \varprojlim M_2 \to \varprojlim M_3 \to 0$$

is exact.

Proof. This is a standard fact on inductive and projective limits. □

2.3 (φ, ψ)-modules

Let M be an A-module with two endomorphisms ψ, φ such that $\psi \circ \varphi = 1$. Then ψ is surjective, φ is injective, the endomorphism $\varphi \circ \psi$ is a projector of M giving the direct decomposition

$$M = \varphi(M) \oplus M^{\psi=0} \quad , \quad m \;=\; (\varphi \circ \psi)(m) \;+\; m^{\psi=0} \tag{3}$$

for $m \in M$ and $m^{\psi=0} \in M^{\psi=0}$ the kernel of ψ. We consider the representation of $\mathbb{Z}$ induced by (M, ψ) as in (2.2),

$$\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M) \;\simeq\; \varprojlim_{\psi} M \;.$$

On the induced representation ψ is an isomorphism and we introduce $\varphi := \psi^{-1}$. As ψ is surjective on M, the map $\mathrm{ev}_0 \;:\; \mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M) \to M$, corresponding to the map

$$\varprojlim_{\psi} M \;\to\; M, \quad (x_m)_{m \in \mathbb{N}} \mapsto x_0$$

is surjective. A splitting is the map $\sigma_0 \;:\; M \to \mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$ corresponding to

$$M \;\to\; \varprojlim_{\psi} M \;, \quad x \mapsto (\varphi^m(x))_{m \in \mathbb{N}} \quad . \tag{4}$$

Obviously ev_0 is ψ-equivariant, σ_0 is φ-equivariant, $\mathrm{ev}_0 \circ \sigma_0 = \mathrm{id}_M$, and

$$R_0 \quad := \quad \sigma_0 \circ \mathrm{ev}_0 \;\in \mathrm{End}_A(\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M))$$

is an idempotent of image $\sigma_0(M)$.

Definition 2.2. *The representation of $\mathbb{Z}$ compactly induced from (M, ψ) is the subrepresentation $\text{c-}\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$ of $\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$ generated by the image of $\sigma_0(M)$.*

We note that, for any $k \geq 1$, the endomorphisms ψ^k, φ^k satisfy the same properties as ψ, φ because $\psi^k \circ \varphi^k = 1$. For any integer $k \geq 0$, the value at k is a surjective map $\mathrm{ev}_k \;:\; \mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M) \to M$, corresponding to the map

$$\varprojlim_{\psi} M \;\to\; M, \quad (x_m)_{m \in \mathbb{N}} \mapsto x_k \tag{5}$$

of splitting $\sigma_k \;:\; M \to \mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$ corresponding to the map

$$M \;\to\; \varprojlim_{\psi} M \;, \quad x \mapsto (\psi^k(x), \ldots, \psi(x), x, \varphi(x), \varphi^2(x), \ldots) \quad . \tag{6}$$

The following relations are immediate:

$$\mathrm{ev}_k \ = \ \mathrm{ev}_0 \circ \varphi^k \ = \ \psi \circ \mathrm{ev}_{k+1} \ = \ \mathrm{ev}_{k+1} \circ \psi \ ,$$

$$\sigma_k \ = \ \psi^k \circ \sigma_0 \ = \ \sigma_{k+1} \circ \varphi \ = \ \varphi \circ \sigma_{k+1} \ .$$

We deduce that $\sigma_k(M) \subset \sigma_{k+1}(M)$. Since $\sigma_k(M)$ is φ-invariant we have

$$\text{(7)} \qquad \text{c-Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M) \ = \ \sum_{k\in\mathbb{N}} \psi^k(\sigma_0(M)) \ = \ \sum_{k\in\mathbb{N}} \sigma_k(M) \ = \ \bigcup_{k\in\mathbb{N}} \sigma_k(M) \ .$$

In $\varprojlim_{\psi}(M)$ the subspace of $(x_m)_{m\in\mathbb{N}}$ such that $x_{k+r} = \varphi^k(x_r)$ for all $k \in \mathbb{N}$ and for some $r \in \mathbb{N}$, is equal to $\text{c-Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$. The definition of $\text{c-Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$ is functorial. We get a functor $\text{c-Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}$ from the category of A-modules with two endomorphisms ψ, φ such that $\psi \circ \varphi = 1$ (a morphism commutes with ψ and with φ) to the category of $A[\mathbb{Z}]$-modules.

Proposition 2.3. *The map*

$$\begin{aligned} A[\mathbb{Z}] \otimes_{A[\mathbb{N}],\varphi} M \ &\to \ \mathrm{Hom}_{A[\mathbb{N}],\psi}(A[\mathbb{Z}], M) \ = \ \mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M) \\ [k] \otimes m \ &\mapsto \ (\varphi^k \circ \sigma_0)(m) \end{aligned}$$

induces an isomorphism from the tensor product $A[\mathbb{Z}] \otimes_{A[\mathbb{N}],\varphi} M$ to the compactly induced representation $\text{c-Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$ *(note that ψ and φ appear).*

Proof. From (3) and the relations between the σ_k we have for $m \in M, k \in \mathbb{N}, k \geq 1$,

$$\sigma_k(m) = \sigma_{k-1}(\psi(m)) + \sigma_k(m^{\psi=0}) \ .$$

By induction $\sum_{k\in\mathbb{N}} \sigma_k(M) = \sigma_0(M) + \sum_{k\geq 1} \sigma_k(M^{\psi=0})$. Using (6) one checks that the sum is direct, hence by (7),

$$\text{c-Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M) \ = \ \sigma_0(M) \oplus (\oplus_{k\geq 1} \sigma_k(M^{\psi=0})) \ .$$

On the other hand, one deduces from (3) that

$$A[\mathbb{Z}] \otimes_{A[\mathbb{N}],\varphi} M \quad = \quad ([0] \otimes M) \oplus (\oplus_{k\geq 1}([-k] \otimes M^{\psi=0})) \ .$$

$\square$

With the lemma 2.1 we deduce:

Corollary 2.4. *The functor* $\text{c-Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}$ *is exact.*

We have two kinds of idempotents in $\mathrm{End}_A(\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M))$, for $k \in \mathbb{N}$, defined by

$$\text{(8)} \qquad R_k \ := \ \sigma_0 \circ \varphi^k \circ \psi^k \circ \mathrm{ev}_0 \quad , \quad R_{-k} \ := \ \psi^k \circ R_0 \circ \varphi^k = \sigma_k \circ \mathrm{ev}_k \quad .$$

The first ones are the images of the idempotents $r_k := \varphi^k \circ \psi^k \in \mathrm{End}_A(M)$ via the ring homomorphism

$$\text{(9)} \qquad \mathrm{End}_A(M) \ \to \ \mathrm{End}_A \mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M) \quad , \quad f \mapsto \sigma_0 \circ f \circ \mathrm{ev}_0 \quad .$$

The second ones give an isomorphism from $\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$ to the limit of the projective system $(\sigma_k(M), R_{-k} : \sigma_{k+1}(M) \to \sigma_k(M))$.

Lemma 2.5. *The map* $f \mapsto (R_{-k}(f))_{k\in\mathbb{N}}$ *is an isomorphism from* $\mathrm{Ind}_{\mathbb{N},\psi}^{\mathbb{Z}}(M)$ *to*

$$\varprojlim_{R_{-k}}(\sigma_k(M)) \quad := \quad \{(f_k)_{k\in\mathbb{N}} \mid f_k \in \sigma_k(M)\ ,\ f_k = R_{-k}(f_{k+1}) \quad \text{for } k\in\mathbb{N}\ \}$$

of inverse $(f_k)_{k\in\mathbb{N}} \ \to f$ *with* $\mathrm{ev}_k(f) = \mathrm{ev}_k(f_k)$.

Remark 2.6. As φ is injective, its restriction to $\cap_{n\in\mathbb{N}}\varphi^n(M)$ is an isomorphism and the following $A[\mathbb{Z}]$-modules are isomorphic (section 2.2):

$$\mathrm{Ind}_{\mathbb{N},\varphi}^{\mathbb{Z}}(M) \quad \simeq \quad \varprojlim_{\varphi} M \quad \simeq \quad \cap_{n\in\mathbb{N}}\varphi^n(M)\ .$$

As ψ is surjective, its action on the quotient $M/M^{\psi^\infty=0}$ is bijective and the following $A[\mathbb{Z}]$-modules are isomorphic (section 2.2):

$$A[\mathbb{Z}]\otimes_{A[\mathbb{N}],\psi} M \quad \simeq \quad \varinjlim_{\psi} M \quad \simeq \quad M/M^{\psi^\infty=0}.$$

Remark 2.7. *When the A-module M is noetherian, a ψ-stable submodule of M which generates M as a φ-module is equal to M.*

Proof. Let N be a submodule of M. As M is noetherian there exists $k\in\mathbb{N}$ such that the φ-stable submodule of M generated by N is the submodule $N_k \subset M$ generated by $N, \varphi(N), \ldots, \varphi^k(N)$. When N is ψ-stable we have $\psi^k(N_k) = N$ and when N generates M as a φ-module we have $M = N_k$. In this case, $M = \psi^k(M) = \psi^k(N_k) = N$. □

3 Etale P_+-modules

Let $P = N \rtimes L$ be a semi-direct product of an invariant subgroup N and of a group L and let $N_0 \subset N$ be a subgroup of N. For any subgroups $V \subset U \subset N$, the symbol $J(U/V) \subset U$ denotes a set of representatives for the cosets in U/V.

The group P acts on N by

$$(b = nt, x) \to b.x = ntxt^{-1}$$

for $n, x \in N$ and $t \in L$. The P-stabilizer $\{b\in P \mid b.N_0 \subset N_0\}$ of N_0 is a monoid

$$P_+ \ = \ N_0L_+$$

where $L_+ \subset L$ is the L-stabilizer of N_0. Its maximal subgroup $\{b\in P \mid b.N_0 = N_0\}$ is the intersection $P_0 = N_0 \rtimes L_0$ of P_+ with the inverse monoid $P_- = L_-N_0$ where L_- is the inverse monoid of L_+ and L_0 is the maximal subgroup of L_+.

We suppose that the subgroup $t.N_0 = tN_0t^{-1} \subset N_0$ has a finite index, for all $t\in L_+$. Let A be a commutative ring and let M be an $A[P_+]$-module, equivalently an $A[N_0]$-module with a semilinear action of L_+.

The action of $b\in P_+$ on M is denoted by φ_b. If $b\in P_0$ then φ_b is invertible and we also write $\varphi_b(m) = bm\ ,\ \varphi_b^{-1}(m) = b^{-1}m$ for $m\in M$. The action $\varphi_t \in \mathrm{End}_A(M)$ of $t\in L_+$ is $A[N_0]$-semilinear:

$$\varphi_t(xm) = \varphi_t(x)\varphi_t(m) \quad \text{for} \quad x\in A[N_0],\ m\in M\ . \tag{10}$$

3.1 Etale module M

The group algebra $A[N_0]$ is naturally an $A[P_+]$-module. For $t \in L_+$ the map φ_t is injective of image $A[tN_0t^{-1}]$, and

$$A[N_0] = \oplus_{u \in J(N_0/tN_0t^{-1})} uA[tN_0t^{-1}] \ .$$

Definition 3.1. *We say that M is étale if, for any $t \in L_+$, the map φ_t is injective and*

$$M = \oplus_{u \in J(N_0/tN_0t^{-1})} \ u \ \varphi_t(M) \ . \tag{11}$$

An equivalent formulation is that, for any $t \in L_+$, the linear map

$$A[N_0] \otimes_{A[N_0],\varphi_t} M \to M \quad , \quad x \otimes m \mapsto x\varphi_t(m)$$

is bijective. For M étale and $t \in L_+$, let $\psi_t \in \mathrm{End}_A(M)$ be the unique canonical left inverse of φ_t of kernel

$$M^{\psi_t=0} = \sum_{u \in (N_0 - tN_0t^{-1})} u\varphi_t(M) \ .$$

The trivial action of P_+ on M is not étale, and obviously the restriction to P_+ of a representation of P is not always étale.

Lemma 3.2. *Let M be an étale $A[P_+]$-module. For $t \in L_+$, the kernel $M^{\psi_t=0}$ is an $A[tN_0t^{-1}]$-module, the idempotents in $\mathrm{End}_A M$*

$$(u \circ \varphi_t \circ \psi_t \circ u^{-1})_{u \in J(N_0/tN_0t^{-1})}$$

are orthogonal of sum the identity. Any $m \in M$ can be written

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(m_{u,t}) \tag{12}$$

for unique elements $m_{u,t} \in M$, equal to $m_{u,t} = \psi_t(u^{-1}m)$.

Proof. The kernel $M^{\psi_t=0}$ is an $A[tN_0t^{-1}]$-module because $N_0 - tN_0t^{-1}$ is stable by left multiplication by tN_0t^{-1}. The endomorphism $\varphi_t \circ \psi_t$ is an idempotent because $\psi_t \circ \varphi_t = \mathrm{id}_M$. Then apply (11) and notice that $m \in M$ is equal to

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} (u \circ \varphi_t \circ \psi_t \circ u^{-1})(m) \ .$$

$\square$

Remark 3.3. 1) An $A[P_+]$-module M is étale when, for any $t \in L_+$, the action φ_t of t admits a left inverse $f_t \in \mathrm{End}_A M$ such that the idempotents $(u \circ \varphi_t \circ f_t \circ u^{-1})_{u \in J(N_0/tN_0t^{-1})}$ are orthogonal of sum the identity. The endomorphism f_t is the canonical left inverse ψ_t.

2) The $A[P_+]$-module $A[N_0]$ is étale. As $A[N_0]$ is a left and right free $A[tN_0t^{-1}]$-module of rank $[N_0 : tN_0t^{-1}]$ we have for $x \in A[N_0]$,

$$x = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(x_{u,t}) = \sum_{u \in J(N_0/tN_0t^{-1})} \varphi_t(x'_{u,t})u^{-1}$$

where $x_{u,t} = \psi_t(u^{-1}x)$, $x'_{u,t} = \psi_t(xu)$ and ψ_t is the left inverse of φ_t of kernel

$$\sum_{u \in N_0 - tN_0t^{-1}} uA[tN_0t^{-1}] = \sum_{u \in N_0 - tN_0t^{-1}} A[tN_0t^{-1}]u^{-1}.$$

Let M be an étale $A[P_+]$-module and $t \in L_+$. We denote $m \mapsto m^{\psi_t=0} : M \to M^{\psi_t=0}$ the projector $\mathrm{id}_M - \varphi_t \circ \psi_t$ along the decomposition $M = \varphi_t(M) \oplus M^{\psi_t=0}$.

Lemma 3.4. *Let $x \in A[N_0]$ and $m \in M$. We have*

$$\psi_t(\varphi_t(x)m) \ = \ x\psi_t(m) \ , \quad \psi_t(x\varphi_t(m)) \ = \ \psi_t(x)m \ ,$$
$$(\varphi_t(x)m)^{\psi_t=0} \ = \ \varphi_t(x)(m^{\psi_t=0}) \quad , \quad (x\varphi_t(m))^{\psi_t=0} \ = \ x^{\psi_t=0}\varphi_t(m) \quad .$$

Proof. We multiply $m = (\varphi_t \circ \psi_t)(m) + m^{\psi_t=0}$ on the left by $\varphi_t(x)$. By the $A[N_0]$-semilinearity of φ_t we have $\varphi_t(x)m = \varphi_t(x\psi_t(m)) + \varphi_t(x)(m^{\psi_t=0})$. As $M^{\psi_t=0}$ is an $A[tN_0t^{-1}]$-module, the uniqueness of the decomposition implies $\psi_t(\varphi_t(x)m) = x\psi_t(m)$ and $(\varphi_t(x)m)^{\psi_t=0} = \varphi_t(x)(m^{\psi_t=0})$.

We multiply $x = (\varphi_t \circ \psi_t)(x) + x^{\psi_t=0}$ on the right by $\varphi_t(m)$. By the semilinearity of φ_t we have $x\varphi_t(m) = \varphi_t(\psi_t(x)m) + x^{\psi_t=0}\varphi_t(m)$. As $A[N_0]^{\psi_t=0}\varphi_t(M) = M^{\psi_t=0}$ the uniqueness of the decomposition implies $\psi_t(x\varphi_t(m)) = \psi_t(x)m$, $(x\varphi_t(m))^{\psi_t=0} = x^{\psi_t=0}\varphi_t(m)$. $\square$

Lemma 3.5. *Let $x \in A[N_0]$ and $m \in M$. We have*

$$\psi_t(xm) = \sum_{u \in J(N_0/tN_0t^{-1})} \psi_t(xu)\psi_t(u^{-1}m) \ .$$

Proof. Using (12), replace m by $\sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(m_{u,t})$ in $\psi_t(xm)$. We get

$$\begin{aligned}\psi_t(xm) = \psi_t(\sum_{u \in J(N_0/tN_0t^{-1})} xu\varphi_t(m_{u,t})) &= \sum_{u \in J(N_0/tN_0t^{-1})} \psi_t(xu)m_{u,t} \\ &= \sum_{u \in J(N_0/tN_0t^{-1})} \psi_t(xu)\psi_t(u^{-1}m)\end{aligned}$$

using the first line of Lemma 3.4. $\square$

Proposition 3.6. *Let M be an étale $A[P_+]$-module. The map*

$$b^{-1} = (ut)^{-1} \ \mapsto \ \psi_b := \psi_t \circ u^{-1} \ : \ P_- \ \to \ \mathrm{End}_A(M) \quad \text{for} \quad t \in L_+ \ , \ u \in N_0 \quad ,$$

defines a canonical action of P_- on M.

Proof. We check that $\psi_{b_1b_2} = \psi_{b_2} \circ \psi_{b_1}$ for $b_1 = u_1t_1, b_2 = u_2t_2 \in P_+$. We have $\psi_{b_1b_2} = \psi_{t_1t_2} \circ (u_1t_1u_2t_1^{-1})^{-1}$ and $\psi_{b_2} \circ \psi_{b_1} = \psi_{t_2} \circ u_2^{-1} \circ \psi_{t_1} \circ u_1^{-1}$. As $u_2^{-1} \circ \psi_{t_1} = \psi_{t_1} \circ t_1u_2^{-1}t_1^{-1}$, it remains only to show $\psi_{t_2}\psi_{t_1} = \psi_{t_1t_2}$. For the sake of simplicity, we note $\varphi_i = \varphi_{t_i}, \psi_i = \psi_{t_i}$. For $m \in M$ we have $m = \varphi_1(\varphi_2 \circ \psi_2(\psi_1(m)) + \psi_1(m)^{\psi_2=0}) + m^{\psi_1=0}$. This is also

$$m = (\varphi_{t_1t_2} \circ \psi_2 \circ \psi_1)(m) + \varphi_1(\psi_1(m)^{\psi_2=0}) + m^{\psi_1=0}$$

because $\varphi_1 \circ \varphi_2 = \varphi_{t_1t_2}$. By the uniqueness of the decomposition $m = (\varphi_{t_1t_2} \circ \psi_{t_1t_2})(m) + m^{\psi_{t_1t_2}=0}$ we are reduced to show that

$$M^{\psi_{t_1t_2}=0} \quad = \quad \varphi_1(M^{\psi_2=0}) + M^{\psi_1=0} \quad .$$

It is enough to prove the inclusion $M^{\psi_{t_1t_2}=0} \ \subset \ \varphi_1(M^{\psi_2=0}) + M^{\psi_1=0}$ to get the equality because $M = \varphi_{t_1t_2}(M) \oplus V$ with V equal to any of them. Hence we want to show
(13)
$$\sum_{u \in N_0 - t_1t_2N_0(t_1t_2)^{-1}} u\varphi_{t_1t_2}(M) \ \subset \ \varphi_1(\sum_{u \in N_0 - t_2N_0t_2^{-1}} u\varphi_2(M)) \ + \sum_{u \in N_0 - t_1N_0t_1^{-1}} u\varphi_1(M) \ .$$

As $\varphi_1 \circ u \circ \varphi_2 = t_1 u t_1^{-1} \circ \varphi_{t_1 t_2}$ the right side of (13) is

$$\sum_{u \in t_1 N_0 t_1^{-1} - t_1 t_2 N_0 (t_1 t_2)^{-1}} u\varphi_{t_1 t_2}(M) + \sum_{u \in N_0 - t_1 N_0 t_1^{-1}} u\varphi_1(M) \quad .$$

As $\varphi_{t_1 t_2} = \varphi_1 \circ \varphi_2$ we have $\varphi_{t_1 t_2}(M) \subset \varphi_1(M)$. Hence (13) is true. $\square$

Lemma 3.7. *Let $f : M \to M'$ be an A-morphism between two étale $A[P_+]$-modules M and M'. Then f is P_+-equivariant if and only if f is P_--equivariant (for the canonical action of P_-).*

Proof. Let $t \in L_+$. We suppose that f is N_0-equivariant and we show that $f \circ \varphi_t = \varphi_t \circ f$ is equivalent to $f \circ \psi_t = \psi_t \circ f$. Our arguments follow the proof of ([5] Prop. II.3.4).

a) We suppose $f \circ \varphi_t = \varphi_t \circ f$. Then $f(\varphi_t(M)) = \varphi_t(f(M))$ is contained in $\varphi_t(M')$ and $f(M^{\psi_t=0}) = \sum_{u \in N_0 - tN_0t^{-1}} u\varphi_t(f(M))$ is contained in $M'^{\psi_t=0}$. By Lemma 3.2, this implies $f \circ \varphi_t \circ \psi_t = \varphi_t \circ \psi_t \circ f$. As $f \circ \varphi_t = \varphi_t \circ f$ and φ_t is injective this is equivalent to $f \circ \psi_t = \psi_t \circ f$.

b) We suppose $f \circ \psi_t = \psi_t \circ f$. Let $m \in M$. Then $f(\varphi_t(m))$ belongs to $\varphi_t(M)$ because $\varphi_t(M)$ is the subset of $x \in M$ such that $\psi_t(u^{-1}x) = 0$ for all $u \in N_0 - tN_0t^{-1}$ and we have

$$\psi_t(u^{-1} f(\varphi_t(m))) = f(\psi_t(u^{-1}(\varphi_t(m)))) \ .$$

Let $x(m) \in M$ be the element such that $f(\varphi_t(m)) = \varphi_t(x(m))$. We have

$$x(m) = \psi_t \varphi_t(x(m)) = \psi_t(f(\varphi_t(m))) = f(\psi_t \varphi_t(m)) = f(m) \ .$$

Therefore $f(\varphi_t(m)) = \varphi_t(f(m))$. $\square$

Proposition 3.8. *The category $\mathfrak{M}_A(P_+)^{et}$ of étale $A[P_+]$-modules is abelian and has a natural fully faithful functor into the abelian category $\mathfrak{M}_A(P_-)$ of $A[P_-]$-modules.*

Proof. From the proposition 3.6 and the lemma 3.7, it suffices to show that the kernel and the image of a morphism $f : M \to M'$ between two étale modules M, M', are étale. Since the ring homomorphism φ_t is flat, for $t \in L_+$, the functor $\Phi_t := A[N_0] \otimes_{A[N_0], \varphi_t} -$ sends the exact sequence

$$(14) \qquad (E) \qquad 0 \to \operatorname{Ker} f \to M \to M' \to \operatorname{Coker} f \to 0$$

to an exact sequence

$$(15) \qquad (\Phi_t(E)) \qquad 0 \to \Phi_t(\operatorname{Ker} f) \to \Phi_t(M) \to \Phi_t(M') \to \Phi_t(\operatorname{Coker} f) \to 0 \ ,$$

and the natural maps $j_- : \Phi_t(-) \to -$ define a map $\Phi_t(E) \to (E)$. The maps j_M and $j_{M'}$ are isomorphisms because M et M' are étale, hence the maps $j_{\operatorname{Ker} f}$ and $j_{\operatorname{Coker} f}$ are isomorphisms, i.e. $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are étale. $\square$

Note that a subrepresentation of an étale representation of P_+ is not necessarily étale nor stable by P_-.

Remark 3.9. *An arbitrary direct product or a projective limit of étale $A[P_+]$-modules is étale.*

Proof. Since the $A[tN_0t^{-1}]$-module $A[N_0]$ is free of finite rank, for $t \in L_+$, the tensor product $A[N_0] \otimes_{A[tN_0t^{-1}]} -$ commutes with arbitrary projective limits. $\square$

3.2 Induced representation M^P

Let P be a locally profinite group, semi-direct product $P = N \rtimes L$ of closed subgroups N, L, let $N_0 \subset N$ be an open profinite subgroup, and let s be an element of the centre $Z(L)$ of L such that $L = L_- s^{\mathbb{Z}}$ (notation of the section 3) and $(N_k := s^k N_0 s^{-k})_{k\in\mathbb{Z}}$ is a decreasing sequence of union N and trivial intersection.

As the conjugation action $L \times N \to N$ of L on N is continuous and N_0 is compact open in N, the subgroups $L_0 \subset L, P_0 \subset P$ are open and the monoids P_+, P_- are open in P.

We have

$$P = P_- s^{\mathbb{Z}} \quad = \quad s^{\mathbb{Z}} P_+$$

because, for $n \in N$ and $t \in L$, there exists $k \in \mathbb{N}$ and $n_0 \in N_0$ such that $n = s^{-k} n_0 s^k$ and $ts^{-k} \in L_-$. Thus $tn = ts^{-k} n_0 s^k \in P_- s^k$ and $(tn)^{-1} \in s^{-k} P_+$. In particular P is generated by P_+ and by its inverse P_-.

Let M be an étale left $A[P_+]$-module. We denote by φ the action of s on M and by ψ the canonical left inverse of φ, by

$$M^P \ := \ \mathrm{Ind}_{P_-}^{P}(M)$$

the $A[P]$-module induced from the canonical action of P_- on M (section 2.1).

When $f : P \to M$ is an element of M^P, the values of f on $s^{\mathbb{N}}$ determine the values of f on N and reciprocally because, for any $u \in N_0, k \in \mathbb{N}$,

$$\begin{aligned} f(s^{-k} u s^k) &= (\psi^k \circ u)(f(s^k)) \ , \\ f(s^k) &= \sum_{v \in J(N_0/N_k)} (v \circ \varphi^k)(f(s^{-k} v^{-1} s^k)) \ . \end{aligned} \tag{16}$$

The first equality is obvious from the definition of $\mathrm{Ind}_{P_-}^{P}$, the second equality is obvious by the first equality as the idempotents $(v \circ \varphi^k \circ \psi^k \circ v^{-1})_{v \in J(N_0/N_k)}$ are orthogonal of sum the identity, by the lemma 3.2.

Proposition 3.10. *a) The restriction to $s^{\mathbb{Z}}$ is an $A[s^{\mathbb{Z}}]$-equivariant isomorphism*

$$M^P \quad \to \quad \mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M) \quad .$$

b) The restriction to N is an N-equivariant bijection from M^P to $\mathrm{Ind}_{N_0}^{N}(M)$.

Proof. a) As $P = P_- s^{\mathbb{Z}}$ and $s^{-\mathbb{N}} \subset P_- \cap s^{\mathbb{Z}}$ (it is an equality if N is not trivial), the restriction to $s^{\mathbb{Z}}$ is a $s^{\mathbb{Z}}$-equivariant injective map $M^P \to \mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M)$. To show that the map is surjective, let $\phi \in \mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M)$ and $b \in P$. Then, for $b = b_- s^r$ with $b_- \in P_-, r \in \mathbf{Z}$,

$$f(b) \quad := \quad b_- \phi(s^r)$$

is well defined because the right side depends only on b, and not on the choice of (b_-, r). Indeed for two choices $b = b_- s^r = b'_- s^{r'}$ with $b_-, b'_- \in P_-, r \geq r'$ in $\mathbf{Z}$, we have

$$b_- \phi(s^r) = b'_- s^{r'-r} \phi(s^r) = b'_- \phi(s^{r'}) \quad .$$

The well defined function $b \mapsto f(b)$ on P belongs obviously to M^P and its restriction to $s^{\mathbb{Z}}$ is equal to ϕ.

b) As $P_- \cap N = N_0$ the restriction to N is an N-equivariant map $M^P \to \mathrm{Ind}_{N_0}^{N}(M)$. The map is injective because the restriction to N of $f \in M^P$ determines the restriction of f to $s^{\mathbb{N}}$ by (16) which determines f by a). We have the natural injective map

$$f \ \mapsto \ \phi_f \quad : \quad \mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M) \to M^P \to \mathrm{Ind}_{N_0}^{N}(M) \tag{17}$$

$$\phi_f(s^{-k}us^k) = (\psi^k \circ u)(f(s^k)) \quad \text{for} \quad k \in \mathbb{N}, u \in N_0 \quad ,$$

and we have the map

$$\phi \;\mapsto\; f_\phi \quad : \quad \mathrm{Ind}_{N_0}^{N}(M) \quad \to \quad \mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M)$$

defined by

$$f_\phi(s^k) \;=\; \sum_{v\in J(N_0/N_k)} (v\circ\varphi^k)(\phi(s^{-k}v^{-1}s^k)) \quad \text{for} \quad k\in\mathbb{N}.$$

Indeed the function f_ϕ satisfies $\psi(f_\phi(s^{k+1})) = f_\phi(s^k)$: since $\psi\circ u\circ\varphi^{k+1} = s^{-1}us\circ\varphi^k$ when $u\in N_1$ and is 0 otherwise, we have

$$\begin{aligned}\psi(f_\phi(s^{k+1})) &= \psi(\sum_{v\in J(N_0/N_{k+1})} (v\circ\varphi^{k+1})(\phi(s^{-k-1}v^{-1}s^{k+1}))\\ &= \sum_{v\in N_1\cap J(N_0/N_{k+1})} (s^{-1}vs\circ\varphi^k)(\phi(s^{-k-1}v^{-1}s^{k+1}))\ .\end{aligned}$$

The last term is

$$\sum_{v\in J(N_0/N_k)} (v\circ\varphi^k)(\phi(s^{-k}v^{-1}s^k)) \;=\; f_\phi(s^k)$$

because $s^{-1}(N_1\cap J(N_0/N_{k+1}))s$ is a system of representatives of N_0/N_k and each term of the sum does not depend on the representative. Indeed for $u\in N_0$,

$$\begin{aligned}&(vs^kus^{-k}\circ\varphi^k)(\phi(s^{-k}(vs^kus^{-k})^{-1}s^k)\\ &\qquad = (v\circ\varphi^k\circ u)(\phi(u^{-1}s^{-k}v^{-1}s^k)) = (v\circ\varphi^k)(\phi(s^{-k}v^{-1}s^k))\ .\end{aligned}$$

For $u\in N_0, k\in\mathbb{N}$, we have

$$\begin{aligned}\phi_{f_\phi}(s^{-k}us^k) \;&=\; (\psi^k\circ u)f_\phi(s^k)\\ &= \sum_{v\in J(N_0/N_k)} (\psi^k\circ uv\circ\varphi^k)(\phi(s^{-k}v^{-1}s^k)) \;=\; \phi(s^{-k}us^k)\end{aligned}$$

where the last equality comes from $\operatorname{Ker}\psi^k = \sum_{u\in N_0-N_k} u\varphi^k(M)$. Moreover, we have $f_{\phi_f} = f$ as a consequence of Lemma 3.2. □

Proposition 3.11. *The induction functor*

$$\mathrm{Ind}_{P_-}^{P} \quad : \quad \mathcal{M}_A(P_+)^{et} \to \mathcal{M}_A(P_-) \to \mathcal{M}_A(P)$$

is exact.

Proof. The canonical action of any element of P_- on an étale $A[P_+]$-module is surjective. Apply Lemma 2.1. □

Proposition 3.12. *Let $f\in M^P$. Let $n,n'\in N$ and $t\in L_+$ and denote by $k(n)$ the smallest integer $k\in\mathbb{N}$ such that $n\in N_{-k}$. We have :*

$$(nf)(s^m) = (s^mns^{-m})(f(s^m)) \quad \textit{for all } m\geq k(n),$$
$$(t^{-1}f)(s^m) = \psi_t(f(s^m)) \quad \textit{and} \quad (sf)(s^m) = f(s^{m+1}) \quad \textit{for all } m\in\mathbb{Z},$$
$$(s^kf)(n') = \sum_{v\in J(N_0/N_k)} v\varphi^k(f(s^{-k}v^{-1}n's^k)) \quad \textit{for all } k\geq 1,$$
$$(t^{-1}f)(n') = \psi_t(f(tn't^{-1})) \quad \textit{and} \quad (nf)(n') = f(n'n)\ .$$

Proof. The formulas $(sf)(s^m) = f(s^{m+1}), (nf)(n') = f(n'n)$ are obvious. It is clear that

$$(t^{-1}f)(s^m) = f(s^m t^{-1}) = f(t^{-1}s^m) = t^{-1}(f(s^m)) = \psi_t(f(s^m)) \ ,$$

$$(t^{-1}f)(n') = f(nt^{-1}) = f(t^{-1}tn't^{-1}) = t^{-1}(f(tn't^{-1})) = \psi_t(f(tn't^{-1})) \ .$$

$$nf(s^m) = f(s^m n) = f(s^m n s^{-m} s^m) = (s^m n s^{-m}) f(s^m) \ .$$

Using Lemma 3.2, we write

$$(s^k f)(n') = \sum_{v \in J(N_0/N_k)} v\varphi^k(\psi^k(v^{-1}((s^k f)(n')))) \ ,$$

$$\psi^k(v^{-1}((s^k f)(n'))) = \psi^k(v^{-1}(f(n's^k))) = \psi^k(f(v^{-1}n's^k)) = f(s^{-k}v^{-1}n's^k) \ .$$

We obtain $(s^k f)(n') = \sum_{v \in J(N_0/N_k)} v\varphi(f(s^{-k}v^{-1}n's^k))$. ∎

Definition 3.13. *The s-model and the N-model of M^P are the spaces $\mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M) \simeq \varprojlim_{\psi} M$ and $\mathrm{Ind}_{N_0}^{N}(M)$, respectively, with the action of P described in proposition 3.12.*

3.3 Compactly induced representation M_c^P

The map

$$\mathrm{ev}_0 \ : \ M^P \ \to \ M \quad , \quad f \ \mapsto f(1) \quad ,$$

admits a splitting

$$\sigma_0 : M \to M^P$$

For $m \in M$, $\sigma_0(m)$ vanishes on $N - N_0$ and is equal to nm on $n \in N_0$ and to $\varphi^k(m)$ on s^k for $k \in \mathbb{N}$. In particular, by proposition 3.10.b, σ_0 is independent of the choice of s.

Lemma 3.14. *The map ev_0 is P_--equivariant, the map σ_0 is P_+-equivariant, the $A[P_+]$-modules $\sigma_0(M)$ and M are isomorphic.*

Proof. It is clear on the definition of M^P that ev_0 is P_--equivariant. We show that σ_0 is L_+-equivariant using the s-model. Let $t \in L_+$. We choose $t' \in L_+, r \in \mathbb{N}$ with $t't = s^r$. Then $\varphi_{t'}\varphi_t = \varphi^r$ and $\varphi_t = \psi_{t'}\varphi^r$. We obtain for $t\sigma_0(m)(s^k) = \sigma_0(m)(s^k t)$ the following expression

$$\begin{aligned}\sigma_0(m)(t'^{-1}s^{k+r}) &= \psi_{t'}(\sigma_0(m)(s^{k+r})) \\ &= \psi_{t'}\varphi^{r+k}(m) = \varphi_t\varphi^k(m) = \varphi^k\varphi_t(m) = \sigma_0(tm)(s^k) \ .\end{aligned}$$

Hence $t\sigma_0(m) = \sigma_0(tm)$. We show that σ_0 is N_0-equivariant using the N-model. Let $n_0 \in N_0$ and $m \in M$. Then $n_0\sigma_0(m) = \sigma_0(n_0 m)$, because for $k \in \mathbb{N},\ u \in N_0$,

$$\begin{aligned}n_0\sigma_0(m)(s^{-k}us^k) &= \sigma_0(m)(s^{-k}us^k n_0) = \sigma_0(m)(s^{-k}us^k n_0 s^{-k} s^k) \\ &= (\psi^k \circ us^k n_0 s^{-k} \circ \varphi^k)(m) = (\psi^k \circ u \circ \varphi^k)(n_0 m) = \sigma_0(n_0 m)(s^{-k}us^k) \ .\end{aligned}$$

∎

The compact induction of M from P_- to P is defined to be the $A[P]$-submodule

$$\text{c-Ind}_{P_-}^P(M) := M_c^P$$

of M^P generated by $\sigma_0(M)$. The space M^P_c is the subspace of functions $f \in M^P$ with compact restriction to N, equivalently such that $f(s^{k+r}) = \varphi^k(f(s^r))$ for all $k \in \mathbb{N}$ and some $r \in \mathbb{N}$. The restriction to $s^{\mathbb{Z}}$ is an $s^{\mathbb{Z}}$-isomorphism (proposition 3.10)

$$M^P_c \quad \to \quad \text{c-Ind}^{s^{\mathbb{Z}}}_{s^{-\mathbb{N}},\psi}(M) \quad .$$

By proposition 2.3, the map

$$\begin{aligned} A[P] \otimes_{A[P_+]} M \ &\to \ \text{c-Ind}^P_{P_-}(M) \\ [s^{-k}] \otimes m \ &\mapsto \ (\varphi^{-k} \circ \sigma_0)(m) \end{aligned}$$

is an isomorphism.

Lemma 3.15. *The compact induction functor from P_- to P is isomorphic to*

$$\text{c-Ind}^P_{P_-} \simeq A[P] \otimes_{A[P_+]} \ : \ \mathcal{M}_A(P_+)^{et} \ \to \ \mathcal{M}_A(P) \ , \tag{18}$$

and is exact.

Proof. For the exactness see Corollary 2.4. ☐

3.4 P-equivariant map $\text{Res} : C^\infty_c(N, A) \to \text{End}_A(M^P)$

Let $C^\infty_c(N, A)$ be the A-module of locally constant compactly supported functions on N with values in A, with the usual product of functions and with the natural action of P,

$$P \times C^\infty_c(N, A) \ \to \ C^\infty_c(N, A) \quad , \quad (b, f) \mapsto (bf)(x) = f(b^{-1}.x) \quad .$$

For any open compact subgroup $U \subset N$, the subring $C^\infty(U, A) \subset C^\infty_c(N, A)$ of functions f supported in U, has a unit equal to the characteristic function 1_U of U, and is stable by the P-stabilizer P_U of U. We have $b1_U = 1_{b.U}$. The $A[P_U]$-module $C^\infty(U, A)$ and the $A[P]$-module $C^\infty_c(N, A)$ are cyclic generated by 1_U. The monoid $P_+ = N_0L_+$ acts on $\text{End}_A(M)$ by

$$\begin{aligned} P_+ \times \text{End}_A(M) \ &\to \ \text{End}_A(M) \\ (b, F) &\mapsto \varphi_b \circ F \circ \psi_b \quad . \end{aligned}$$

Note that we have $\psi_{ut} = \psi_t \circ u^{-1}$.

Proposition 3.16. *There exists a unique P_+-equivariant A-linear map*

$$\text{res} \ : \ C^\infty(N_0, A) \quad \to \quad \text{End}_A(M)$$

respecting the unit. It is a homomorphism of A-algebras.

Proof. If the map res exists, it is unique because the $A[P_+]$-module $C^\infty(N_0, A)$ is generated by the unit 1_{N_0}. The existence of res is equivalent to lemma 3.2 as we will show below. For $b \in P_+$ we have the idempotent

$$\text{res}(1_{b.N_0}) := \varphi_b \circ \psi_b \ \in \ \text{End}_A(M) \quad . \tag{19}$$

We claim that for any finite disjoint union $b.N_0 = \coprod_{i \in I} b_i.N_0$ with $b_i \in P_+$, the idempotents $\text{res}(1_{b_i.N_0})$ are orthogonal of sum $\text{res}(1_{b.N_0})$. We may assume that $b = 1$, since the inclusion $b_i.N_0 \subset b.N_0$ yields $b^{-1}b_i \in P_+$. Write $b_i = u_it_i$ with $u_i \in N_0$ and $t_i \in L_+$, and choose $t' \in L_+$ such that $t' \in t_iL_+$, say $t' = t_il_i$ (with $l_i \in L_+$). Let $(n_{ij})_j$ be a system of representatives for $N_0/l_i.N_0$. Since M is étale, lemma 3.2 shows that, for each i, the idempotents $(\varphi_{n_{ij}l_i} \circ \psi_{n_{ij}l_i})_j$ are orthogonal, with sum id_M. Note that $(v_{ij} := u_it_in_{ij}t_i^{-1})_{(i,j)}$ form a system of representatives for $N_0/t'.N_0$, so again by lemma 3.2 the idempotents

$(\varphi_{v_{ij}t'} \circ \psi_{v_{ij}t'})_{(i,j)}$ are orthogonal with sum id_M. The claim follows, since $v_{ij}t' = b_i(n_{ij}l_i)$, so

$$\varphi_{v_{ij}t'} \circ \psi_{v_{ij}t'} = \varphi_{b_i} \circ \varphi_{n_{ij}l_i} \circ \psi_{n_{ij}l_i} \circ \psi_{b_i} \ .$$

The claim being proved, we get an A-linear map $\mathrm{res} : C^\infty(N_0, A) \to \mathrm{End}_A(M)$ which is clearly P_+-equivariant and respects the unit. It respects the product because, for $f_1, f_2 \in C^\infty(N_0, A)$, there exists $t \in L_+$ such that f_1 and f_2 are constant on each coset $utN_0t^{-1} \subset N_0$. Hence $\mathrm{res}(f_1 f_2) = \sum_{v \in J(N_0/tN_0t^{-1})} f_1(v) f_2(v) \,\mathrm{res}(1_{vt.N_0}) = \mathrm{res}(f_1) \circ \mathrm{res}(f_2)$. $\square$

The group $P = NL$ acts on $\mathrm{End}_A(M^P)$ by conjugation. We have the canonical injective algebra map

$$F \ \mapsto \ \sigma_0 \circ F \circ \mathrm{ev}_0 \quad : \quad \mathrm{End}_A M \ \to \ \mathrm{End}_A(M^P) \ . \tag{20}$$

It is P_+-equivariant since, by the lemma 3.14 for $b \in P_+$, we have

$$b \circ \sigma_0 \circ F \circ \mathrm{ev}_0 \circ b^{-1} \ = \ \sigma_0 \circ \varphi_b \circ F \circ \psi_b \circ \mathrm{ev}_0 \ . \tag{21}$$

We consider the composite P_+-equivariant algebra homomorphism

$$C^\infty(N_0, A) \ \xrightarrow{\mathrm{res}} \ \mathrm{End}_A(M) \ \longrightarrow \ \mathrm{End}_A(M^P) \ .$$

sending 1_{N_0} to $R_0 := \sigma_0 \circ \mathrm{ev}_0$ and, more generally, $1_{b.N_0}$ to $b \circ R_0 \circ b^{-1}$ for $b \in P_+$.

For $f \in M^P$, $R_0(f) \in M^P$ vanishes on $N - N_0$ and $R_0(f)(s^k) \ = \ \varphi^k(f(1))$. In the N-model, R_0 is the restriction to N_0.

We show now that the composite morphism extends to $C_c^\infty(N, A)$.

Proposition 3.17. *There exists a unique P-equivariant A-linear map*

$$\mathrm{Res} \ : \ C_c^\infty(N, A) \ \to \ \mathrm{End}_A(M^P)$$

such that $\mathrm{Res}(1_{N_0}) \ = \ R_0$. The map Res *is an algebra homomorphism.*

Proof. If the map Res exists, it is unique because the $A[P]$-module $C_c^\infty(N, A)$ is generated by 1_{N_0}.

For $b \in P$ we define

$$\mathrm{Res}(1_{b.N_0}) \ := \ b \circ R_0 \circ b^{-1} \ .$$

We prove that $b \circ R_0 \circ b^{-1}$ depends only on the subset $b.N_0 \subset N$, and that for any finite disjoint decomposition of $b.N_0 = \sqcup_{i \in I} b_i.N_0$ with $b_i \in P$, the idempotents $b_i \circ R_0 \circ b_i^{-1}$ are orthogonal of sum $b \circ R_0 \circ b^{-1}$.

The equivalence relation $b.N_0 = b'.N_0$ for $b, b' \in P$ is equivalent to $b'P_0 = bP_0$ because the normalizer of N_0 in P is P_0. We have $b \circ R_0 \circ b^{-1} \ = \ R_0$ when $b \in P_0$ because $\mathrm{res}(1_{b.N_0}) = \mathrm{res}(1_{N_0}) = \mathrm{id}$ (proposition 3.16). Hence $b \circ R_0 \circ b^{-1}$ depends only on $b.N_0$. By conjugation by b^{-1}, we reduce to prove that the idempotents $b_i \circ R_0 \circ b_i^{-1}$ are orthogonal of sum R_0 for any disjoint decomposition of $N_0 = \sqcup_{i \in I} b_i.N_0$ and $b_i \in P$. The b_i belong to P_+, and the proposition 3.16 implies the equality.

To prove that the A-linear map Res respects the product it suffices to check that, for any $t \in L_+, k \in \mathbb{N}$, the endomorphisms $\mathrm{Res}(1_{vtN_0t^{-1}}) \in \mathrm{End}_A(M^P)$ are orthogonal idempotents, for $v \in J(N_{-k}/tN_0t^{-1})$. We already proved this for $k = 0$ and for all $t \in L_+$, and $s^k J(N_{-k}/tN_0t^{-1}) s^{-k} = J(N_0/s^k tN_0t^{-1}s^{-k})$. Hence we know that

$$(s^k \circ \mathrm{Res}(1_{vtN_0t^{-1}}) \circ s^{-k})_{v \in J(N_{-k}/tN_0t^{-1})}$$

are orthogonal idempotents. This implies that $(\mathrm{Res}(1_{vtN_0t^{-1}}))_{v \in J(N_{-k}/tN_0t^{-1})}$ are orthogonal idempotents. $\square$

Remark 3.18. *(i) The map* Res *is the restriction of an algebra homomorphism*

$$C^\infty(N,A) \ \to \ \mathrm{End}_A(M^P) \ ,$$

where $C^\infty(N,A)$ is the algebra of all locally constant functions on N. For this we observe

1. *The $A[P_+]$-module $C^\infty(N_0,A)$ is étale. For $t \in L_+$, the corresponding ψ_t satisfies $(\psi_t f)(x) = f(txt^{-1})$.*
2. *The map $(f,m) \mapsto \mathrm{res}(f)(m) \ : \ C^\infty(N_0,A) \times M \to M$ is ψ_t-equivariant, hence induces to a pairing $C^\infty(N_0,A)^P \times M^P \to M^P$.*
3. *The $A[P]$-module $C^\infty(N_0,A)^P$ is canonically isomorphic to $C^\infty(N,A)$.*

(ii) The monoid $P_+ \times P_+$ acts on $\mathrm{End}_A(M)$ by $\varphi_{(b_1,b_2)}F := \varphi_{b_1} \circ F \circ \psi_{b_2}$. For this action $\mathrm{End}_A(M)$ is an étale $A[P_+ \times P_+]$-module, and we have $\psi_{(b_1,b_2)}F = \psi_{b_1} \circ F \circ \varphi_{b_2}$.

Definition 3.19. *For any compact open subsets $V \subset U \subset N_0$ and $m \in M$, we denote*

$$\mathrm{res}_U := \mathrm{res}(1_U) \ , \ M_U := \mathrm{res}_U(M) \ , \ m_U := \mathrm{res}_U(m) \ , \ \mathrm{res}_V^U := \mathrm{res}_V|_{M_U} : M_U \to M_V \ .$$

For any compact open subsets $V \subset U \subset N$ and $f \in M^P$

$$\mathrm{Res}_U := \mathrm{Res}(1_U) \ , \ M_U := \mathrm{Res}_U(M^P) \ , \ f_U := \mathrm{Res}_U(f) \ , \ \mathrm{Res}_V^U := \mathrm{Res}_V|_{M_U} : M_U \to M_V .$$

Remark 3.20. The notations are coherent for $U \subset N_0$, as follows from the following properties. For $b \in P_+$ we have

– $\mathrm{res}_{b.U} = \varphi_b \circ \mathrm{res}_U \circ \psi_b$ (proposition 3.16) ;

– $b \circ \mathrm{Res}_U \ = \ \sigma_0 \circ \varphi_b \circ \mathrm{res}_U \circ \mathrm{ev}_0$ and $\mathrm{Res}_U \circ b^{-1} \ = \ \sigma_0 \circ \mathrm{res}_U \circ \psi_b \circ \mathrm{ev}_0$;

– $(\mathrm{Res}_U\, f)(1) \ = \ \mathrm{res}_U(f(1))$.

We note also that the proposition 3.17 implies:

Corollary 3.21. *For any compact open subset $U \subset N$ equal to a finite disjoint union $U = \sqcup_{i \in I} U_i$ of compact open subsets $U_i \subset N$, the idempotents Res_{U_i} are orthogonal of sum Res_U.*

Corollary 3.22. *For $u \in N$, the projector Res_{uN_0} is the restriction to $N_0 u^{-1}$ in the N-model.*

Proof. We have $\mathrm{Res}_{uN_0} = u \circ \mathrm{Res}_{N_0} \circ u^{-1}$ and Res_{N_0} is the restriction to N_0 in the N-model. Hence for $x \in N$, $(\mathrm{Res}_{uN_0}\, f)(x) = (\mathrm{Res}_{N_0}\, u^{-1} f)(xu)$ vanishes for $x \in N - N_0 u^{-1}$ and for $v \in N_0$, $(\mathrm{Res}_{uN_0}\, f)(vu^{-1}) = (u^{-1} f)(v) = f(vu^{-1})$. ☐

The constructions are functorial. A morphism $f : M \to M'$ of $A[P_+]$-modules, being also $A[P_-]$-equivariant induces a morphism $\mathrm{Ind}_{P_-}^P(f) : M^P \to M'^P$ of $A[P]$-modules. On the other hand, M^P is a module over the non unital ring $C_c^\infty(N,A)$ through the map Res. The morphism $\mathrm{Ind}_{P_-}^P(f)$ is $C_c^\infty(N,A)$-equivariant. Since Res is P-equivariant , it suffices to prove that $\mathrm{Ind}_{P_-}^P(f)$ respects $R_0 = \sigma_0 \circ \mathrm{ev}_0$ which is obvious.

3.5 P-equivariant sheaf on N

We formulate now the proposition 3.17 in the language of sheaves.

Theorem 3.23. *One can associate to an étale $A[P_+]$-module M, a P-equivariant sheaf $\mathcal{S}_M$ of A-modules on the compact open subsets $U \subset N$, with*

- *sections M_U on U,*
- *restrictions Res_V^U for any open compact subset $V \subset U$,*
- *action $f \mapsto bf = \mathrm{Res}_{b.U}(bf) \ : \ M_U \ \to \ M_{b.U}$ of $b \in P$.*

Proof. a) Res_U^U is the identity on $M_U = \mathrm{Res}_U(M)$ because Res_U is an idempotent.

b) $\mathrm{Res}_W^V \circ \mathrm{Res}_V^U = \mathrm{Res}_W^U$ for compact open subsets $W \subset V \subset U \subset N$. Indeed, we have $\mathrm{Res}_W \circ \mathrm{Res}_V = \mathrm{Res}_W$ on M_U.

c) If U is the union of compact open subsets $U_i \subset U$ for $i \in I$, and $f_i \in M_{U_i}$ satisfying $\mathrm{Res}^{U_i}_{U_i \cap U_j}(f_i) = \mathrm{Res}^{U_j}_{U_i \cap U_j}(f_j)$ for $i, j \in I$, there exists a unique $f \in M_U$ such that $\mathrm{Res}^U_{U_i}(f) = f_i$ for all $i \in I$.

c1) True when $(U_i)_{i \in I}$ is a partition of U because I is finite and Res_U is the sum of the orthogonal idempotents Res_{U_i}.

c2) True when I is finite because the finite covering defines a finite partition of U by open compact subsets V_j for $j \in J$, such that $V_j \cap U_i$ is empty or equal to V_j for all $i \in I, j \in J$. By hypothesis on the f_i, if $V_j \subset U_i$, then the restriction of f_i to V_j does not depend on the choice of i, and is denoted by ϕ_j. Applying c1), there is a unique $f \in M_U$ such that $\mathrm{Res}_{V_j}(f) = \phi_j$ for all $j \in J$. Note also that the V_j contained in U_i form a finite partition of U_i and that f_i is the unique element of M_{U_i} such that $\mathrm{Res}_{V_j}(f_i) = \phi_j$ for those j. We deduce that f is the unique element of M_U such that $\mathrm{Res}_{U_i}(f) = f_i$ for all $i \in I$.

c3) In general, U being compact, there exists a finite subset $I' \subset I$ such that U is covered by U_i for $i \in I'$. By c2), there exists a unique $f_{I'} \in M_U$ such that $f_i = \mathrm{Res}_{U_i}(f_{I'})$ for all $i \in I'$. Let $i' \in I$ not belonging to I'. Then the non empty intersections $U_{i'} \cap U_j$ for $j \in I'$ form a finite covering of $U_{i'}$ by compact open subsets. By c2), $f_{i'}$ is the unique element of $M_{U_{i'}}$ such that $\mathrm{Res}_{U_{i'} \cap U_j}(f_j) = \mathrm{Res}_{U_{i'} \cap U_j}(f_{i'})$ for all non empty $U_{i'} \cap U_j$. The element $\mathrm{Res}_{U_{i'}}(f_{I'})$ has the same property, we deduce by uniqueness that $f_{i'} = \mathrm{Res}_{U_{i'}}(f_{I'})$.

d) Let $f \in M_U$. When $b = 1$ we have clearly $1(f) = f$. For $b, b' \in P$, we have $(bb')(f) = \mathrm{Res}_{(bb').U}((bb')f) = \mathrm{Res}_{b.(b'.U)}(b(b'f)) = b(b'f)$. For a compact open subset $V \subset U$, we have $b \circ \mathrm{Res}_V \circ \mathrm{Res}_U = \mathrm{Res}_{bV} \circ b \circ \mathrm{Res}_U$ in $\mathrm{End}_A M^P$ hence $b \, \mathrm{Res}^U_V = \mathrm{Res}_{b.V} \, b$. □

Proposition 3.24. *Let H be a topological group acting continuously on a locally compact totally disconnected space X. Any H-equivariant sheaf $\mathcal{F}$ (of A-modules) on the compact open subsets of X extends uniquely to a H-equivariant sheaf on the open subsets of X.*

Proof. This is well known. See [4] §9.2.3 Prop. 1.

□

Remark 3.25. *The space of sections on an open subset $U \subset X$ is the projective limit of the sections $\mathcal{F}(V)$ on the compact open subsets V of U for the restriction maps $\mathcal{F}(V) \to \mathcal{F}(V')$ for $V' \subset V$.*

By this general result, the P-equivariant sheaf defined by M on the compact open subsets of N (theorem 3.23), extends uniquely to a P-equivariant sheaf $\mathcal{S}_M$ on (arbitrary open subsets of) N. We extend the definitions 3.19 to arbitrary open subsets $U \subset N$. We denote by Res^U_V the restriction maps for open subsets $V \subset U$ of N, by $\mathrm{Res}_U = \mathrm{Res}^N_U$ and by $M_U = \mathrm{Res}_U(M^P)$. In this way we obtain an exact functor $M \to (M_U)_U$ from $\mathcal{M}_A(P_+)^{et}$ to the category of P-equivariant sheaves of A-modules on N. Note that for a compact open subset U even the functor $M \to M_U$ is exact.

Proposition 3.26. *The representation of P on the global sections of the sheaf $\mathcal{S}_M$ is canonically isomorphic to M^P.*

Proof. We have the obvious P-equivariant homomorphism

$$M^P \xrightarrow{(\mathrm{Res}_U)_U} M_N = \varprojlim_U M_U \ .$$

The group N is the union of $s^{-k}.N_0 = s^{-k}N_0s^k$ for $k \in \mathbb{N}$. Hence $M_N = \varprojlim_k M_{N_{-k}}$. In the s-model of M^P we have $\mathrm{Res}_{s^{-k}.N_0} = R_{-k}$ and by the lemma 2.5 the morphism

$$f \mapsto (\mathrm{Res}_{s^{-k}.N_0}(f))_{k\in\mathbb{N}} \ : \ M^P \quad \to \quad M_N$$

is bijective. $\square$

Corollary 3.27. *The restriction* $\mathrm{Res}^N_U : M_N \to M_U$ *from the global sections to the sections on an open compact subset* $U \subset N$ *is surjective with a natural splitting.*

Proof. It corresponds to an idempotent $\mathrm{Res}_U = \mathrm{Res}(1_U) \in \mathrm{End}_A(M^P)$. $\square$

3.6 Independence of N_0

Let $U \subset N$ be a compact open subgroup. For $n \in N$ and $t \in L$, the inclusion $ntUt^{-1} \subset U$ is obviously equivalent to $n \in U$ and $tUt^{-1} \subset U$. Hence the P-stabilizer $P_U := \{b \in P \mid b.U \subset U\}$ of U is the semi-direct product of U by the L-stabilizer L_U of U. As the decreasing sequence $(N_k = s^kN_0s^{-k})_{k\in\mathbb{N}}$ forms a basis of neighborhoods of 1 in N and $N = \cup_{r\in\mathbb{Z}}N_{-r}$, the compact open subgroup $U \subset N$ contains some N_k and is contained in some N_{-r}. This implies that the intersection $L_U \cap s^{\mathbb{N}}$ is not empty hence is equal to $s_U^{\mathbb{N}}$ where $s_U = s^{k_U}$ for some $k_U \geq 1$. The monoid $P_U = UL_U$ and the central element s_U of L satisfy the same conditions as $(P_+ = N_0L_+, s)$, given at the beginning of the section 3.2. Our theory associates to each étale $A[P_U]$-module a P-equivariant sheaf on N.

The subspace $M_U \subset M^P$ (definition 3.19) is stable by P_U because $b \circ \mathrm{Res}_U = \mathrm{Res}_{b.U} \circ b$ for $b \in P$ and $M_{b.U} = \mathrm{Res}_{b.U}(M) \subset \mathrm{Res}_U(M) = M_U$. As $M_U = \oplus_{u\in J(U/t.U)} uM_{t.U}$ for $t \in L_U$ the $A[P_U]$-module M_U is étale.

Proposition 3.28. *The P-equivariant sheaf $\mathcal{S}_M$ on N associated to the étale $A[P_+]$-module M is equal to the P-equivariant sheaf on N associated to the étale $A[P_U]$-module M_U.*

Proof. For $b \in P_U$ we denote by $\varphi_{U,b}$ the action of b on M_U and by $\psi_{U,b}$ the left inverse of $\varphi_{U,b}$ with kernel $M_{U-b.U}$. We have $M_U = M_{b.U} \oplus M_{U-b.U}$ and for $f_U \in M_U$,

$$(22)\ \ \varphi_{U,b}(f_U) \ = \ bf_U \ , \ \psi_{U,b}(f_U) \ = \ b^{-1}\,\mathrm{Res}_{b.U}(f_U) \ , \ (\varphi_{U,b} \circ \psi_{U,b})(f_U) \ = \ \mathrm{Res}_{b.U}(f_U) \ .$$

By the last formula and the remark 3.20, the sections on $b.U$ and the restriction maps from M_U to $M_{b.U}$ in the two sheaves are the same for any $b \in P_U$. This implies that the two sheaves are equal on (the open subsets of) U. By symmetry they are also equal on (the open subsets of) N_0. The same arguments for arbitrary compact open subgroups $U, U' \subset N$ imply that the P-equivariant sheaves on N associated to the étale $A[P_U]$-module M_U and to the étale $A[P_{U'}]$-module $M_{U'}$ are equal on (the open subsets of) U and on (the open subsets of) U'. Hence all these sheaves are equal on (the open subsets of) the compact open subsets of N and also on (the open subsets of) N. $\square$

3.7 Etale $A[P_+]$-module and P-equivariant sheaf on N

Proposition 3.29. *Let M be an $A[P_+]$-module such that the action φ of s on M is étale. Then M is an étale $A[P_+]$-module.*

Proof. Let $t \in L_+$. We have to show that the action φ_t of t on M is étale. As $L = L_+s^{-\mathbb{N}}$ with s is central in L, there exists $k \in \mathbb{N}$ such that $s^kt^{-1} \in L_+$. This implies $\varphi^k = \varphi_{s^kt^{-1}} \circ \varphi_t$ in $\mathrm{End}_A(M)$ and $s^kN_0s^{-k} \subset tN_0t^{-1}$. As φ is injective, φ_t is also injective. For any representative system $J(tN_0t^{-1}/s^kN_0s^{-k})$ of $tN_0t^{-1}/s^kN_0s^{-k}$ and any representative system $J(N_0/tN_0t^{-1})$ of N_0/tN_0t^{-1}, the set of uv for $u \in J(N_0/tN_0t^{-1})$

and $v \in J(tN_0t^{-1}/s^kN_0s^{-k})$ is a representative system $J(N_0/s^kN_0s^{-k})$ of $N_0/s^kN_0s^{-k}$. Let ψ be the canonical left inverse of φ. We have

$$\mathrm{id} = \sum_{u\in J(N_0/tN_0t^{-1})} u \circ \sum_{v\in J(tN_0t^{-1}/s^kN_0s^{-k})} v \circ \varphi^k \circ \psi^k \circ v^{-1} \circ u^{-1}$$

$$= \sum_{u\in J(N_0/tN_0t^{-1})} u \circ \sum_{v\in J(tN_0t^{-1}/s^kN_0s^{-k})} v \circ \varphi_t \circ \varphi_{t^{-1}s^k} \circ \psi^k \circ v^{-1} \circ u^{-1}$$

$$= \sum_{u\in J(N_0/tN_0t^{-1})} u \circ \varphi_t \circ (\sum_{v\in J(N_0/t^{-1}s^kN_0s^{-k}t)} v \circ \varphi_{t^{-1}s^k} \circ \psi^k \circ v^{-1}) \circ u^{-1} \ .$$

We deduce that φ_t is étale of canonical left inverse ψ_t the expression between parentheses. □

Corollary 3.30. *An $A[P_+]$-submodule $M' \subset M$ of an étale $A[P_+]$-module M is étale if and only if it is stable by the canonical inverse ψ of φ.*

Proof. If M' is ψ-stable, for $m' \in M'$ every $m'_{u,s}$ belongs to M' in (12). Hence the action of s on M' is étale, and M' is étale by Proposition 3.29. □

Corollary 3.31. *The space $\mathcal{S}(N_0)$ of global sections of a P_+-equivariant sheaf $\mathcal{S}$ on N_0 is an étale representation of P_+, when the action φ of s on $\mathcal{S}(N_0)$ is injective.*

Proof. By proposition 3.29 it suffices to show that $\mathcal{S}(N_0) = \oplus_{u\in J(N_0/sN_0s^{-1})} us(\mathcal{S}(N_0))$. But this equality is true because N_0 is the disjoint sum of the open subsets $usN_0s^{-1} = us.N_0$ and $\mathcal{S}(us.N_0) = us(\mathcal{S}(N_0))$. □

The canonical left inverse ψ of the action φ of s on $\mathcal{S}(N_0)$ vanishes on $\mathcal{S}(usN_0s^{-1})$ for $u \neq 1$ and on $\mathcal{S}(sN_0s^{-1})$ is equal to the isomorphism $\mathcal{S}(sN_0s^{-1}) \to \mathcal{S}(N_0)$ induced by s^{-1}.

Theorem 3.32. *The functor $M \mapsto \mathcal{S}_M$ is an equivalence of categories from the abelian category of étale $A[P_+]$-modules to the abelian category of P-equivariant sheaves of A-modules on N, of inverse the functor $\mathcal{S} \mapsto \mathcal{S}(N_0)$ of sections over N_0.*

Proof. Let $\mathcal{S}$ be a P-equivariant sheaf on N. By the corollary 3.31, the space $\mathcal{S}(N_0)$ of sections on N_0 is an étale representation of P_+ because the action φ of s on $\mathcal{S}(N_0)$ is injective.

We show now that the representation of P on the space $\mathcal{S}(N)_c$ of compact sections on N depends uniquely of the representation of P_+ on $\mathcal{S}(N_0)$. The representation of N on $\mathcal{S}(N)_c$ is defined by the representation of N_0 on $\mathcal{S}(N_0)$, because $\mathcal{S}(N)_c = \oplus_{u\in J(N/N_0)}\mathcal{S}(uN_0)$ and $\mathcal{S}(uN_0) = u\mathcal{S}(N_0)$ for $u \in N$. The group P is generated by N and L_+. For $t \in L_+$, the action of t on $\mathcal{S}(N)_c$ is defined by the action of N on $\mathcal{S}(N)_c$ and by the action of t on $\mathcal{S}(N_0)$, because $t\mathcal{S}(uN_0) = tut^{-1}t\mathcal{S}(N_0)$ with $tut^{-1} \in N$ for $u \in N$.

We deduce that the $A[P]$-module $\mathcal{S}(N)_c$ is equal to the compact induced representation $\mathcal{S}(N_0)_c^P$, and that the sheaves $\mathcal{S}$ and $\mathcal{S}_{\mathcal{S}(N_0)}$ are equal.

Conversely, let M be an étale $A[P_+]$-module. The $A[P_+]$-module $\mathcal{S}_M(N_0)$ of sections on N_0 of the sheaf $\mathcal{S}_M$ is equal to M (Theorem 3.23). □

4 Topology

4.1 Topologically étale $A[P_+]$-modules

We add to the hypothesis of section 3.2 the following

a) A is a linearly topological commutative ring (the open ideals form a basis of neighborhoods of 0).

b) M is a linearly topological A-module (the open A-submodules form a basis of neighborhoods of 0), with a continuous action of P_+

$$\begin{aligned} P_+ \times M &\to M \\ (b, x) &\mapsto \varphi_b(x) \ . \end{aligned}$$

We call such an M a continuous $A[P_+]$-module. If M is also étale in the algebraic sense (definition 3.1) and the maps ψ_t, for $t \in L_+$, are continuous we call M a topologically étale $A[P_+]$-module.

Lemma 4.1. *Let M be a continuous $A[P_+]$-module which is algebraically étale, then:*

(i) The maps ψ_t for $t \in L_+$ are open.

(ii) If $\psi = \psi_s$ is continuous then M is topologically étale.

Proof. (i) The projection of $M = M_0 \oplus M_1$ onto the algebraic direct summand M_0 (with the submodule topology) is open. Indeed let $V \subset M$ be an open subset, then $M_0 \cap (V + M_1)$ is open in M_0 and is equal to the projection of V. We apply this to $M = \varphi_t(M) \oplus \operatorname{Ker} \psi_t$ and to the projection $\varphi_t \circ \psi_t$. Then we note that $\psi_t(V) = \varphi_t^{-1}((\varphi_t \circ \psi_t)(V))$.

(ii) Given any $t \in L_+$ we find $t' \in L_+$ and $n \in \mathbb{N}$ such that $t't = s^n$. Hence $\psi_{t't} = \psi_t \circ \psi_{t'} = \psi^n$ is continuous by assumption. As $\psi_{t'}$ is surjective and open, for any open subset $V \subset M$ we have $\psi_t^{-1}(V) = \psi_{t'}((\psi_t \circ \psi_{t'})^{-1}(V))$ which is open. □

Lemma 4.2. *(i) A compact algebraically étale $A[P_+]$-module is topologically étale.*

(ii) Let M be a topologically étale $A[P_+]$-module. The P_--action $(b^{-1}, m) \mapsto \psi_b(m)$: $P_- \times M \to M$ on M is continuous.

Proof. (i) The compactness of M implies that

$$M = \varphi_t(M) \oplus \bigoplus_{u \in (N_0 - tN_0t^{-1})} u\varphi_t(M)$$

is a topological decomposition of M as the direct sum of finitely many closed submodules. It suffices to check that the restriction of ψ_t to each summand is continuous. On all summands except the first one ψ_t is zero. By compactness of M the map φ_t is a homeomorphism between M and the closed submodule $\varphi_t(M)$. We see that $\psi_t | \varphi_t(M)$ is the inverse of this homeomorphism and hence is continuous.

(ii) Since P_0 is open in $P_- = L_+^{-1} P_0$ we only need to show that the restriction of the P_--action to $t^{-1}P_0 \times M \to M$, for any $t \in L_+$, is continuous. We contemplate the commutative diagram

$$\begin{tikzcd} t^{-1}P_0 \times M \arrow[r] \arrow[d, "t\cdot \times \mathrm{id}"'] & M \\ P_0 \times M \arrow[r] & M \arrow[u, "\psi_t"'] \end{tikzcd}$$

where the horizontal arrows are given by the P_--action. The P_0-action on M induced by P_- coincides with the one induced by the P_+-action. Therefore the bottom horizontal arrow is continuous. The left vertical arrow is trivially continuous, and ψ_t is continuous by assumption. □

Lemma 4.3. *For any compact subgroup $C \subset P_+$, the open C-stable A-submodules of M form a basis of neighborhoods of* 0.

Proof. We have to show that any open A-submodule $\mathcal{M}$ of M contains an open C-stable A-submodule. By continuity of the action of P_+ on M, there exists for each $c \in C$, an open A-submodule $\mathcal{M}_c$ of M and an open neighborhood $H_c \subset P_+$ of c such that $\varphi_x(\mathcal{M}_c) \subset \mathcal{M}$ for all $x \in H_c$. By the compactness of C, there exists a finite subset $I \subset C$ such that $C = \cup_{c \in I}(H_c \cap C)$. By finiteness of I, the intersection $\mathcal{M}'' := \cap_{c \in I}\mathcal{M}_c \subset M$ is an open A-submodule such that $\mathcal{M}' := \sum_{c \in C} \varphi_c(\mathcal{M}'') \subset \mathcal{M}$. The A-submodule $\mathcal{M}'$ is C-stable and, since $\mathcal{M}'' \subset \mathcal{M}' \subset \mathcal{M}$, also open. $\square$

Let M be a topologically étale $A[P_+]$-module. Since P_0 is open in P the A-module M^P is a submodule of the A-module $C(P, M)$ of all continuous maps from P to M. We equip $C(P, M)$ with the compact-open topology which makes it a linear-topological A-module. A basis of neighborhoods of zero is given by the submodules $\mathcal{C}(C, \mathcal{M}) := \{f \in C(P, M) \mid f(C) \subset \mathcal{M}\}$ with C and $\mathcal{M}$ running over all compact subsets in P and over all open submodules in M, respectively. With M also $C(P, M)$ is Hausdorff. It is well known that the regular action of P on $C(P, M)$ is continuous (see for instance Proposition 4.5(ii) for a proof). Therefore M^P is characterized inside $C(P, M)$ by closed conditions and hence is a closed submodule. Similarly, $\mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M)$ and $\mathrm{Ind}_{N_0}^{N}(M)$ are closed submodules of $C(s^{\mathbb{Z}}, M)$ and $C(N, M)$, respectively, for the compact-open topologies. Clearly the homomorphisms of restricting maps (proposition 3.10) $M^P \to \mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M)$ and $M^P \to \mathrm{Ind}_{N_0}^{N}(M)$ are continuous.

Lemma 4.4. *The restriction maps $M^P \to \mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}}(M)$ and $M^P \to \mathrm{Ind}_{N_0}^{N}(M)$ are topological isomorphisms.*

Proof. The topology on M^P induced by the compact-open topology on the s-model $\mathrm{Ind}_{s^{-\mathbb{N}}}^{s^{\mathbb{Z}}} M$ is the topology with basis of neighborhoods of zero

$$B_{k,\mathcal{M}} \ := \ \{f \in M^P \mid f(s^m) \in \mathcal{M} \text{ for all } -k \leq m \leq k\} \ ,$$

for all $k \in \mathbb{N}$ and all open A-submodules $\mathcal{M}$ of M. One can replace $B_{k,\mathcal{M}}$ by

$$C_{k,\mathcal{M}} \ := \ \{f \in M^P \mid f(s^k) \in \mathcal{M}\} \ ,$$

because $B_{k,\mathcal{M}} \subset C_{k,\mathcal{M}}$ and conversely given $(k, \mathcal{M})$ there exists an open A-submodule $\mathcal{M}' \subset \mathcal{M}$ such that $\psi^m(\mathcal{M}') \subset \mathcal{M}$ for all $0 \leq m \leq 2k$ as ψ is continuous (lemma 4.2), hence $C_{k,\mathcal{M}'} \subset B_{k,\mathcal{M}}$.

The topology on M^P induced by the compact-open topology on the N-model $\mathrm{Ind}_{N_0}^{N} M$ is the topology with basis of neighborhoods of zero

$$D_{k,\mathcal{M}} \ := \ \{f \in M^P \mid f(N_{-k}) \subset \mathcal{M}\} \ ,$$

for all $(k, \mathcal{M})$ as above.

We fix an auxiliary compact open subgroup $P_0' \subset P_0$. It then suffices, by Lemma 4.3, to let $\mathcal{M}$ run, in the above families, over the open $A[P_0']$-submodules $\mathcal{M}$ of M.

Let $C \subset P$ be any compact subset and let $\mathcal{M}$ be an open $A[P_0']$-submodule of M. We choose $k \in \mathbb{N}$ large enough so that $Cs^{-k} \subset P_-$. Since Cs^{-k} is compact and P_0' is an open subgroup of P we find finitely many $b_1, \ldots, b_m \in P_+$ such that $Cs^{-k} \subset b_1^{-1}P_0' \cup \ldots \cup b_m^{-1}P_0'$. The continuity of the maps ψ_{b_i} implies the existence of an open $A[P_0']$-submodule $\mathcal{M}'$ of M such that $\psi_{b_i}(\mathcal{M}') \subset \mathcal{M}$ for any $1 \leq i \leq m$. We deduce that

$$C_{k,\mathcal{M}'} \subset \mathcal{C}(\bigcup_i b_i^{-1}P_0's^k, \mathcal{M}) \subset \mathcal{C}(C, \mathcal{M}) \ .$$

Furthermore, by the continuity of the action of P_+ on M, there exists an open submodule $\mathcal{M}''$ such that $\sum_{v \in J(N_0/N_k)} v\varphi^k(\mathcal{M}'') \subset \mathcal{M}'$. The second part of the formula (16) then implies that

$$D_{k,\mathcal{M}''} \subset C_{k,\mathcal{M}} \ .$$

□

The maps $\mathrm{ev}_0 : M^P \to M$ and $\sigma_0 : M \to M^P$ are continuous (section 3.3). We denote by $\mathrm{End}_A^{cont}(M) \subset \mathrm{End}_A(M)$ and $E^{cont} \subset E := \mathrm{End}_A(M^P)$ the subalgebra of continuous endomorphisms. We have the canonical injective algebra map (20)

$$f \mapsto \sigma_0 \circ f \circ \mathrm{ev}_0 \quad : \quad \mathrm{End}_A^{cont}(M) \ \to \ E^{cont} \ .$$

Proposition 4.5. *Let M be a topologically étale $A[P_+]$-module.*

(i) If M is complete, resp. compact, the A-module M^P is complete, resp. compact.

(ii) The natural map $P \times M^P \ \to \ M^P$ is continuous.

(iii) $\mathrm{Res}(f) \in E^{cont}$ *for each $f \in C_c^\infty(N,A)$ (proposition 3.17).*

Proof. (i) If M is complete, by [3] TG X.9 Cor. 3 and TG X.25 Th. 2, the compact-open topology on $C(P,M)$ is complete because P is locally compact. Hence, M^P as a closed submodule is complete as well.

If M is compact, the s-model of M^P is compact as a closed subset of the compact space $M^{\mathbb{N}}$. Hence by Lemma 4.4, M^P is compact.

(ii) It suffices to show that the right translation action of P on $C(P,M)$ is continuous. This is well known: the map in question is the composite of the following three continuous maps

$$\begin{aligned} P \times C(P,M) &\longrightarrow P \times C(P \times P, M) \\ (b,f) &\longmapsto (b, (x,y) \mapsto f(yx)) \ , \end{aligned}$$

$$\begin{aligned} P \times C(P \times P, M) &\longrightarrow P \times C(P, C(P,M)) \\ (b,F) &\longmapsto (b, x \mapsto [y \mapsto F(x,y)]) \ , \end{aligned}$$

and

$$\begin{aligned} P \times C(P, C(P,M)) &\longrightarrow C(P,M) \\ (b,\Phi) &\longmapsto \Phi(b) \ , \end{aligned}$$

where the continuity of the latter relies on the fact that P is locally compact.

(iii) It suffices to consider functions of the form $f = 1_{b.N_0}$ for some $b \in P$. But then $\mathrm{Res}(f) = b \circ \sigma_0 \circ \mathrm{ev}_0 \circ b^{-1}$ is the composite of continuous endomorphisms. □

4.2 Integration on N with value in $\mathrm{End}_A^{cont}(M^P)$

We suppose that M is a complete topologically étale $A[P_+]$-module.

We denote by E^{cont} the ring of continuous A-endomorphisms of the complete A-module M^P with the topology defined by the right ideals

$$E_{\mathcal{L}}^{cont} \ := \ \mathrm{Hom}_A^{cont}(M^P, \mathcal{L})$$

for all open A-submodules $\mathcal{L} \subset M^P$.

Lemma 4.6. *E^{cont} is a complete topological ring.*

Proof. It is clear that the maps $(x,y) \mapsto x - y$ and $(x,y) \mapsto x \circ y$ from $E^{cont} \times E^{cont}$ to E^{cont} are continuous, i.e. that E^{cont} is a topological ring. The composite of the natural morphisms

$$E^{cont} \;\to\; \varprojlim_{\mathcal{L}} E^{cont}/E^{cont}_{\mathcal{L}} \;\to\; \varprojlim_{\mathcal{L}} \mathrm{Hom}^{cont}_A(M^P, M^P/\mathcal{L})$$

is an isomorphism (the natural map $M^P \to \varprojlim_{\mathcal{L}} M^P/\mathcal{L}$ is an isomorphism), hence the two morphisms are isomorphisms since the kernel of the map $E^{cont} \to \mathrm{Hom}^{cont}_A(M^P, M^P/\mathcal{L})$ is $E^{cont}_{\mathcal{L}}$. We deduce that E^{cont} is complete. □

Definition 4.7. *An A-linear map $C^\infty_c(N,A) \to E^{cont}$ is called a measure on N with values in E^{cont}.*

The map Res is a measure on N with values in E^{cont} (proposition 4.5).

Let $C_c(N, E^{cont})$ be the space of compactly supported **continuous** maps from N to E^{cont}. We will prove that one can "integrate" a function in $C_c(N, E^{cont})$ with respect to a measure on N with values in E^{cont}.

Proposition 4.8. *There is a natural bilinear map*

$$\begin{aligned} C_c(N, E^{cont}) \times \mathrm{Hom}_A(C^\infty_c(N,A), E^{cont}) \;&\to\; E^{cont} \\ (f,\lambda) \;&\mapsto\; \int_N f \; d\lambda \;. \end{aligned}$$

Proof. a) Every compact subset of N is contained in a compact open subset. It follows that $C_c(N, E^{cont})$ is the union of its subspaces $C(U, E^{cont})$ of functions with support contained in U, for all compact open subsets $U \subset N$.

b) For any open A-submodule $\mathcal{L}$ of M^P, a function in $C(U, E^{cont}/E^{cont}_{\mathcal{L}})$ is locally constant because $E^{cont}/E^{cont}_{\mathcal{L}}$ is discrete. An upper index ∞ means that we consider locally constant functions hence

$$C(U, E^{cont}/E^{cont}_{\mathcal{L}}) \;=\; C^\infty(U, E^{cont}/E^{cont}_{\mathcal{L}}) \;=\; C^\infty(U,A) \otimes_A E^{cont}/E^{cont}_{\mathcal{L}} \;.$$

There is a natural linear pairing

$$\begin{aligned} (C^\infty(U,A) \otimes_A E^{cont}/E^{cont}_{\mathcal{L}}) \times \mathrm{Hom}_A(C^\infty(U,A), E^{cont}) \;&\to\; E^{cont}/E^{cont}_{\mathcal{L}} \\ (f \otimes x, \lambda) \;&\mapsto\; x\lambda(f) \;. \end{aligned}$$

Note that $E^{cont}/E^{cont}_{\mathcal{L}}$ is a right E^{cont}-module.

c) Let $f \in C_c(N, E^{cont})$ and let $\lambda \in \mathrm{Hom}_A(C^\infty_c(N,A), E^{cont})$. Let $U \subset N$ be an open compact subset containing the support of f. For any open A-submodule L of M^P let $f_{\mathcal{L}} \in C^\infty_c(U, E^{cont}/E^{cont}_{\mathcal{L}})$ be the map induced by f. Let

$$\int_U f_{\mathcal{L}} \; d\lambda \quad \in E^{cont}/E^{cont}_{\mathcal{L}}$$

be the image of $(f_{\mathcal{L}}, \lambda)$ by the natural pairing of b). The elements $\int_U f_{\mathcal{L}} \; d\lambda$ combine in the projective limit $E^{cont} = \varprojlim_{\mathcal{L}} E^{cont}/E^{cont}_{\mathcal{L}}$ to give an element $\int_U f \; d\lambda \in E^{cont}$. One checks easily that $\int_U f \; d\lambda$ does not depend on the choice of U. We define

$$\int_N f \; d\lambda \quad := \quad \int_U f \; d\lambda \quad .$$

□

We recall that $J(N/V)$ is a system of representatives of N/V when $V \subset N$ is a compact open subgroup.

Corollary 4.9. *Let $f \in C_c(N, E^{cont})$ and let λ be a measure on N with values in E^{cont}. Then*

$$\lim_{V \to \{1\}} \sum_{v \in J(N/V)} f(v) \ \lambda(1_{vV}) \quad = \quad \int_N f \ d\lambda \quad .$$

limit on compact open subgroups $V \subset N$ shrinking to $\{1\}$.

Proof. We choose an open compact subset $U \subset N$ containing the support of f. Let L be an open o-submodule of M^P and a compact open subgroup $V \subset N$ such that $uV \subset U$ and $f_{\mathcal{L}}$ (proof of the proposition 4.8) is constant on uV for all $u \in U$. Then $\int_U f_{\mathcal{L}} \ d\lambda$ is the image of

$$\sum_{v \in J(N/V)} f(v) \ \lambda(1_{vV})$$

by the quotient map $E^{cont} \to E^{cont}/E^{cont}_{\mathcal{L}}$. □

Lemma 4.10. *Let $f \in C_c(N, E^{cont})$ be a continuous map with support in the compact open subset $U \subset N$, let λ be a measure on N with values in E^{cont}, and let $\mathcal{L} \subset M^P$ be any open A-submodule. There is a compact open subgroup $V_{\mathcal{L}} \subset N$ such that $UV_{\mathcal{L}} = U$ and*

$$\int_N f 1_{uV} d\lambda - f(u)\lambda(1_{uV}) \in E^{cont}_{\mathcal{L}}$$

for any open subgroup $V \subset V_{\mathcal{L}}$ and any $u \in U$.

Proof. The integral in question is the limit (with respect to open subgroups $V' \subset V$) of the net

$$\sum_{v \in J(V/V')} (f(uv) - f(u))\lambda(1_{uvV'}) \ .$$

Since $E^{cont}_{\mathcal{L}}$ is a right ideal it therefore suffices to find a compact open subgroup $V_{\mathcal{L}} \subset N$ such that $UV_{\mathcal{L}} = U$ and

$$f(uv) - f(u) \in E^{cont}_{\mathcal{L}} \qquad \text{for any } u \in U \text{ and } v \in V_{\mathcal{L}}.$$

We certainly find a compact open subgroup $\tilde{V} \subset N$ such that $U\tilde{V} = U$. The map

$$\begin{aligned} U \times \tilde{V} \ &\to \ E^{cont} \\ (u, v) \ &\mapsto \ f(uv) - f(u) \end{aligned}$$

is continuous and maps any $(u, 1)$ to zero. Hence, for any $u \in U$, there is an open neighborhood $U_u \subset U$ of u and a compact open subgroup $V_u \subset \tilde{V}$ such that $U_u \times V_u$ is mapped to $E^{cont}_{\mathcal{L}}$. Since U is compact we have $U = U_{u_1} \cup \ldots \cup U_{u_s}$ for finitely many appropriate $u_i \in U$. The compact open subgroup $V_{\mathcal{L}} := V_{u_1} \cap \ldots \cap V_{u_s}$ then is such that $U \times V_{\mathcal{L}}$ is mapped to $E^{cont}_{\mathcal{L}}$. □

Let $C(N, E^{cont})$ be the space of continuous functions from N to E^{cont}. For any continuous function $f \in C(N, E^{cont})$, for any compact open subset $U \subset N$ and for any measure λ on N with values in E^{cont} we denote

$$\int_U f \ d\lambda \ := \int_N f \ 1_U \ d\lambda$$

where $1_U \in C^\infty(U, A)$ is the characteristic function of U hence $f1_U \in C_c(N, E^{cont})$ is the restriction of f to U. The "integral of f on U" (with respect to the measure λ) is equal to the "integral of the restriction of f to U".

Remark 4.11. *For $f \in C_c(N, E^{cont})$ and $\phi \in C_c^\infty(N, A)$ we have*

$$\int_N f\phi d\,\mathrm{Res} \;=\; \int_N \phi f d\,\mathrm{Res} \;=\; \int_N f d\,\mathrm{Res} \circ \mathrm{Res}(\phi)\ .$$

Proof. This is immediate from the construction of the integral and the multiplicativity of Res. □

5 G-equivariant sheaf on G/P

Let G be a locally profinite group containing $P = N \rtimes L$ as a closed subgroup satisfying the assumptions of section 3.2 such that

a) G/P is compact.

b) There is a subset W in the G-normalizer $N_G(L)$ of L such that

- *the image of W in $N_G(L)/L$ is a subgroup,*
- *G is the disjoint union of PwP for $w \in W$.*

We note that $PwP = NwP = PwN$.

c) There exists $w_0 \in W$ such that Nw_0P is an open dense subset of G. We call

$$\mathcal{C} := Nw_0P/P$$

the open cell of G/P.

d) The map $(n, b) \mapsto nw_0b$ from $N \times P$ onto Nw_0P is a homeomorphism.

Remark 5.1. *These conditions imply that*

$$G = P\overline{P}P = C(w_0)C(w_0^{-1})$$

where $\overline{P} := w_0Pw_0^{-1}$ and $C(g) = PgP$ for $g \in G$.

Proof. The intersection of the two dense open subsets $g\mathcal{C}$ and $\mathcal{C}$ in G/P is open and not empty, for any $g \in G$. □

The group G acts continuously on the topological space G/P,

$$\begin{aligned} G \;\times\; G/P \;&\to\; G/P \\ (g, xP) \;&\mapsto\; gxP\ . \end{aligned}$$

For $n, x \in N$ and $t \in L$ we have $ntxw_0P = ntxt^{-1}w_0P = (nt.x)w_0P$ hence the action of P on the open cell corresponds to the action of P on N introduced before proposition 3.17, ie. the homeomorphism

$$N \to \mathcal{C}, \quad u \mapsto x_u := uw_0P$$

is P-equivariant.

When M is an étale $A[P_+]$-module, this allows us to systematically view the map Res in the following as a P-equivariant homomorphism of A-algebras

$$\mathrm{Res} : C_c^\infty(\mathcal{C}, A) \;\to\; \mathrm{End}_A(M^P)$$

and the corresponding sheaf (theorem 3.23) as a sheaf on $\mathcal{C}$. Our purpose is to show that this sheaf extends naturally to a G-equivariant sheaf on G/P for certain étale $A[P_+]$-modules. When M is a complete topologically étale $A[P_+]$-module we note that also integration with respect to the measure Res (proposition 4.8) will be viewed in the following as a map

$$\begin{aligned} C_c(\mathcal{C}, E^{cont}) &\rightarrow E^{cont} \\ f &\mapsto \int_{\mathcal{C}} f \, d\operatorname{Res} \end{aligned}$$

on the space $C_c(\mathcal{C}, E^{cont})$ of compactly supported continuous maps from $\mathcal{C}$ to E^{cont}.

5.1 Topological G-space G/P and the map α

Definition 5.2. *An open subset $\mathcal{U}$ of G/P is called standard if there is a $g \in G$ such that $g\mathcal{U}$ is contained in the open cell $\mathcal{C}$.*

The inclusion $g\mathcal{U} \subset Nw_0P/P$ is equivalent to $\mathcal{U} = g^{-1}Uw_0P/P$ for a unique open subset $U \subset N$. An open subset of a standard open subset is standard. The translates by G of N_0w_0P/P form a basis of the topology of G/P.

Proposition 5.3. *A compact open subset $\mathcal{U} \subset G/P$ is a disjoint union*

$$\mathcal{U} = \bigsqcup_{g \in I} g^{-1}Vw_0P/P$$

where $V \subset N$ is a compact open subgroup and $I \subset G$ a finite subset.

Proof. We first observe that any open covering of $\mathcal{U}$ can be refined into a disjoint open covering. In our case, this implies that $\mathcal{U}$ has a finite disjoint covering by standard compact open subsets. Let $g^{-1}Uw_0P/P \subset G/P$ be a standard compact open subset. Then $U = \sqcup_{u \in J} uV$ (disjoint union) with a finite set $J \subset U$ and $V \subset N$ is a compact open subgroup. Then $g^{-1}Uw_0P/P = \sqcup_{h \in I} h^{-1}Vw_0P/P$ (disjoint union) where $I = \{u^{-1}g \mid u \in J\}$. $\square$

For $g \in G$ and x in the non empty open subset $g^{-1}\mathcal{C} \cap \mathcal{C}$ of G/P (remark 5.1), there is a unique element $\alpha(g, x) \in P$ such that, if $x = uw_0P/P$ with $u \in N$, then

$$guw_0N \ = \ \alpha(g, x)uw_0N \ .$$

We give some properties of the map α.

Lemma 5.4. *Let $g \in G$. Then*
(i) $g^{-1}\mathcal{C} \cap \mathcal{C} = \mathcal{C}$ if and only if $g \in P$.
(ii) The map $\alpha(g, .) : g^{-1}\mathcal{C} \cap \mathcal{C} \to P$ is continuous.
(iii) We have $gx = \alpha(g, x)x$ for $x \in g^{-1}\mathcal{C} \cap \mathcal{C}$ and we have $\alpha(b, x) = b$ for $b \in P$ and $x \in \mathcal{C}$.

Proof. (i) We have $g^{-1}\mathcal{C} \cap \mathcal{C} = \mathcal{C}$ if and only if $gNw_0P \subset Nw_0P$ if and only if $g \in P$. Indeed, the condition $hPw_0P \ \subset \ Pw_0P$ on $h \in G$ depends only on PhP and for $w \in W$, the condition $w\,Pw_0P \ \subset \ Pw_0P$ implies $ww_0 \in Pw_0P$ hence $ww_0 \in w_0L$ by the hypothesis b) hence $w \in L$.

(ii) Let $N_g \subset N$ be such that $N_gw_0P/P = g^{-1}\mathcal{C} \cap \mathcal{C}$. It suffices to show that the map $u \to \alpha(g, uw_0P)u : N_g \to P$ is continuous. This follows from the continuity of the maps $u \mapsto guw_0N : N_g \to Pw_0P/N = Pw_0N/N$ and $bw_0N \mapsto b : Pw_0N/N \to P$.

(iii) Obvious. $\square$

Lemma 5.5. *Let $g, h \in G$ and $x \in (gh)^{-1}\mathcal{C} \cap h^{-1}\mathcal{C} \cap \mathcal{C}$. Then $hx \in g^{-1}\mathcal{C} \cap \mathcal{C}$ and we have*

$$\alpha(gh, x) = \alpha(g, hx)\alpha(h, x) \ .$$

Proof. The first part of the assertion is obvious. Let $x = uw_0P$ and $hx = vw_0P$ with $u, v \in N$. We have

$$huw_0N = \alpha(h,x)uw_0N, \ gvw_0N = \alpha(g,hx)vw_0N, \ \text{and } \alpha(gh,x)uw_0N = ghuw_0N \ .$$

The first identity implies $\alpha(h,x)uw_0P = vw_0P$, hence $v^{-1}\alpha(h,x)u \in P \cap w_0Pw_0^{-1}$. Hypothesis d) easily yields $P \cap w_0Pw_0^{-1} = L$, hence $\alpha(h,x)u = vt$ for some $t \in L$. Multiplying the second identity on the right by $w_0^{-1}tw_0$ we obtain $gvtw_0N = \alpha(g,hx)vtw_0N = \alpha(g,hx)\alpha(h,x)uw_0N$. Finally, by inserting the first identity into the right hand side of the third identity we get

$$\alpha(gh,x)uw_0N = g\alpha(h,x)uw_0N = gvtw_0N = \alpha(g,hx)\alpha(h,x)uw_0N$$

which is the assertion. $\square$

It will be technically convenient later to work on N instead of $\mathcal{C}$. For $g \in G$ let therefore N_g be the open subset of N such that $\mathcal{C} \cap g^{-1}\mathcal{C} = N_gw_0P/P$. We have $N_g = N$ if and only if $g \in P$ (lemma 5.4 (i)). We have the homeomorphism $u \mapsto x_u := uw_0P/P : N \xrightarrow{\sim} \mathcal{C}$ and the continuous map (lemma 5.4 (ii))

$$\begin{aligned} N_g &\longrightarrow P \\ u &\longmapsto \alpha(g, x_u) \end{aligned}$$

such that

$$\begin{aligned} gu &= \alpha(g,x_u)u\bar{n}(g,u) && \text{for some } \bar{n}(g,u) \in \overline{N} := w_0Nw_0^{-1}, \\ \alpha(g,x_u)u &= n(g,u)t(g,u) && \text{for some } \ n(g,u) \in N, t(g,u) \in L \ . \end{aligned} \tag{23}$$

Lemma 5.6. *Fix $g \in G$ and let $V \subset g^{-1}\mathcal{C} \cap \mathcal{C}$ be any compact open subset. There exists a disjoint covering $V = V_1 \dot\cup \ldots \dot\cup V_m$ by compact open subsets V_i and points $x_i \in V_i$ such that*

$$\alpha(g,x_i)V_i \subset gV \qquad \textit{for any } 1 \leq i \leq m.$$

Proof. We denote the inverse of the homeomorphism $u \mapsto x_u : N \xrightarrow{\sim} \mathcal{C}$ by $x \mapsto u_x$. The image $C \subset P$ of V under the continuous map $x \mapsto \alpha(g,x)u_x \ : \ V \to P$ is compact. As (lemma 5.4 (iii)) $\alpha(g,x)x = gx \in gV$ for any $x \in V$, under the continuous action of P on $\mathcal{C}$, every element in the compact set C maps the point w_0P into gV. It follows that there is an open neighborhood $V_0 \subset \mathcal{C}$ of w_0P such that $CV_0 \subset gV$. This means that

$$\alpha(g,x)u_xV_0 \subset gV \qquad \text{for any } x \in V.$$

Using the proposition 5.3 we find, by appropriately shrinking V_0, a disjoint covering of V of the form $V = u_1V_0 \dot\cup \ldots \dot\cup u_mV_0$ with $u_i \in N$. We put $x_i := u_iw_0P$. $\square$

We denote by $G_X := \{x \in G \mid xX \subset X\}$ the G-stabilizer of a subset $X \subset G/P$ and by

$$G_X^\dagger := \{g \in G \mid g \in G_X \ , \ g^{-1} \in G_X\} \ = \ \{x \in G \mid xX \ = \ X\}$$

the subgroup of invertible elements of G_X. If G_X is open then its inverse monoid is open hence $G_X^\dagger$ is open (and conversely).

Lemma 5.7. *The G-stabilizers $G_{\mathcal{U}}$ and $G_{\mathcal{U}}^\dagger$ are open in G, for any compact open subset $\mathcal{U} \subset G/P$.*

Proof. By proposition 5.3 it suffices to consider the case where $\mathcal{U} = Uw_0P/P$ for some compact open subgroup $U \subset N$. As $Uw_0P \subset G$ is an open subset containing w_0 there exists an open subgroup $K \subset G$ such that $Kw_0 \subset Uw_0P$. The set $U/(K \cap U)$ is finite because U is compact and $(K \cap U) \subset U$ is an open subgroup. The finite intersection $K' := \bigcap_{u \in U/(U \cap K)} uKu^{-1} = \bigcap_{u \in U} uKu^{-1}$ is an open subgroup of K which is normalized by U. But $K'U = UK'$ implies that $K'Uw_0P = UK'w_0P \subset U(Uw_0P)P = Uw_0P$, and hence that $K' \subset G_{\mathcal{U}}$. We deduce that $G_{\mathcal{U}}$ is open. Hence $G^\dagger_{\mathcal{U}}$ is open. □

Remark 5.8. *The G-stabilizer of the open cell $\mathcal{C}$ is the group P.*

Proof. Lemma 5.4 (i). □

For $\mathcal{U} \subset \mathcal{C}$ the map

$$G_{\mathcal{U}} \times \mathcal{U} \to P \quad , \quad (g, x) \mapsto \alpha(g, x) \tag{24}$$

is continuous because, if $\mathcal{U} = Uw_0P/P$ with U open in N, then the map $(g, u) \mapsto guw_0N : G_{\mathcal{U}} \times U \to Pw_0P/N = Pw_0N/N$ is continuous (cf. the proof of lemma 5.4 (ii)).

5.2 Equivariant sheaves and modules over skew group rings

Our construction of the sheaf on G/P will proceed through a module theoretic interpretation of equivariant sheaves. The ring $C^\infty_c(\mathcal{C}, A)$ has no unit element. But it has sufficiently many idempotents (the characteristic functions 1_V of the compact open subsets $V \subset \mathcal{C}$). A (left) module Z over $C^\infty_c(\mathcal{C}, A)$ is called nondegenerate if for any $z \in Z$ there is an idempotent $e \in C^\infty_c(\mathcal{C}, A)$ such that $ez = z$.

It is well known that the functor

$$\text{sheaves of } A\text{-modules on } \mathcal{C} \ \to \ \text{nondegenerate } C^\infty_c(\mathcal{C}, A)\text{-modules}$$

which sends a sheaf $\mathcal{S}$ to the A-module of global sections with compact support $\mathcal{S}_c(\mathcal{C}) := \bigcup_V \mathcal{S}(V)$, with V running over all compact open subsets in $\mathcal{C}$, is an equivalence of categories. In fact, as we have discussed in the proof of the theorem 3.23 a quasi-inverse functor is given by sending the module Z to the sheaf whose sections on the compact open subset $V \subset \mathcal{C}$ are equal to $1_V Z$.

In order to extend this equivalence to equivariant sheaves we note that the group P acts, by left translations, from the right on $C^\infty_c(\mathcal{C}, A)$ which we write as $(f, b) \mapsto f^b(.) := f(b.)$. This allows to introduce the skew group ring

$$\mathcal{A}_{\mathcal{C}} := C^\infty_c(\mathcal{C}, A)\#P = \oplus_{b \in P}\, bC^\infty_c(\mathcal{C}, A)$$

in which the multiplication is determined by the rule

$$(b_1 f_1)(b_2 f_2) = b_1 b_2 f_1^{b_2} f_2 \qquad \text{for } b_i \in P \text{ and } f_i \in C^\infty_c(\mathcal{C}, A).$$

It is easy to see that the above functor extends to an equivalence of categories

$$P\text{-equivariant sheaves of } A\text{-modules on } \mathcal{C} \ \xrightarrow{\simeq} \ \text{nondegenerate } \mathcal{A}_{\mathcal{C}}\text{-modules}.$$

We have the completely analogous formalism for the G-space G/P. The only small difference is that, since G/P is assumed to be compact, the ring $C^\infty(G/P, A)$ of locally constant A-valued functions on G/P is unital. The skew group ring

$$\mathcal{A}_{G/P} := C^\infty(G/P, A)\#G = \oplus_{g \in G}\, gC^\infty(G/P, A)$$

therefore is unital as well, and the equivalence of categories reads

$$G\text{-equivariant sheaves of } A\text{-modules on } G/P \xrightarrow{\simeq} \text{unital } \mathcal{A}_{G/P}\text{-modules.}$$

For any open subset $\mathcal{U} \subset G/P$ the A-algebra $C_c^\infty(\mathcal{U}, A)$ of A-valued locally constant and compactly supported functions on $\mathcal{U}$ is, by extending functions by zero, a subalgebra of $C^\infty(G/P, A)$. It follows in particular that $\mathcal{A}_\mathcal{C}$ is a subring of $\mathcal{A}_{G/P}$. There is a for our purposes very important ring in between these two rings which is defined to be

$$\mathcal{A} := \mathcal{A}_{\mathcal{C}\subset G/P} := \oplus_{g\in G}\, gC_c^\infty(g^{-1}\mathcal{C} \cap \mathcal{C}, A)\ .$$

That $\mathcal{A}$ indeed is multiplicatively closed is immediate from the following observation. If $\mathrm{supp}(f)$ denotes the support of a function $f \in C^\infty(G/P, A)$ then we have the formula

$$\text{(25)}\qquad \mathrm{supp}(f_1^g f_2) = g^{-1}\,\mathrm{supp}(f_1) \cap \mathrm{supp}(f_2) \qquad \text{for } g \in G \text{ and } f_1, f_2 \in C^\infty(G/P, A).$$

In particular, if $f_i \in C_c^\infty(g_i^{-1}\mathcal{C} \cap \mathcal{C}, A)$ then

$$\mathrm{supp}(f_1^{g_2} f_2) \subset g_2^{-1}(g_1^{-1}\mathcal{C} \cap \mathcal{C}) \cap (g_2^{-1}\mathcal{C} \cap \mathcal{C}) \subset (g_1 g_2)^{-1}\mathcal{C} \cap \mathcal{C}\ .$$

We also have the A-submodule

$$\mathcal{Z} := \oplus_{g\in G}\, gC_c^\infty(\mathcal{C}, A)$$

of $\mathcal{A}_{G/P}$. Using (25) again one sees that $\mathcal{Z}$ actually is a left ideal in $\mathcal{A}_{G/P}$ which at the same time is a right $\mathcal{A}$-submodule. This means that we have the well defined functor

$$\begin{aligned} \text{nondegenerate } \mathcal{A}\text{-modules} &\ \to\ \text{unital } \mathcal{A}_{G/P}\text{-modules} \\ Z &\ \mapsto\ \mathcal{Z} \otimes_\mathcal{A} Z\ . \end{aligned}$$

Remark 5.9. *The functor of restricting G-equivariant sheaves on G/P to the open cell $\mathcal{C}$ is faithful and detects isomorphisms.*

Proof. Any sheaf homomorphism which is the zero map, resp. an isomorphism, on sections on any compact open subset of $\mathcal{C}$ has, by G-equivariance, the same property on any standard compact open subset and hence, by the proposition 5.3, on any compact open subset of G/P. □

Proposition 5.10. *The above functor $Z \mapsto \mathcal{Z} \otimes_\mathcal{A} Z$ is an equivalence of categories; a quasi-inverse functor is given by sending the $\mathcal{A}_{G/P}$-module Y to $\bigcup_{V\subset\mathcal{C}} 1_V Y$ where V runs over all compact open subsets in $\mathcal{C}$.*

Proof. We abbreviate the asserted candidate for the quasi-inverse functor by $R(Y) := \bigcup_{V\subset\mathcal{C}} 1_V Y$. It immediately follows from the remark 5.9 that the functor R, which in terms of sheaves is the functor of restriction, is faithful and detects isomorphisms.

By a slight abuse of notation we identify in the following a function $f \in C^\infty(G/P, A)$ with the element $1f \in \mathcal{A}_{G/P}$, where $1 \in G$ denotes the unit element. Let $V \subset \mathcal{C}$ be a compact open subset. Then $1_V \mathcal{A}_{G/P} 1_V$ is a subring of $\mathcal{A}_{G/P}$ (with the unit element 1_V), which we compute as follows:

$$\begin{aligned} 1_V \mathcal{A}_{G/P} 1_V &= \sum_{g\in G} 1_V g C^\infty(V, A) = \sum_{g\in G} g 1_{g^{-1}V} C^\infty(V, A) \\ &= \sum_{g\in G} g C^\infty(g^{-1}V \cap V, A)\ . \end{aligned}$$

We note:

– If $U \subset V \subset \mathcal{C}$ are two compact open subsets then $1_V \mathcal{A}_{G/P} 1_V \supset 1_U \mathcal{A}_{G/P} 1_U$.

– Let $f \in C_c^\infty(g^{-1}\mathcal{C} \cap \mathcal{C}, A)$ be supported on the compact open subset $U \subset g^{-1}\mathcal{C} \cap \mathcal{C}$. Then $V := U \cup gU$ is compact open in $\mathcal{C}$ as well, and $U \subset g^{-1}V \cap V$. This shows that $C_c^\infty(g^{-1}\mathcal{C} \cap \mathcal{C}, A) = \bigcup_{V \subset \mathcal{C}} C^\infty(g^{-1}V \cap V, A)$.

We deduce that

$$\bigcup_{V \subset \mathcal{C}} 1_V \mathcal{A}_{G/P} 1_V = \mathcal{A}_{\mathcal{C} \subset G/P} = \mathcal{A} \ .$$

A completely analogous computation shows that

$$1_V \mathcal{Z} = 1_V \mathcal{A} \ .$$

Given a nondegenerate $\mathcal{A}$-module Z the map

$$\begin{aligned} 1_V(\mathcal{Z} \otimes_{\mathcal{A}} Z) = (1_V \mathcal{Z}) \otimes_{\mathcal{A}} Z = (1_V \mathcal{A}) \otimes_{\mathcal{A}} Z \ &\rightarrow \ 1_V Z \\ 1_V a \otimes z = 1_V \otimes 1_V az \ &\mapsto \ 1_V az \end{aligned}$$

therefore is visibly an isomorphism of $1_V \mathcal{A}_{G/P} 1_V$-modules. In the limit with respect to V we obtain a natural isomorphism of $\mathcal{A}$-modules

$$R(\mathcal{Z} \otimes_{\mathcal{A}} Z) \xrightarrow{\cong} Z \ .$$

On the other hand, for any unital $\mathcal{A}_{G/P}$-module Y there is the obvious natural homomorphism of $\mathcal{A}_{G/P}$-modules

$$\begin{aligned} \mathcal{Z} \otimes_{\mathcal{A}} R(Y) \ &\rightarrow \ Y \\ a \otimes z \ &\mapsto \ az \ . \end{aligned}$$

It is an isomorphism because applying the functor R, which detects isomorphisms, to it gives the identity map. $\square$

Remark 5.11. *Let Z be a nondegenerate $\mathcal{A}$-module. Viewed as an $\mathcal{A}_{\mathcal{C}}$-module it corresponds to a P-equivariant sheaf $\widetilde{Z}$ on $\mathcal{C}$. On the other hand, the $\mathcal{A}_{G/P}$-module $Y := \mathcal{Z} \otimes_{\mathcal{A}} Z$ corresponds to a G-equivariant sheaf $\widetilde{Y}$ on G/P. We have $\widetilde{Y}|\mathcal{C} = \widetilde{Z}$, i. e., the sheaf $\widetilde{Y}$ extends the sheaf $\widetilde{Z}$.*

We have now seen that the step of going from $\mathcal{A}$ to $\mathcal{A}_{G/P}$ is completely formal. On the other hand, for any topologically étale $A[P_+]$-module M, the P-equivariance of Res together with the proposition 4.5 imply that Res extends to the A-algebra homomorphism

$$\begin{aligned} \mathrm{Res} \ : \qquad \mathcal{A}_{\mathcal{C}} \ &\rightarrow \ \mathrm{End}_A^{cont}(M^P) \\ \sum_{b \in P} b f_b \ &\mapsto \ \sum_{b \in P} b \circ \mathrm{Res}(f_b) \ . \end{aligned}$$

When M is compact it is relatively easy, as we will show in the next section, to further extend this map from $\mathcal{A}_{\mathcal{C}}$ to $\mathcal{A}$. This makes crucially use of the full topological module M^P and not only its submodule M_c^P of sections with compact support. When M is not compact this extension problem is much more subtle and requires more facts about the ring $\mathcal{A}$.

We introduce the compact open subset $\mathcal{C}_0 := N_0 w_0 P/P$ of $\mathcal{C}$, and we consider the unital subrings

$$\mathcal{A}_0 := 1_{\mathcal{C}_0} \mathcal{A}_{G/P} 1_{\mathcal{C}_0} = \sum_{g \in G} g C^\infty(g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, A)$$

and

$$\mathcal{A}_{\mathcal{C}0} := 1_{\mathcal{C}_0} \mathcal{A}_{\mathcal{C}} 1_{\mathcal{C}_0} = \sum_{b \in P} b C^\infty(b^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, A)$$

of $\mathcal{A}$ and $\mathcal{A}_\mathcal{C}$, respectively. Obviously $\mathcal{A}_{\mathcal{C}0} \subseteq \mathcal{A}_0$ with the same unit element $1_{\mathcal{C}_0}$. Since $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$ is nonempty if and only if $g \in N_0\overline{P}N_0$ we in fact have

$$\mathcal{A}_0 = \sum_{g \in N_0\overline{P}N_0} gC^\infty(g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, A) \ .$$

The map $A[G] \longrightarrow \mathcal{A}_{G/P}$ sending g to $g1_{G/P}$ is a unital ring homomorphism. Hence we may view $\mathcal{A}_{G/P}$ as an $A[G]$-module for the adjoint action

$$\begin{aligned} G \times \mathcal{A}_{G/P} &\longrightarrow \mathcal{A}_{G/P} \\ (g, y) &\longmapsto (g1_{G/P})y(g1_{G/P})^{-1} \ . \end{aligned}$$

One checks that $\mathcal{A}_\mathcal{C} \subseteq \mathcal{A}$ are $A[P]$-submodules, that $\mathcal{A}_{\mathcal{C}0} \subseteq \mathcal{A}_0$ are $A[P_+]$-submodules, and that the map $\mathrm{Res} : \mathcal{A}_\mathcal{C} \longrightarrow E^{cont}$ is a homomorphism of $A[P]$-modules.

Proposition 5.12. *The homomorphism of $A[P]$-modules*

$$\begin{aligned} A[P] \otimes_{A[P_+]} \mathcal{A}_0 &\xrightarrow{\cong} \mathcal{A} \\ b \otimes y &\longmapsto (b1_{G/P})y(b1_{G/P})^{-1} \end{aligned}$$

is bijective; it restricts to an isomorphism $A[P] \otimes_{A[P_+]} \mathcal{A}_{\mathcal{C}0} \xrightarrow{\cong} \mathcal{A}_\mathcal{C}$.

Proof. Since $P = s^{-\mathbb{N}}P_+$ the assertion amounts to the claim that

$$\mathcal{A} = \bigcup_{n \geq 0} (s^{-n}1_{G/P})\mathcal{A}_0(s^n 1_{G/P})$$

and correspondingly for $\mathcal{A}_\mathcal{C}$. But we have

$$(s^{-n}1_{G/P})\big(gC^\infty(g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, A)\big)(s^n 1_{G/P}) = s^{-n}gs^n C^\infty((s^{-n}g^{-1}s^n)s^{-n}\mathcal{C}_0 \cap s^{-n}\mathcal{C}_0, A)$$

for any $n \geq 0$ and any $g \in G$. □

Suppose that we may extend the map $\mathrm{Res} : \mathcal{A}_{\mathcal{C}0} \longrightarrow \mathrm{End}_A^{cont}(M^P)$ to an $A[P_+]$-equivariant (unital) A-algebra homomorphism

$$\mathcal{R}_0 : \mathcal{A}_0 \longrightarrow \mathrm{End}_A(M^P) \ .$$

By the above proposition 5.12 it further extends uniquely to an $A[P]$-equivariant map $\mathcal{R} : \mathcal{A} \longrightarrow \mathrm{End}_A(M^P)$.

Lemma 5.13. *The map $\mathcal{R}$ is multiplicative.*

Proof. Using proposition 5.12 we have that two arbitrary elements $y, z \in \mathcal{A}$ are of the form $y = (s^{-m}1_{G/P})y_0(s^m 1_{G/P}), z = (s^{-n}1_{G/P})z_0(s^n 1_{G/P})$ with $m, n \in \mathbb{N}$ and $y_0, z_0 \in \mathcal{A}_0$. We can choose $m = n$. It follows that

$$yz = (s^{-m}1_{G/P})y_0z_0(s^m 1_{G/P}) = (s^{-m}1_{G/P})x_0(s^m 1_{G/P})$$

with $x_0 := y_0z_0 \in \mathcal{A}_0$, and that

$$\begin{aligned} \mathcal{R}(yz) &= \mathcal{R}((s^{-m}1_{G/P})x_0(s^m 1_{G/P})) = s^{-m} \circ \mathcal{R}_0(x_0) \circ s^m \\ &= s^{-m} \circ \mathcal{R}_0(y_0) \circ \mathcal{R}_0(z_0) \circ s^m \\ &= (s^{-m} \circ \mathcal{R}_0(y_0) \circ s^m) \circ (s^{-m} \circ \mathcal{R}_0(z_0) \circ s^m) \\ &= \mathcal{R}(y) \circ \mathcal{R}(z) \ . \end{aligned}$$

□

Note that the images $\mathrm{Res}(\mathcal{A}_{\mathcal{C}0})$ and $\mathcal{R}_0(\mathcal{A}_0)$ necessarily lie in the image of $\mathrm{End}_A(M) = \mathrm{End}_A(\mathrm{Res}(1_{\mathcal{C}_0})(M^P))$ by the natural embedding into $\mathrm{End}_A(M^P)$. This reduces us to search for an $A[P_+]$-equivariant (unital) A-algebra homomorphism

$$\mathcal{R}_0 : \mathcal{A}_0 \longrightarrow \mathrm{End}_A(M)$$

which extends $\mathrm{Res}\,|\mathcal{A}_{\mathcal{C}0}$. In fact, since for $g \in N_0\overline{P}N_0$ and $f \in C^\infty(g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0, A)$ we have $gf = (g1_{g^{-1}\mathcal{C}_0\cap\mathcal{C}_0})(1f)$ with $1f \in \mathcal{A}_{\mathcal{C}0}$ it suffices to find the elements

$$\mathcal{H}_g = \mathcal{R}_0(g1_{g^{-1}\mathcal{C}_0\cap\mathcal{C}_0}) \in \mathrm{End}_A(M) \qquad \text{for } g \in N_0\overline{P}N_0\ .$$

Note that $P_+ = N_0L_+$ is contained in $N_0\overline{P}N_0 = N_0L\overline{N}N_0$.

Proposition 5.14. *We suppose given, for any $g \in N_0\overline{P}N_0$, an element $\mathcal{H}_g \in \mathrm{End}_A(M)$. Then the map*

$$\begin{aligned} \mathcal{R}_0 : \qquad \mathcal{A}_0 &\longrightarrow \mathrm{End}_A(M) \\ \sum_{g\in N_0\overline{P}N_0} gf_g &\longmapsto \sum_{g\in N_0\overline{P}N_0} \mathcal{H}_g \circ \mathrm{res}(f_g) \end{aligned}$$

is an $A[P_+]$-equivariant (unital) A-algebra homomorphism which extends $\mathrm{Res}\,|\mathcal{A}_{\mathcal{C}0}$ if and only if, for all $g, h \in N_0\overline{P}N_0$, $b \in P \cap N_0\overline{P}N_0$, and all compact open subsets $\mathcal{V} \subset \mathcal{C}_0$, the relations

H1. $\mathrm{res}(1_\mathcal{V}) \circ \mathcal{H}_g = \mathcal{H}_g \circ \mathrm{res}(1_{g^{-1}\mathcal{V}\cap\mathcal{C}_0})$,

H2. $\mathcal{H}_g \circ \mathcal{H}_h = \mathcal{H}_{gh} \circ \mathrm{res}(1_{(gh)^{-1}\mathcal{C}_0\cap h^{-1}\mathcal{C}_0\cap\mathcal{C}_0})$,

H3. $\mathcal{H}_b = b \circ \mathrm{res}(1_{b^{-1}\mathcal{C}_0\cap\mathcal{C}_0})$.

hold true. When H1 is true, H2 can equivalently be replaced by

$$\mathcal{H}_g \circ \mathcal{H}_h = \mathcal{H}_{gh} \circ \mathrm{res}(1_{h^{-1}\mathcal{C}_0\cap\mathcal{C}_0})\ .$$

Proof. Necessity of the relations is easily checked. Vice versa, the first two relations imply that $\mathcal{R}_0$ is multiplicative. The third relation says that $\mathcal{R}_0$ extends $\mathrm{Res}\,|\mathcal{A}_{\mathcal{C}0}$.

The last sentence of the assertion derives from the fact that we have

$$\begin{aligned} \mathcal{H}_{gh} \circ \mathrm{res}(1_{(gh)^{-1}\mathcal{C}_0\cap h^{-1}\mathcal{C}_0\cap\mathcal{C}_0}) &= \mathcal{H}_{gh} \circ \mathrm{res}(1_{(gh)^{-1}\mathcal{C}_0\cap\mathcal{C}_0}) \circ \mathrm{res}(1_{h^{-1}\mathcal{C}_0\cap\mathcal{C}_0}) \\ &= \mathcal{H}_{gh} \circ \mathrm{res}(1_{h^{-1}\mathcal{C}_0\cap\mathcal{C}_0}) \end{aligned}$$

since $\mathcal{H}_{gh} \circ \mathrm{res}(1_{(gh)^{-1}\mathcal{C}_0\cap\mathcal{C}_0}) = \mathcal{H}_{gh}$ by the first relation.

The P_+-equivariance is equivalent to the identity

$$\mathcal{R}_0((c1_{G/P})gf_g(c1_{G/P})^{-1}) = \varphi_c \circ \mathcal{R}_0(gf_g) \circ \psi_c$$

where $c \in P_+$ and f_g is any function in $C^\infty(g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0)$. By the definition of $\mathcal{R}_0$ and the P_+-equivariance of res the left hand side is equal to

$$\mathcal{H}_{cgc^{-1}} \circ \varphi_c \circ \mathrm{res}(f_g) \circ \psi_c$$

whereas the right hand side is

$$\varphi_c \circ \mathcal{H}_g \circ \mathrm{res}(f_g) \circ \psi_c\ .$$

Since ψ_c is surjective and $\mathrm{res}(f_g) = \mathrm{res}(1_{g^{-1}C_0\cap\mathcal{C}_0})\circ\mathrm{res}(f_g)$ we see that the P_+-equivariance of $\mathcal{R}_0$ is equivalent to the identity

$$\mathcal{H}_{cgc^{-1}} \circ \varphi_c \circ \mathrm{res}(1_{g^{-1}C_0\cap\mathcal{C}_0}) = \varphi_c \circ \mathcal{H}_g \circ \mathrm{res}(1_{g^{-1}C_0\cap\mathcal{C}_0})\ .$$

But as a special case of the first relation we have $\mathcal{H}_g \circ \mathrm{res}(1_{g^{-1}C_0\cap\mathcal{C}_0}) = \mathcal{H}_g$. Hence the latter identity coincides with the relation

$$\mathcal{H}_{cgc^{-1}} \circ \varphi_c \circ \mathrm{res}(1_{g^{-1}C_0\cap\mathcal{C}_0}) = \varphi_c \circ \mathcal{H}_g\ .$$

This relation holds true because $\varphi_c = \mathcal{H}_c$ and by the second relation $\mathcal{H}_{cgc^{-1}} \circ \mathcal{H}_c = \mathcal{H}_{cg}$ and $\mathcal{H}_c \circ \mathcal{H}_g = \mathcal{H}_{cg} \circ \mathrm{res}(1_{g^{-1}C_0\cap\mathcal{C}_0})$. $\square$

5.3 Integrating α when M is compact

Let M be a compact topologically étale $A[P_+]$-module. Then M^P is compact, hence the continuous action of P on M^P (proposition 4.5) induces a continuous map $P \to E^{cont}$.

We will construct an extension $\widetilde{\mathrm{Res}}$ of Res to $\mathcal{A}_{\mathcal{C}\subset G/P}$ by integration. For any $g \in G$, we consider the continuous map

$$\alpha_g \ : \ g^{-1}\mathcal{C} \cap \mathcal{C} \xrightarrow{\alpha(g,.)} P \ \to \ E^{cont} \ .$$

We introduce the A-linear maps

$$\begin{aligned} \rho \ : \ \mathcal{A} = \mathcal{A}_{\mathcal{C}\subset G/P} \ &\to \ C_c(\mathcal{C}, E^{cont}) \\ \sum_{g\in G} g f_g \ &\mapsto \ \sum_{g\in G} \alpha_g f_g \ . \end{aligned}$$

and

$$\begin{aligned} \widetilde{\mathrm{Res}} \ : \ \mathcal{A} = \mathcal{A}_{\mathcal{C}\subset G/P} \ &\to \ E^{cont} \\ a \ &\mapsto \ \int_{\mathcal{C}} \rho(a) d\,\mathrm{Res} \ . \end{aligned}$$

For $b \in P$ the map α_b is the constant map on $\mathcal{C}$ with value b (lemma 5.3 iii). It follows that

$$\widetilde{\mathrm{Res}} \,|\, \mathcal{A}_{\mathcal{C}} = \mathrm{Res} \ .$$

is an extension as we want it.

Theorem 5.15. $\widetilde{\mathrm{Res}}$ *is a homomorphism of A-algebras.*

Proof. Let $g, h \in G$ and let V_g and V_h be compact open subsets of $g^{-1}\mathcal{C} \cap \mathcal{C}$ and $h^{-1}\mathcal{C} \cap \mathcal{C}$, respectively. We have to show that

$$\widetilde{\mathrm{Res}}((g 1_{V_g})(h 1_{V_h})) = \widetilde{\mathrm{Res}}(g 1_{V_g}) \circ \widetilde{\mathrm{Res}}(h 1_{V_h})$$

holds true. This is, by definition of $\widetilde{\mathrm{Res}}$, equivalent to the identity

$$\int_{\mathcal{C}} \alpha_{gh} 1_{h^{-1}V_g \cap V_h} d\mathrm{Res} = \int_{\mathcal{C}} \alpha_g 1_{V_g} d\mathrm{Res} \circ \int_{\mathcal{C}} \alpha_h 1_{V_h} d\mathrm{Res} \ .$$

Let U_g, U_h be the open compact subsets of N corresponding to V_g, V_h and let f be the map $\alpha_g 1_{V_g}$, seen as a map on N with support on U_g. Let $\mathcal{L} \subset M^P$ be an open A-submodule and let $V_{\mathcal{L}}$ be chosen as in lemma 4.10, with $\lambda = \mathrm{Res}$. If we let $N_{k,v} = \alpha(h, x_v).(vN_k)$, then the P-equivariance of Res combined with remark 4.11 yield, for $v \in U_h$ and $k \geq 1$

$$\int_{\mathcal{C}} \alpha_g 1_{V_g} d\mathrm{Res} \circ \alpha(h, x_v) \circ \mathrm{Res}(1_{vN_k}) = \int_N f d\mathrm{Res} \circ \mathrm{Res}(1_{N_{k,v}}) \circ \alpha(h, x_v)$$

$$= \int_N (f 1_{N_{k,v}}) d\mathrm{Res} \circ \alpha(h, x_v) \ .$$

Writing $\alpha(h, x_v) = n_v t_v$ with $n_v \in N$ and $t_v \in L$, for k large enough we have $U_g t_v N_k t_v^{-1} \subset U_g$ and $t_v N_k t_v^{-1} \subset V_{\mathcal{L}}$ for all $v \in U_h$ (by compactness of $(t_v)_{v \in U_h}$). Since $N_{k,v} = (\alpha(h, x_v).v) t_v N_k t_v^{-1}$, we deduce that $N_{k,v} \cap U_g \neq \emptyset \Leftrightarrow \alpha(h, x_v).v \in U_g \Leftrightarrow h x_v \in V_g$ and hence, by lemma 4.10 for all sufficiently large k we have, uniformly in $v \in U_h$,

$$\int_N f 1_{N_{k,v}} d\mathrm{Res} \equiv 1_{x_v \in h^{-1}V_g \cap V_h} f(\alpha(h, x_v).v) \circ \mathrm{Res}(1_{N_{k,v}}) =$$

$$= 1_{x_v \in h^{-1}V_g \cap V_h} \alpha(g, h x_v) \circ \mathrm{Res}(1_{N_{k,v}}) \pmod{E_{\mathcal{L}}^{\mathrm{cont}}} \ .$$

Combining the last two relations with lemma 5.5, and using again the P-equivariance of Res, we obtain for k large enough and for all $v \in U_h$

$$\int_{\mathcal{C}} \alpha_g 1_{V_g} d\mathrm{Res} \circ \alpha(h, x_v) \circ \mathrm{Res}(1_{vN_k}) \equiv 1_{x_v \in h^{-1}V_g \cap V_h} \alpha(gh, x_v) \circ \mathrm{Res}(1_{vN_k}) \quad (\mathrm{mod}\ E_{\mathcal{L}}^{\mathrm{cont}}) \ .$$

The result follows by summing over v and letting $k \to \infty$ (Cor. 4.9). $\square$

5.4 G-equivariant sheaf on G/P

Let M be a compact topologically étale $A[P_+]$-module. We briefly survey our construction of a G-equivariant sheaf on G/P functorially associated with M.

From proposition 3.17 we obtain an A-algebra homomorphism

$$\mathrm{Res} \ : \ C_c^\infty(\mathcal{C}, A)\#P \ \to \ E^{cont}$$

which gives rise to a P-equivariant sheaf on $\mathcal{C}$ as described in detail in the theorem 3.23. By theorem 5.15, it extends to an A-algebra homomorphism

$$\widetilde{\mathrm{Res}} \ : \ \mathcal{A}_{\mathcal{C} \subset G/P} \ \to \ E^{cont} \ .$$

This homomorphism defines on the global sections with compact support M_c^P of the sheaf on $\mathcal{C}$ the structure of a nondegenerate $\mathcal{A}_{\mathcal{C} \subset G/P}$-module. The latter leads, by proposition 5.10, to the unital $C_c^\infty(G/P, A)\#G$-module $\mathcal{Z} \otimes_{\mathcal{A}} M_c^P$ which corresponds to a G-equivariant sheaf on G/P extending the earlier sheaf on $\mathcal{C}$ (remark 3.24). We will denote the sections of this latter sheaf on an open subset $\mathcal{U} \subset G/P$ by $M \boxtimes \mathcal{U}$. The restriction maps in this sheaf, for open subsets $\mathcal{V} \subset \mathcal{U} \subset G/P$, will simply be written as $\mathrm{Res}_{\mathcal{V}}^{\mathcal{U}} \ : \ M \boxtimes \mathcal{U} \ \to \ M \boxtimes \mathcal{V}$.

We observe that for a standard compact open subset $\mathcal{U} \subset G/P$ with $g \in G$ such that $g\mathcal{U} \subset \mathcal{C}$ the action of the element g on the sheaf induces an isomorphism of A-modules $M \boxtimes \mathcal{U} \xrightarrow{\cong} M \boxtimes g\mathcal{U} = M_{g\mathcal{U}}$. Being the image of a continuous projector on M^P (proposition 4.5), $M_{g\mathcal{U}}$ is naturally a compact topological A-module. We use the above isomorphism to transport this topology to $M \boxtimes \mathcal{U}$. The result is independent of the choice of g since, if $g\mathcal{U} = h\mathcal{U}$ for some other $h \in G$, then $h\mathcal{U} \subset (gh^{-1})^{-1}\mathcal{C} \cap \mathcal{C}$ and, by construction, the endomorphism gh^{-1} of $M \boxtimes h\mathcal{U}$ is given by the continuous map $\widetilde{\mathrm{Res}}(gh^{-1}1_{h\mathcal{U}})$.

A general compact open subset $\mathcal{U} \subset G/P$ is the disjoint union $\mathcal{U} = \mathcal{U}_1 \dot\cup \ldots \dot\cup \mathcal{U}_m$ of standard compact open subsets $\mathcal{U}_i$ (proposition 5.3). We equip $M \boxtimes \mathcal{U} = M \boxtimes \mathcal{U}_1 \oplus \ldots \oplus M \boxtimes \mathcal{U}_m$ with the direct product topology. One easily verifies that this is independent of the choice of the covering.

Finally, for an arbitrary open subset $\mathcal{U} \subset G/P$ we have $M \boxtimes \mathcal{U} = \varprojlim M \boxtimes \mathcal{V}$, where $\mathcal{V}$ runs over all compact open subsets $\mathcal{V} \subset \mathcal{U}$, and we equip $M \boxtimes \mathcal{U}$ with the corresponding projective limit topology.

It is straightforward to check that all restriction maps are continuous and that any $g \in G$ acts by continuous homomorphisms. We see that $(M \boxtimes \mathcal{U})_{\mathcal{U}}$ is a G-equivariant sheaf of compact topological A-modules.

Lemma 5.16. *For any compact open subset $\mathcal{U} \subset G/P$ the action $G_{\mathcal{U}}^\dagger \times (M \boxtimes \mathcal{U}) \ \to \ M \boxtimes \mathcal{U}$ of the open subgroup $G_{\mathcal{U}}^\dagger$ (lemma 5.7) on the sections on $\mathcal{U}$ is continuous.*

Proof. Using proposition 5.3, it suffices to consider the case that $\mathcal{U} \subset \mathcal{C}$. Note that $G_{\mathcal{U}}^\dagger$ acts by continuous automorphisms on $M \boxtimes \mathcal{U} = M_{\mathcal{U}}$. By (24) the map

$$\begin{aligned} G_{\mathcal{U}}^\dagger \times \mathcal{U} \ &\to \ E^{cont} \\ (g, x) \ &\mapsto \ \alpha_g(x) \end{aligned}$$

is continuous. Hence ([3] TG X.28 Th. 3) the corresponding map

$$G^\dagger_{\mathcal{U}} \;\to\; C(\mathcal{U}, E^{cont})$$

is continuous, where we always equip the module $C(\mathcal{U}, E^{cont})$ of E^{cont}-valued continuous maps on $\mathcal{U}$ with the compact-open topology. On the other hand it is easy to see that, for any measure λ on $\mathcal{C}$ with values in E^{cont}, the map

$$\int_{\mathcal{U}} .\, d\lambda \;:\; C(\mathcal{U}, E^{cont}) \;\to\; E^{cont}$$

is continuous. It follows that the map

$$\begin{aligned} G^\dagger_{\mathcal{U}} \;&\to\; E^{cont} \\ g \;&\mapsto\; \widetilde{\mathrm{Res}}(g1_{\mathcal{U}}) \end{aligned}$$

is continuous. The direct decomposition $M^P = M_{\mathcal{U}} \oplus M_{\mathcal{C}-\mathcal{U}}$ gives a natural inclusion map $\mathrm{End}_A^{cont}(M_{\mathcal{U}}) \to E^{cont}$ through which the above map factorizes. The resulting map

$$G^\dagger_{\mathcal{U}} \;\to\; \mathrm{End}_A^{cont}(M_{\mathcal{U}})$$

is continuous and coincides with the $G^\dagger_{\mathcal{U}}$-action on $M_{\mathcal{U}}$. As $M_{\mathcal{U}}$ is compact this continuity implies the continuity of the action $G^\dagger_{\mathcal{U}} \times M_{\mathcal{U}} \to M_{\mathcal{U}}$. □

The same construction can be done, starting from the compact topologically étale $A[P_U]$-module M_U, for any compact open subgroup $U \subset N$.

Proposition 5.17. *Let $U \subset N$ be a compact open subgroup. The G-equivariant sheaves on G/P associated to (N_0, M) and to (U, M_U) are equal.*

Proof. As the P-equivariant sheaves on the open cell associated to (N_0, M) and to (U, M_U) are equal by proposition 3.28, and as the function α_g depends only on the open cell, our formal construction gives the same G-equivariant sheaf. □

6 Integrating α when M is non compact

Recall that we have chosen a certain element $s \in Z(L)$ such that $L = L_- s^{\mathbb{Z}}$ and $(N_k = s^k N_0 s^{-k})_{k\in\mathbb{Z}}$ is a decreasing sequence with union N and trivial intersection. *We now suppose in addition that $(\overline{N}_k := s^{-k} w_0 N_0 w_0^{-1} s^k)_{k\in\mathbb{Z}}$ is a decreasing sequence with union $\overline{N} = w_0 N w_0^{-1}$ and trivial intersection.*

We have chosen A and M in section 4.1. *We suppose now in addition that M is a topologically étale $A[P_+]$-module which is Hausdorff and complete.*

Definition 6.1. *A special family of compact sets in M is a family $\mathfrak{C}$ of compact subsets of M satisfying :*

$\mathfrak{C}(1)$ *Any compact subset of a compact set in $\mathfrak{C}$ also lies in $\mathfrak{C}$.*

$\mathfrak{C}(2)$ *If $C_1, C_2, \ldots, C_n \in \mathfrak{C}$ then $\bigcup_{i=1}^n C_i$ is in $\mathfrak{C}$, as well.*

$\mathfrak{C}(3)$ *For all $C \in \mathfrak{C}$ we have $N_0 C \in \mathfrak{C}$.*

$\mathfrak{C}(4)$ *$M(\mathfrak{C}) := \bigcup_{C\in\mathfrak{C}} C$ is an étale $A[P_+]$-submodule of M.*

Note that M is the union of its compact subsets, and that the family of all compact subsets of M satisfies these four properties.

Let $\mathfrak{C}$ be a special family of compact sets in M. A map from $M(\mathfrak{C})$ to M is called $\mathfrak{C}$-continuous if its restriction to any $C \in \mathfrak{C}$ is continuous. We equip the A-module

$\mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$ of $\mathfrak{C}$-continuous A-linear homomorphisms from $M(\mathfrak{C})$ to M with the $\mathfrak{C}$-open topology. The A-submodules

$$E(C, \mathcal{M}) := \{f \in \mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M) \colon f(C) \subseteq \mathcal{M}\} \ ,$$

for any $C \in \mathfrak{C}$ and any open A-submodule $\mathcal{M} \subseteq M$, form a fundamental system of open neighborhoods of zero in $\mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$. Indeed, this system is closed for finite intersection by $\mathfrak{C}(2)$. Since N_0 is compact the $E(C, \mathcal{M})$ for C such that $N_0 C \subseteq C$ and $\mathcal{M}$ an $A[N_0]$-submodule still form a fundamental system of open neighborhoods of zero. (Lemma 4.3 and $\mathfrak{C}(3)$). We have:

- $\mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$ is a topological A-module.
- $\mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$ is Hausdorff, since $\mathfrak{C}$ covers $M(\mathfrak{C})$ by $\mathfrak{C}(4)$ and M is Hausdorff.
- $\mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$ is complete ([3] TG X.9 Cor.2).

6.1 $(s, \mathrm{res}, \mathfrak{C})$-integrals

We have the P_+-equivariant measure $\mathrm{res} : C^\infty(N_0, A) \longrightarrow \mathrm{End}_A^{cont}(M)$ on N_0. If M is not compact then it is no longer possible to integrate any map in the A-module $C(N_0, \mathrm{End}_A^{cont}(M))$ of all continuous maps on N_0 with values in $\mathrm{End}_A^{cont}(M)$ against this measure. This forces us to introduce a notion of integrability with respect to a special family of compact sets in M.

Definition 6.2. *A map $F \colon N_0 \to \mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$ is called integrable with respect to (s,* res*, $\mathfrak{C}$) if the limit*

$$\int_{N_0} F d\,\mathrm{res} := \lim_{k\to\infty} \sum_{u \in J(N_0/N_k)} F(u) \circ \mathrm{res}(1_{uN_k}) \ ,$$

where $J(N_0/N_k) \subseteq N_0$, for any $k \in \mathbb{N}$, is a set of representatives for the cosets in N_0/N_k, exists in $\mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$ and does not depend on the choice of the sets $J(N_0/N_k)$.

We suppress $\mathfrak{C}$ from the notation when $\mathfrak{C}$ is the family of all compact subsets of M.

Note that we regard $\mathrm{res}(1_{uN_k})$ as an element of $\mathrm{End}_A^{cont}(M(\mathfrak{C}))$. This makes sense as the algebraically étale submodule $M(\mathfrak{C})$ of the topologically étale module M is topologically étale.

One easily sees that the set $C^{int}(N_0, \mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M))$ of integrable maps is an A-module. The A-linear map

$$\int_{N_0} .d\,\mathrm{res} : C^{int}(N_0, \mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)) \longrightarrow \mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$$

will be called the $(s, \mathrm{res}, \mathfrak{C})$-integral.

We give now a general integrability criterion.

Proposition 6.3. *A map $F : N_0 \longrightarrow \mathrm{Hom}_A^{\mathfrak{C}\,cont}(M(\mathfrak{C}), M)$ is (s,* res*, $\mathfrak{C}$)-integrable if, for any $C \in \mathfrak{C}$ and any open A-submodule $\mathcal{M} \subseteq M$, there exists an integer $k_{C,\mathcal{M}} \geq 0$ such that*

$$(F(u) - F(uv)) \circ \mathrm{res}(1_{uN_{k+1}}) \in E(C, \mathcal{M}) \qquad \textit{for any } k \geq k_{C,\mathcal{M}},\ u \in N_0, \textit{ and } v \in N_k.$$

Proof. Let $J(N_0/N_k)$ and $J'(N_0/N_k)$, for $k \geq 0$, be two choices of sets of representatives. We put

$$s_k(F) := \sum_{u \in J(N_0/N_k)} F(u) \circ \operatorname{res}(1_{uN_k}) \quad \text{and} \quad s'_k(F) := \sum_{u' \in J'(N_0/N_k)} F(u') \circ \operatorname{res}(1_{u'N_k}) \ .$$

Since $\operatorname{Hom}_A^{\mathfrak{C} cont}(M(\mathfrak{C}), M)$ is Hausdorff and complete it suffices to show that, given any neighborhood of zero $E(C, \mathcal{M})$, there exists an integer $k_0 \geq 0$ such that

$$s_k(F) - s_{k+1}(F),\ s_k(F) - s'_k(F) \in E(C, \mathcal{M}) \qquad \text{for any } k \geq k_0.$$

For $u \in J(N_0/N_{k+1})$ let $\bar{u} \in J(N_0/N_k)$ and $u' \in J'(N_0/N_{k+1})$ be the unique elements such that $uN_k = \bar{u}N_k$ and $uN_{k+1} = u'N_{k+1}$, respectively. Then

$$s_k(F) = \sum_{u \in J(N_0/N_{k+1})} F(\bar{u}) \circ \operatorname{res}(1_{uN_{k+1}})$$

and hence

$$s_k(F) - s_{k+1}(F) = \sum_{u \in J(N_0/N_{k+1})} (F(u(u^{-1}\bar{u})) - F(u)) \circ \operatorname{res}(1_{uN_{k+1}}) \ . \tag{26}$$

Since $u^{-1}\bar{u} \in N_k$ it follows from our assumption that the right hand side lies in $E(C, \mathcal{M})$ for $k \geq k_{C,\mathcal{M}}$. Similarly

$$s_{k+1}(F) - s'_{k+1}(F) = \sum_{u \in J(N_0/N_{k+1})} (F(u) - F(u(u^{-1}u'))) \circ \operatorname{res}(1_{uN_{k+1}}) \ ;$$

again, as $u^{-1}u' \in N_{k+1} \subseteq N_k$, the right hand sum is contained in $E(C, \mathcal{M})$ for $k \geq k_{C,\mathcal{M}}$. $\square$

6.2 Integrability criterion for α

Let $U_g \subseteq N_0$ be the compact open subset such that $U_g w_0 P/P = g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$. This intersection is nonempty if and only if $g \in N_0 \overline{P} N_0$, which we therefore assume in the following. We consider the map

$$\begin{aligned} \alpha_{g,0} : N_0 &\longrightarrow \operatorname{End}_A^{cont}(M) \\ u &\longmapsto \begin{cases} \operatorname{Res}(1_{N_0}) \circ \alpha_g(x_u) \circ \operatorname{Res}(1_{N_0}) & \text{if } u \in U_g, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(where we identify $\operatorname{End}_A^{cont}(M)$ with its image in E^{cont} under the natural embedding (20) using that $\operatorname{Res}(1_{N_0}) = \sigma_0 \circ \operatorname{ev}_0$). Restricting $\alpha_{g,0}(u) \in \operatorname{End}_A^{cont}(M)$ to $M(\mathfrak{C})$ for any $u \in N_0$ we may view $\alpha_{g,0}$ as a map from N_0 to $\operatorname{End}_A^{cont}(M(\mathfrak{C}))$ since $M(\mathfrak{C})$ is an étale $A[P_+]$-submodule of M. However, as we do not assume $M(\mathfrak{C})$ to be complete, it will be more convenient for the purpose of integration to regard $\alpha_{g,0}$ as a map into $\operatorname{Hom}_A^{\mathfrak{C} cont}(M(\mathfrak{C}), M)$. We want to establish a criterion for the $(s, \operatorname{res}, \mathfrak{C})$-integrability of the map $\alpha_{g,0}$.

By the argument in the proof of lemma 5.6 (applied to $V = g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$) we may choose an integer $k_g^{(0)} \geq 0$ such that, for any $k \geq k_g^{(0)}$, we have $U_g N_k \subseteq U_g$ and

$$\alpha(g, x_u).uN_k \subseteq gU_g \qquad \text{for any } u \in U_g. \tag{27}$$

Lemma 6.4. *For $u \in U_g$ and $k \geq k_g^{(0)}$ we have*

$$\alpha_{g,0}(u) \circ \operatorname{res}(1_{uN_k}) = \alpha(g, x_u) \circ \operatorname{Res}(1_{uN_k}) \ .$$

Proof. Using the P-equivariance of Res we have

$$\begin{aligned}\alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k}) &= \mathrm{Res}(1_{\alpha(g,x_u).uN_k}) \circ \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k}) \\ &= \mathrm{Res}(1_{N_0}) \circ \mathrm{Res}(1_{\alpha(g,x_u).uN_k}) \circ \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k}) \\ &= \mathrm{Res}(1_{N_0}) \circ \alpha(g, x_u) \circ \mathrm{Res}(1_{N_0}) \circ \mathrm{Res}(1_{uN_k}) \\ &= \alpha_{g,0}(u) \circ \mathrm{res}(1_{uN_k})\end{aligned}$$

where the second identity follows from (27). □

For $u \in U_g$ and $k \geq k_g^{(0)}$ we put

$$\mathcal{H}_{g,J(N_0/N_k)} := \sum_{u \in U_g \cap J(N_0/N_k)} \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k}) \ . \tag{28}$$

By Lemma 6.4, each summand on the right hand side belongs to $\mathrm{End}_A(M(\mathfrak{C}))$. If $\alpha_{g,0}$ is $(s, \mathrm{res}, \mathfrak{C})$-integrable, the limit

$$\mathcal{H}_g := \lim_{k \geq k_g^{(0)}, k \to \infty} \mathcal{H}_{g,J(N_0/N_k)} \tag{29}$$

exists in $\mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ and is equal to the $(s, \mathrm{res}, \mathfrak{C})$-integral of $\alpha_{g,0}$

$$\int_{N_0} \alpha_{g,0} d\,\mathrm{res} = \mathcal{H}_g \ . \tag{30}$$

We investigate the integrability criterion of Prop. 6.3 for the function $\alpha_{g,0}$. We have to consider the elements

$$\Delta_g(u, k, v) := (\alpha_{g,0}(u) - \alpha_{g,0}(uv)) \circ \mathrm{res}(1_{uN_{k+1}}) \ , \tag{31}$$

for $u \in U_g$, $k \geq k_g^{(0)}$, and $v \in N_k$. By Lemma 6.4, we have

$$\begin{aligned}\Delta_g(u, k, v) &= (\alpha_{g,0}(u) \circ \mathrm{res}(1_{uN_k}) - \alpha_{g,0}(uv) \circ \mathrm{res}(1_{uvN_k})) \circ \mathrm{res}(1_{uN_{k+1}}) \\ &= (\alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k}) - \alpha(g, x_{uv}) \circ \mathrm{Res}(1_{uvN_k})) \circ \mathrm{Res}(1_{uN_{k+1}}) \\ &= (\alpha(g, x_u) - \alpha(g, x_{uv})) \circ \mathrm{Res}(1_{uN_{k+1}}) \\ &= (\alpha(g, x_u) - \alpha(g, x_{uv})) \circ u \circ \mathrm{Res}(1_{N_{k+1}}) \circ u^{-1}\end{aligned}$$

Recall that $N_g \subset N$ is the subset such that $N_g w_0 P/P = g^{-1}\mathcal{C} \cap \mathcal{C}$.

Lemma 6.5. *For $u \in N_g$ and $v \in N$ such that $uv \in N_g$ we have:*

i. $v \in N_{\bar{n}(g,u)}$;

ii. $\alpha(g, x_{uv}) = \alpha(g, x_u) u \alpha(\bar{n}(g, u), x_v) u^{-1}$.

Proof. i. Because of $gu = \alpha(g, x_u) u \bar{n}(g, u)$ we have

$$\alpha(g, x_u) u \bar{n}(g, u) v = guv \in \alpha(g, x_{uv}) uv \overline{N}$$

and hence

$$\bar{n}(g, u) v w_0 P = u^{-1} \alpha(g, x_u)^{-1} \alpha(g, x_{uv}) uv w_0 P \in P w_0 P \ .$$

ii. By i. the equation $\bar{n}(g, u) v w_0 N = \alpha(\bar{n}(g, u), x_v) v w_0 N$ holds. Hence

$$guvw_0 N = \alpha(g, x_u) u \bar{n}(g, u) v w_0 N = \alpha(g, x_u) u \alpha(\bar{n}(g, u), x_v) v w_0 N$$

and therefore $\alpha(g, x_{uv}) uv = \alpha(g, x_u) u \alpha(\bar{n}(g, u), x_v) v$. □

Let $f\colon U_g \to P$ be the map $u \mapsto \alpha(g, x_u)u$. The previous computation shows that for all $u \in U_g$ and $v \in N_k$ we have

$$\Delta_g(u, k, v) = (f(u) - f(uv)v^{-1}) \circ \mathrm{Res}(1_{N_{k+1}}) \circ u^{-1} \ . \tag{32}$$

Let $f(u) = n(g, u)t(g, u)$, with $n(g, u) \in N_0$ and $t(g, u) \in L$. Also, write $gu = f(u)\overline{n}(g, u)$ with $\overline{n}(g, u) \in \overline{N}$. Since $t(g, U_g) \subset L$ and $\overline{n}(g, U_g) \subset \overline{N}$ are compact subsets, there is $k_g^{(1)} \geq k_g^{(0)}$ such that

$$\Lambda_g := t(g, U_g)s^{k_g^{(1)}} \subset L_+, \quad \overline{n}(g, U_g) \subset \overline{N}_{-k_g^{(1)}} \ . \tag{33}$$

Proposition 6.6. *For any compact open subgroup P_1 of P_0 there is $k_g^{(2)}(P_1) \geq k_g^{(1)}$ such that for all $k \geq k_g^{(2)}(P_1)$, $u \in U_g$ and $v \in N_k$*

$$f(u) - f(uv)v^{-1} \in N_0 s^{k-k_g^{(1)}}(1 - P_1)\Lambda_g s^{-k}.$$

Proof. We abbreviate $n(u) = n(g, u)$ and similarly for $t(u)$ and $\overline{n}(u)$. Since $f(u)\overline{n}(u)v = guv = f(uv)\overline{n}(uv)$, we have

$$f(u) - f(uv)v^{-1} = f(u)(1 - \overline{n}(u)v\overline{n}(uv)^{-1}v^{-1}) =$$
$$= n(u)(1 - t(u)\overline{n}(u)v\overline{n}(uv)^{-1}v^{-1}t(u)^{-1})t(u).$$

Since $n(u) \in N_0$, $t(u) \in s^{-k_g^{(1)}}\Lambda_g$ and $(t(u))_{u \in U_g}$ is compact, it suffices to prove that for any compact open subgroup P_2 of P_0 we have $\overline{n}(u)v\overline{n}(uv)^{-1}v^{-1} \in s^k P_2 s^{-k}$ for sufficiently large k. But if $v = s^k n_0 s^{-k}$, we can write

$$\overline{n}(u)v\overline{n}(uv)^{-1}v^{-1} = s^k(s^{-k}\overline{n}(u)s^k)n_0(s^{-k}\overline{n}(uv)^{-1}s^k)n_0^{-1}s^{-k}$$
$$\in s^k\overline{N}_{k-k_g^{(1)}}\Big(\bigcup_{n_0 \in N_0} n_0\overline{N}_{k-k_g^{(1)}}n_0^{-1}\Big)s^{-k}.$$

The result follows from the compactness of N_0 and the fact that the $\overline{N}_k$'s shrink to $\{1\}$ as $k \to \infty$.

□

Corollary 6.7. *For any compact open subgroup P_1 of P_0 and $k \geq k_g^{(2)}(P_1)$*

$$\Delta_g(U_g, k, N_k) \subset N_0 s^{k-k_g^{(1)}}(1 - P_1)\Lambda_g s\psi^{k+1}N_0.$$

Proof. Proposition 6.6 and relation (32) show that

$$\Delta_g(U_g, k, N_k) \subset N_0 s^{k-k_g^{(1)}}(1 - P_1)\Lambda_g s \circ s^{-(k+1)} \circ \mathrm{Res}(1_{N_{k+1}}) \circ N_0.$$

The P-equivariance of Res yields $s^{-(k+1)} \circ \mathrm{Res}(1_{N_{k+1}}) = \mathrm{Res}(1_{N_0}) \circ s^{-k-1}$, and this is the image of $\psi^{k+1} \in \mathrm{End}_A^{\mathrm{cont}}(M)$ in $\mathrm{End}_A^{\mathrm{cont}}(M^P)$. The result follows. □

This leads to an integrability criterion for $\alpha_{g,0}$, which depends only on $(s, M, \mathfrak{C})$.

Proposition 6.8. *We suppose that $(s, M, \mathfrak{C})$ satisfies:*

$\mathfrak{C}(5)$ *For any $C \in \mathfrak{C}$ the compact subset $\psi(C) \subseteq M$ also lies in $\mathfrak{C}$.*

$\mathfrak{T}(1)$ *For any $C \in \mathfrak{C}$ such that $C = N_0C$, any open $A[N_0]$-submodule $\mathcal{M}$ of M, and any compact subset $C_+ \subseteq L_+$ there exists a compact open subgroup $P_1 = P_1(C, \mathcal{M}, C_+) \subseteq P_0$ and an integer $k(C, \mathcal{M}, C_+) \geq 0$ such that*

$$s^k(1 - P_1)C_+\psi^k(C) \subseteq \mathcal{M} \qquad \textit{for any } k \geq k(C, \mathcal{M}, C_+) \ . \tag{34}$$

Then the map $\alpha_{g,0}\colon N_0 \to \mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ *is* $(s, \mathrm{res}, \mathfrak{C})$*-integrable for all* $g \in N_0\overline{P}N_0$.

Proof. By the general integrability criterion of Prop. 6.3, the map $\alpha_{g,0}$ is integrable if for any $(C, \mathcal{M})$ as above, there exists $k_{C,\mathcal{M},g} \geq 0$ such that

$$(35) \qquad \Delta_g(u,k,v) \in E(C,\mathcal{M}) \qquad \text{for any } k \geq k_{C,\mathcal{M},g},\ u \in U_g, \text{ and } v \in N_k.$$

By (6.7), this is true if $k_{C,\mathcal{M},g} \geq k_g^{(2)}(P_1)$ and

$$(36) \qquad s^{k-k_g^{(1)}}(1-P_1)\Lambda_g s\psi^{k+1}(C) \subset \mathcal{M}\ ,$$

because $N_0\mathcal{M} = \mathcal{M}$ and $N_0 C = C$.

We note that the set $C_+ = \Lambda_g s$ is contained in L_+ by (33) and is compact, that the set $C' = \psi^{k_g^{(1)}+1}(C) \subset M$ is compact and $N_0 C' = C'$ because the map ψ is continuous and $N_0\psi(C) = \psi(sN_0s^{-1}C) = \psi(C)$, and that (36) is equivalent to

$$s^{k-k_g^{(1)}}(1-P_1)C_+\psi^{k-k_g^{(1)}} \subset E(C',\mathcal{M})\ .$$

By our hypothesis, there exists an open subgroup $P_1 \subset P_0$ such that this inclusion is satisfied when $k \geq k_g^{(1)} + k(C',\mathcal{M},C_+)$. For

$$(37) \qquad k_{C,\mathcal{M},g} := \max(k_g^{(1)} + k(C',\mathcal{M},C_+), k_g^{(2)}(P_1)).$$

(35) is satisfied. By construction, P_1 depends on $\psi^{k_g^{(1)}+1}(C), \mathcal{M}, \Lambda_g s$, hence only on $C, \mathcal{M}, g$. $\square$

Later, under the assumptions of Prop. 6.8, we will use the argument in the previous proof in the following slightly more general form: for $C, \mathcal{M}, C_+$ as in the proposition and an integer $k' \geq 0$ we have

$$(38) \qquad s^{k-k'}(1-P_1(\psi^{k'}(C),\mathcal{M},C_+))C_+\psi^k \subseteq E(C,\mathcal{M})$$

for any $k \geq k' + k(\psi^{k'}(C),\mathcal{M},C_+)$.

6.3 Extension of Res

Proposition 6.9. *Suppose that* $(s, M, \mathfrak{C})$ *satisfies the assumptions of Prop. 6.8 and that the* $(s, \mathrm{res}, \mathfrak{C})$*-integral* $\mathcal{H}_g$ *of* $\alpha_{g,0}$ *is contained in* $\mathrm{End}_A(M(\mathfrak{C}))$ *for all* $g \in N_0\overline{P}N_0$. *In addition we assume that:*

$\mathfrak{C}(6)$ *For any* $C \in \mathfrak{C}$ *the compact subset* $\varphi(C) \subseteq M$ *also lies in* $\mathfrak{C}$.

$\mathfrak{T}(2)$ *Given a set* $J(N_0/N_k) \subset N_0$ *of representatives for cosets in* N_0/N_k, *for* $k \geq 1$, *for any* $x \in M(\mathfrak{C})$ *and* $g \in N_0\overline{P}N_0$ *there exists a compact A-submodule* $C_{x,g} \in \mathfrak{C}$ *and a positive integer* $k_{x,g}$ *such that* $\mathcal{H}_{g,J(N_0/N_k)}(x) \subseteq C_{x,g}$ *for any* $k \geq k_{x,g}$.

Then the $\mathcal{H}_g$ *satisfy the relations H1, H2, H3 of Prop. 5.14.*

Remark 6.10. *The properties* $\mathfrak{C}(3), \mathfrak{C}(5), \mathfrak{C}(6)$ *imply that for any* $u \in N_0, k \geq 1$, *and* $C \in \mathfrak{C}$ *also* $\mathrm{res}(1_{uN_k})(C)$ *lies in* $\mathfrak{C}$. *Indeed,* $\mathrm{res}(1_{uN_k}) = u \circ \varphi^k \circ \psi^k \circ u^{-1}$.

We prove now H1 and H3, which do not use the last assumption. The proof of ii. is postponed to the next subsection.

Proof. For the proof of H1 let $\mathcal{V} \subset \mathcal{C}_0$ be a compact open subset and let U_1, U_2 be the compact open subsets of N_0 corresponding to $\mathcal{V}$ and $g^{-1}\mathcal{V} \cap \mathcal{C}_0$. To prove that $\mathrm{res}(1_{\mathcal{V}}) \circ \mathcal{H}_g =$

$\mathcal{H}_g \circ \mathrm{res}(1_{g^{-1}\mathcal{V}\cap\mathcal{C}_0})$, it suffices to verify that if k is large enough, then for all $u \in U_g$ we have

$$(39) \qquad \mathrm{Res}(1_{U_1}) \circ \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k}) = \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k}) \circ \mathrm{Res}(1_{U_2}) \ .$$

If $N_{k,u} = \alpha(g, x_u).(uN_k)$, then by P-equivariance of Res, (39) is equivalent to

$$(40) \qquad \mathrm{Res}(1_{U_1\cap N_{k,u}}) \circ \alpha(g, x_u) = \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k\cap U_2}) \ .$$

Write $\alpha(g, x_u) = n_u t_u$ with $n_u \in N$ and $t_u \in L$. If k is large enough, then for all $u \in U_g$ we have $U_2 N_k \subset U_2$ and $U_1 t_u N_k t_u^{-1} \subset U_1$. Since $N_{k,u} = (\alpha(g, x_u).u)t_u N_k t_u^{-1}$, we obtain

$$U_1 \cap N_{k,u} \neq \emptyset \Leftrightarrow \alpha(g, x_u).u \in U_1 \Leftrightarrow g x_u \in \mathcal{V}$$
$$\Leftrightarrow x_u \in g^{-1}\mathcal{V} \cap \mathcal{C}_0 \Leftrightarrow u \in U_2 \Leftrightarrow uN_k \subset U_2 \ .$$

Hence (40) is equivalent to $0 = 0$ or to $\mathrm{Res}(1_{N_{k,u}}) \circ \alpha(g, x_u) = \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k})$, which is true as Res is P-equivariant.

H3. For $b \in P \cap N_0 \overline{P} N_0$ we have

$$\alpha_{b,0} = \text{ constant map on } N_0 \text{ with value } \mathrm{res}(1_{\mathcal{C}_0}) \circ b \circ \mathrm{res}(1_{\mathcal{C}_0})$$

and hence

$$\mathcal{H}_b = \mathrm{res}(1_{\mathcal{C}_0}) \circ b \circ \mathrm{res}(1_{\mathcal{C}_0}) = b \circ \mathrm{res}(1_{b^{-1}\mathcal{C}_0\cap\mathcal{C}_0}) \ .$$

□

6.4 Proof of the product formula

We invoke now the full set of assumptions of Prop. 6.9 and we prove the product formula

$$\mathcal{H}_g \circ \mathcal{H}_h = \mathcal{H}_{gh} \circ \mathrm{res}(1_{h^{-1}\mathcal{C}_0\cap\mathcal{C}_0}) \ .$$

for $g, h \in N_0 \overline{P} N_0$. This suffices by Prop. 5.14.

Let $k_0 := \max(k_g^{(0)}, k_h^{(1)}, k_{gh}^{(0)}) + 1$ and let $k \geq k_0$.

As $k \geq k_h^{(0)}$ (because $k_h^{(1)} \geq k_h^{(0)}$ (33)), the set U_h is a disjoint union of cosets uN_k. We choose a set $J(N_0/N_k) \subset N_0$ of representatives of the cosets in N_0/N_k and for each $u \in J(N_0/N_k) \cap U_h$ a set $J_u(N_0/N_{k-k_0}) \subset N_0$ of representatives of the cosets in N_0/N_{k-k_0} with $n(g, u) \in J_u(N_0/N_{k-k_0})$ (see (23)).

We write $\mathcal{H}_g \circ \mathcal{H}_h - \mathcal{H}_{gh} \circ \mathrm{res}(1_{h^{-1}\mathcal{C}_0\cap\mathcal{C}_0})$ as the sum over $u \in J(N_0/N_k) \cap U_h$ of

$$(41) \qquad (\mathcal{H}_g \circ \mathcal{H}_h - \mathcal{H}_{gh} \circ \mathrm{Res}(1_{U_h})) \circ \mathrm{Res}(1_{uN_k}) = a_{k,u} + b_{k,u} + c_{k,u} \ ,$$

where

$$\begin{aligned}
a_{k,u} :=& (\mathcal{H}_g \circ \mathcal{H}_h - \mathcal{H}_{g,J_u(N_0/N_{k-k_0})} \circ \mathcal{H}_{h,J(N_0/N_k)}) \circ \mathrm{Res}(1_{uN_k}) \\
b_{k,u} :=& (\mathcal{H}_{g,J_u(N_0/N_{k-k_0})} \circ \mathcal{H}_{h,J(N_0/N_k)} - \mathcal{H}_{gh,J(N_0/N_k)}) \circ \mathrm{Res}(1_{U_h}) \circ \mathrm{Res}(1_{uN_k}) \\
c_{k,u} :=& (\mathcal{H}_{gh,J(N_0/N_k)} - \mathcal{H}_{gh}) \circ \mathrm{Res}(1_{U_h}) \circ \mathrm{Res}(1_{uN_k}).
\end{aligned}$$

The product formula follows from the claim that $b_{k,u} = 0$ and that for an arbitrary compact subset $C \in \mathfrak{C}$ such that $N_0 C = C$, and an arbitrary open $A[N_0]$-module $\mathcal{M} \subset M$, $a_{k,u}$ and $c_{k,u}$ lies in $E(C, \mathcal{M})$ when k is very large, independently of u.

The claim results from the following three propositions.

Because $(s, M, \mathfrak{C})$ satisfies Prop. 6.8, we associate to $(C, \mathcal{M}, g)$ the integer $k_{C,\mathcal{M},g}$ defined in (37) which is independent of the choice of the $J(N_0/N_k)$. For the sake of simplicity, we write

$$\mathcal{H}_g^{(k)} := \mathcal{H}_{g,J(N_0/N_k)} \ , \ s_g^{(k)} := \mathcal{H}_g^{(k+1)} - \mathcal{H}_g^{(k)}. \tag{42}$$

By (26), we have, for $k \geq k_g^{(0)}$,

$$s_g^{(k)} = \sum_{u \in U_g \cap J(N_0/N_{k+1})} \Delta_g(u, k, v_u)$$

for some $v_u \in N_k$. It follows from (6.7) that, for any given compact open subgroup $P_1 \subset P_0$, we have

$$s_g^{(k)} \in < N_0 s^{k-k_g^{(1)}} (1 - P_1) \Lambda_g s \psi^{k+1} N_0 >_A \qquad \text{for } k \geq k_g^{(2)}(P_1) \ , \tag{43}$$

where we use the notation $< X >_A$ for the A-submodule in $\mathrm{End}_A(M)$ generated by X. We deduce from the proof of Prop. 6.8, that $s_g^{(k)} \in E(C, \mathcal{M})$ for any $k \geq k_{C,\mathcal{M},g}$.

Proposition 6.11. $(\mathcal{H}_g - \mathcal{H}_{g,J(N_0/N_k)}) \circ \mathrm{Res}(1_{uN_k}) \in E(C, \mathcal{M})$ *for any* $k \geq k_{C,\mathcal{M},g}$.

Proof. When $k \geq 0$, $k_2 \geq \max(k-1, k_g^{(0)})$, $u' \in U_g, v \in N_k$ we have that $\Delta_g(u', k_2, v) \circ \mathrm{Res}(1_{uN_k})$ is equal either to $\Delta_g(u', k_2, v)$ or to 0. If follows that

$$s_g^{(k_2)} \circ \mathrm{Res}(1_{uN_k}) \subseteq E(C, \mathcal{M}) \qquad \text{for any } k_2 \geq \max(k-1, k_{C,\mathcal{M},g}) \text{ and } k \geq 0,$$

Now we fix $k \geq k_{C,\mathcal{M},g}$. Note that $\mathrm{Res}(1_{uN_k})(C)$ is contained in $\mathfrak{C}$ by the stability of $\mathfrak{C}$ by ψ, φ, and $u^{\pm 1}$. Therefore the sequence $(\mathcal{H}_g^{(k_2)} \circ \mathrm{Res}(1_{uN_k}))_{k_2}$ converges to $\mathcal{H}_g \circ \mathrm{Res}(1_{uN_k})$ in $\mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$. In particular, we have

$$(\mathcal{H}_g - \mathcal{H}_g^{(k_2)}) \circ \mathrm{Res}(1_{uN_k}) \subseteq E(C, \mathcal{M}) \qquad \text{for any } k_2 \geq \max(k-1, k_{C,\mathcal{M},g}) \text{ and } k \geq 0.$$

The statement follows by taking $k_2 = k$. $\square$

This establishes that $c_{k,u}$ lies in $E(C, \mathcal{M})$ when $k \geq k_{C,\mathcal{M},gh}$.

Note that the proposition is true also for any other system $J'(N_0/N_k) \subset N_0$ of representatives for the cosets in N_0/N_k for the same integer $k_{C,\mathcal{M},g}$. We write $\mathcal{H}_g'^{(k)}$ and $s_g'^{(k)}$ for the elements defined in (42) for $J'(N_0/N_k)$.

Proposition 6.12. *There exists an integer* $k_{C,\mathcal{M},g,h,k_0} \in \mathbb{N}$*, independent of the choices of* $J(N_0/N_k)$ *and* $J'(N_0/N_k)$*, such that:*

i. $\mathcal{H}_g'^{(k+1-k_0)} \circ \mathcal{H}_h^{(k+1)} - \mathcal{H}_g'^{(k-k_0)} \circ \mathcal{H}_h^{(k)} \in E(C, \mathcal{M})$*, for all* $k \geq k_{C,\mathcal{M},g,h,k_0}$*, and the sequence* $(\mathcal{H}_g'^{(k-k_0)} \circ \mathcal{H}_h^{(k)})$ *converges to* $\mathcal{H}_g \circ \mathcal{H}_h$ *in* $\mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$.

ii. $(\mathcal{H}_g \circ \mathcal{H}_h - \mathcal{H}_g'^{(k-k_0)} \circ \mathcal{H}_h^{(k)}) \circ \mathrm{Res}(1_{uN_k}) \in E(C, \mathcal{M})$*, for all* $k \geq k_{C,\mathcal{M},g,h,k_0}$.

Proof. i. To prove the first assertion, we write

$$\mathcal{H}_g'^{(k+1-k_0)} \circ \mathcal{H}_h^{(k+1)} - \mathcal{H}_g'^{(k-k_0)} \circ \mathcal{H}_h^{(k)} = \mathcal{H}_g'^{(k+1-k_0)} \circ s_h^{(k)} + s_g'^{(k-k_0)} \circ \mathcal{H}_h^{(k)}. \tag{44}$$

Note that, when $k \geq k_g^{(1)}$, the endomorphisms $\mathcal{H}_g'^{(k)}$ and $\mathcal{H}_g^{(k)}$ are contained in the A-module $< N_0 s^{k-k_g^{(1)}} \Lambda_g \psi^k N_0 >_A$, because

$$\alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k}) = n(g,u) t(g,u) u^{-1} u s^k \psi^k u^{-1} \subset N_0 s^{k-k_g^{(1)}} \Lambda_g \psi^k N_0 \qquad \text{for } u \in U_g.$$

We consider any compact open subgroup $P_1 \subset P_0$ and we assume $k \geq \max(k_g^{(2)}(P_1) + k_0, k_h^{(2)}(P_1))$. With (43) we obtain that (44) is contained in

$$< N_0 s^{k+1-k_0-k_g^{(1)}} \Lambda_g \psi^{k+1-k_0} N_0 s^{k-k_h^{(1)}} (1-P_1) \Lambda_h s \psi^{k+1} N_0 >_A$$
$$+ \ < N_0 s^{k-k_0-k_g^{(1)}} (1-P_1) \Lambda_g s \psi^{k-k_0+1} N_0 s^{k-k_h^{(1)}} \Lambda_h \psi^k N_0 >_A \ .$$

Recalling that $\psi^a(N_0 \varphi^{a+b}(m)) = \psi^a(N_0)\varphi^b(m) = N_0 \varphi^b(m)$ for $a, b \in \mathbb{N}$ and $m \in M$, we see that this is contained in

$$< N_0 s^{k+1-k_0-k_g^{(1)}} \Lambda_g N_0 s^{k_0-k_h^{(1)}-1} (1-P_1) \Lambda_h s \psi^{k+1} N_0 >_A$$
$$+ \ < N_0 s^{k-k_0-k_g^{(1)}} (1-P_1) \Lambda_g N_0 s^{k_0-k_h^{(1)}} \Lambda_h \psi^k N_0 >_A \ .$$

As $k+1-k_0-k_g^{(1)} \geq k_g^{(2)}(P_1) + 1 - k_g^{(1)} \geq 1$ and as $\Lambda_g \subset L_+$, we have

$$N_0 s^{k+1-k_0-k_g^{(1)}} \Lambda_g N_0 \subset N_0 s^{k+1-k_0-k_g^{(1)}} \Lambda_g \ ,$$

and this is contained in

$$< N_0 s^{k+1-k_0-k_g^{(1)}} \Lambda_g s^{k_0-k_h^{(1)}-1} (1-P_1) \Lambda_h s \psi^{k+1} N_0 >_A$$
$$+ \ < N_0 s^{k-k_0-k_g^{(1)}} (1-P_1) \Lambda_g s^{k_0-k_h^{(1)}} \Lambda_h \psi^k N_0 >_A \ .$$

We assume, as we may, that the compact open subgroup P_1 of P_0 satisfies $tP_1t^{-1} \subseteq P_1$ for all t in the compact set $\Lambda_g s^{k_0-k_h^{(1)}-1}$ of L_+. Then we finally obtain that (44 is contained in

$$< N_0 s^{k+1-k_0-k_g^{(1)}} (1-P_1) \Lambda_g s^{k_0-k_h^{(1)}} \Lambda_h \psi^{k+1} N_0 >_A$$
$$+ \ < N_0 s^{k-k_0-k_g^{(1)}} (1-P_1) \Lambda_g s^{k_0-k_h^{(1)}} \Lambda_h \psi^k N_0 >_A \ .$$

This subset of $\mathrm{End}_A(M)$ is contained in $E(C, \mathcal{M})$ when

$$s^{k+1-k_0-k_g^{(1)}} (1-P_1) \Lambda_g s^{k_0-k_h^{(1)}} \Lambda_h \psi^{k+1}(C) \quad \text{and} \quad s^{k-k_0-k_g^{(1)}} (1-P_1) \Lambda_g s^{k_0-k_h^{(1)}} \Lambda_h \psi^k(C)$$

are contained in $E(C, \mathcal{M})$ because $N_0 C = C$ and $\mathcal{M}$ is an $A[N_0]$-module. By (38), this is true when P_1 is contained in $P_1(\psi^{k_0+k_g^{(1)}}(C), \mathcal{M}, \Lambda_g s^{k_0-k_h^{(1)}} \Lambda_h)$ and $k \geq k_{C,\mathcal{M},g,h,k_0}$ where

$$k_{C,\mathcal{M},g,h,k_0} := \max(k_g^{(2)}(P_1) + k_0, k_h^{(2)}(P_1), k(\psi^{k_0+k_g^{(1)}}(C), \mathcal{M}, \Lambda_g s^{k_0-k_h^{(1)}} \Lambda_h)). \tag{45}$$

The first assertion of i. is proved. We deduce the second assertion from the following claim and the last assumption of Prop. 6.9:

Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be two convergent sequences in $\mathrm{Hom}_A^{\mathfrak{C}ont}(M(\mathfrak{C}), M)$ with limits A and B, respectively; assume that $(B_n)_{n\in\mathbb{N}}$ and B are in $\mathrm{End}_A(M(\mathfrak{C}))$ and that, for any $x \in \mathfrak{C}$ there exists an A-submodule $C \in \mathfrak{C}$ such that $B_n(x) \in C$ for any large n. Then, if the sequence $(A_n \circ B_n)_{n\in\mathbb{N}}$ is convergent, its limit is $A \circ B$.

Let D be the limit of the sequence $(A_n \circ B_n)_n$. It suffices to show that, for any open A-submodule $\mathcal{M} \subseteq M$ and any element $x \in M(\mathfrak{C})$ we have $(D - A \circ B)(x) \in \mathcal{M}$. We write

$$D - A \circ B = (D - A_n \circ B_n) - (A - A_n) \circ B_n - A \circ (B - B_n) \ .$$

Obviously $(D - A_n \circ B_n)(x) \in \mathcal{M}$ for large n. Secondly, the elements $B_n(x)$ for any large n are contained in some compact A-submodule $C \in \mathfrak{C}$, hence also $(B - B_n)(x)$. Moreover $A - A_n \in E(C, \mathcal{M})$ for large n. Hence $(A - A_n) \circ B_n(x) \in \mathcal{M}$ for large n. Finally, A being $\mathfrak{C}$-continuous there is an open A-submodule $\mathcal{M}' \subseteq M$ such that $A(\mathcal{M}' \cap C) \subseteq \mathcal{M}$. Furthermore $(B - B_n)(x) \in \mathcal{M}' \cap C$ for large n. Hence $A \circ (B - B_n)(x) \in \mathcal{M}$ for large n.

ii. This follows from the second assertion in i. together with remark 6.10. □

We have now proved that $a_{k,u} \in E(C,\mathcal{M})$ when $k \geq k_{C,\mathcal{M},g,h,k_0}$.

Proposition 6.13. *For* $u \in J(N_0/N_k) \cap U_h$, *we have*

$$\mathcal{H}_{g,J_u(N_0/N_{k-k_0})} \circ \mathcal{H}_{h,J(N_0/N_k)} \circ \operatorname{Res}(1_{uN_k}) = \mathcal{H}_{gh,J(N_0/N_k)} \circ \operatorname{Res}(1_{uN_k}). \tag{46}$$

Proof. The left side of (46) is

$$\sum_{v \in U_g \cap J_u(N_0/N_{k-k_0})} \alpha(g,x_v) \circ \operatorname{Res}(1_{vN_{k-k_0}}) \circ \alpha(h,x_u) \circ \operatorname{Res}(1_{uN_k}) \ .$$

The right side of (46) is $\alpha(gh,x_u) \circ \operatorname{Res}(1_{uN_k})$ if $u \in J(N_0/N_k) \cap U_h \cap U_{gh}$ and is 0 if u does not belong to U_{gh}. We recall that

$$\alpha(h,x_u)u = n(h,u)t(h,u) \qquad \text{with } n(h,u) \in N_0 \text{ and } t(h,u) \in L_+ s^{-k_h^{(1)}}.$$

It follows that

$$\alpha(h,x_u)uN_k w_0 P \subseteq n(h,u)N_{k-k_h^{(1)}} w_0 P \subset n(h,u)N_{k-k_0} w_0 P.$$

We obtain

$$\operatorname{Res}(1_{vN_{k-k_0}}) \circ \alpha(h,x_u) \circ \operatorname{Res}(1_{uN_k}) = \begin{cases} \alpha(h,x_u) \circ \operatorname{Res}(1_{uN_k}) & \text{if } vN_{k-k_0} = n(h,u)N_{k-k_0}, \\ 0 & \text{otherwise.} \end{cases}$$

We check now that $u \in U_{gh} \cap U_h$ if and only if $n(h,u) \in U_g$. Indeed $x_u = uw_0P/P$ belongs to $h^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 = U_h w_0 P/P$,

$$x_u \in (gh)^{-1}\mathcal{C}_0 \cap h^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 \quad \text{ if and only if } \quad hx_u \in g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$$

and $hx_u = \alpha(h,x_u)x_u = n(h,u)w_0P/P$. It follows that $u \in U_{gh} \cap U_h$ if and only if $n(h,u) \in U_g$. As $J_u(N_0/N_{k-k_0})$ contains $n(h,u)$, we have $v = n(h,u)$ when $vN_{k-k_0} = n(h,u)N_{k-k_0}$. We deduce that the left side of (46) is 0 when u does not belong to U_{gh} and otherwise is equal to

$$\alpha(g,hx_u) \circ \alpha(h,x_u) \circ \operatorname{Res}(1_{uN_k}) = \alpha(gh,x_u) \circ \operatorname{Res}(1_{uN_k}) \ ,$$

where the last equality follows from the product formula for α (Lemma 5.5). ☐

We have proved that $b_{k,u} = 0$, therefore ending the proof of the product formula.

6.5 Reduction modulo p^n

We investigate now the situation that will appear for generalized (φ,Γ)-modules M, where the reduction modulo a power of p allows to reduce to the simpler case where M is killed by a power of p. We will use later this section to get a special family $\mathfrak{C}_s$ in M such that the $(s,\operatorname{res},\mathfrak{C}_s)$-integrals $\mathcal{H}_g$ exist for all $g \in N_0\overline{P}N_0$ and satisfy the relations H1, H2, H3 of Prop. 5.14.

We assume now that (A,M) satisfies:

a. *A is a commutative ring with the p-adic topology (the ideals p^nA for $n \geq 1$ form a basis of neighborhoods of 0) and is Hausdorff.*

b. *M is a linearly topological A-module with a topology weaker than the p-adic topology (a neighborhood of 0 contains some p^nM) and M is a Hausdorff and topological $A[P_+]$-module as in section 6 (we do not suppose that M is complete).*

c. The submodules $p^n M$, for $n \geq 1$, are closed in M.

d. M is p-adically complete: the linear map $M \to \varprojlim_{n\geq 1}(M/p^n M)$ is bijective.

For all $n \geq 1$, we equip $M/p^n M$ with the quotient topology so that the quotient map $p_n : M \to M/p^n M$ is continuous. The natural homomorphism

$$M \xrightarrow{\cong} \varprojlim_{n\geq 1}(M/p^n M)$$

is a homeomorphism, and the natural homomorphism

$$\mathrm{End}_A^{cont}(M) \xrightarrow{\cong} \varprojlim_{n\geq 1} \mathrm{End}_A^{cont}(M/p^n M)$$

is bijective. We have:

- For a subset C of M, let $\overline{C}$ be the closure of C. Then $\overline{C} = \varprojlim_{n\geq 1} \overline{p_n(C)}$ and if C is closed, $C = \varprojlim_{n\geq 1} p_n(C)$. If C is p-compact (i.e. $p_n(C)$ are compact for all $n \geq 1$), then C is compact, and conversely ([2] I.29 Cor. and I.64 Prop.8).
- An endomorphism f of M which is p-continuous (i.e. the endomorphism f_n induced by f on $M/p^n M$ is continuous for all $n \geq 1$) is continuous, and conversely.
- An action of a topological group H on M which is p-continuous (i.e. the induced action of H on $M/p^n M$ is continuous for all $n \geq 1$) is continuous, and conversely.
- If the $M/p^n M$ are complete for all $n \geq 1$, then M is complete.
- The image $\mathfrak{C}_n$ in $M/p^n M$, for all $n \geq 1$, of a special family $\mathfrak{C}$ of compact subsets in M such that, for all positive integers n,

  $$p^n M \cap M(\mathfrak{C}) = p^n M(\mathfrak{C})$$

  is a special family. In this case, one has $M(\mathfrak{C}_n) = M(\mathfrak{C})/p^n M(\mathfrak{C})$.
- M is a topologically étale $A[P_+]$-module if and only if $M/p^n M$ is a topologically étale $A[P_+]$-module, for all $n \geq 1$. If we replace "topologically" by "algebraically", this is the same proof as for classical (φ, Γ)-modules (see subsection 7.3). The canonical inverse ψ_s of the action φ_s of s is continuous if and only if it is p-continuous.

We introduce now our setting which will be discussed in this section.

We suppose that :

- *M is a topologically étale $A[P_+]$-module, and $M/p^n M$ is complete for all $n \geq 1$.*
- *We are given, for $n \geq 1$, a special family $\mathfrak{C}_n$ of compact subsets in $M_n = M/p^n M$ such that $\mathfrak{C}_n$ contains the image of $\mathfrak{C}_{n+1}$ in M_n for all $n \geq 1$.*

Let $\mathfrak{C}$ be the set of compact subsets $C \subset M$ such that $p_n(C) \in \mathfrak{C}_n$ for all $n \geq 1$.

Lemma 6.14. *$\mathfrak{C}$ is a special family in M and $M(\mathfrak{C}) = \varprojlim_{n\geq 1} M(\mathfrak{C}_n)$.*

Proof. $\mathfrak{C}(1)$ It is obvious that a compact subset C' of $C \in \mathfrak{C}$ is in $\mathfrak{C}$ because p_n is continuous and $p_n(C')$ is compact.

$\mathfrak{C}(2)$ p_n commutes with finite union hence $\mathfrak{C}$ is stable by finite union.

$\mathfrak{C}(3)$ p_n commutes with the action of N_0 hence $C \in \mathfrak{C}$ implies $N_0 C \in \mathfrak{C}$.

$\mathfrak{C}(4)$ By definition $x \in M(\mathfrak{C})$ if and only if $p_n(x) \in M(\mathfrak{C}_\mathfrak{n})$ for all $n > 1$. The compatibility of the $\mathfrak{C}_n$ implies that the $M(\mathfrak{C}_n)$ form a projective system. We deduce $M(\mathfrak{C}) = \varprojlim_{n\geq 1} M(\mathfrak{C}_n)$. As the latter ones are topologically étale, the topological $A[P_+]$-module $M(\mathfrak{C})$ is topologically étale by Remark 3.9. $\square$

We have the natural map

$$\varprojlim_n \mathrm{Hom}_A(M(\mathfrak{C}_n), M/p^nM) \to \mathrm{Hom}_A(\varprojlim_n M(\mathfrak{C}_n), \varprojlim_n M/p^nM) = \mathrm{Hom}_A(M(\mathfrak{C}), M) \ .$$

Lemma 6.15. *The above map induces a continuous map*

$$\varprojlim_n \mathrm{Hom}_A^{\mathfrak{C}_n cont}(M(\mathfrak{C}_n), M/p^nM) \to \mathrm{Hom}_A^{\mathfrak{C} cont}(M(\mathfrak{C}), M) \ , \tag{47}$$

for the projective limit of the $\mathfrak{C}_n$-open topologies on the left hand side.

Proof. Let $f = \varprojlim f_n$ be a map in the image, and let $C \in \mathfrak{C}$. Then $f|_C$ is the projective limit of the $f_n|_{p_n(C)}$ hence is continuous. This means that the map in the assertion is well defined. For the continuity, let $C \in \mathfrak{C}$ and $\mathcal{M} \subset M$ be an open A-submodule. The preimage of $E(C, \mathcal{M})$ is equal to

$$\big(\varprojlim_n \ \mathrm{Hom}_A^{\mathfrak{C}_n cont}(M(\mathfrak{C}_n), M/p^nM)\big) \ \cap \ \big(\prod_n E(p_n(C), \mathcal{M} + p^nM/p^nM)\big) \ .$$

Since $\mathcal{M}$ contains some $p^{n_o}M$, this intersection is equal to the open submodule

$$\{(f_n) \in \varprojlim_n \ \mathrm{Hom}_A^{\mathfrak{C}_n cont}(M(\mathfrak{C}_n), M/p^nM) : f_n \in E(p_n(C), \mathcal{M} + p^nM/p^nM) \text{ for } n \leq n_0\}.$$

$\square$

Proposition 6.16. *In the above setting assume that all the assumptions of Prop. 6.9 are satisfied for $(s, M/p^nM, \mathfrak{C}_n)$ and for all $n \geq 1$. Then, for all $g \in N_0\overline{P}N_0$, the functions*

$$\alpha_{g,0} : N_0 \to \mathrm{Hom}_A^{\mathfrak{C} ont}(M(\mathfrak{C}), M)$$

are $(s, res, \mathfrak{C})$-integrable, their $(s, res, \mathfrak{C})$-integrals $\mathcal{H}_g$ belong to $\mathrm{End}_A(M(\mathfrak{C}))$ *and satisfy the relations H1, H2, H3 of Prop. 5.14.*

Proof. In the following we indicate with an extra index n that the corresponding notation is meant for the module M/p^nM with the special family $\mathfrak{C}_n$. Then $\alpha_{g,0}(u)$ is the image of $(\alpha_{g,0,n}(u))_n$ by the map (47), for $u \in N_0$. It follows that $\mathcal{H}_{g,J(N_0/N_k)}$ is the image of $(\mathcal{H}_{g,J(N_0/N_k),n})_n$ for $g \in N_0\overline{P}N_0$. By assumption the integral $\mathcal{H}_{g,n} = \lim_{k\to\infty} \mathcal{H}_{g,J(N_0/N_k),n}$ exists, lies in $\mathrm{Hom}_A^{\mathfrak{C}_n ont}(M(\mathfrak{C}_n), M/p^nM)$, and satisfies the relations H1, H2, H3 of Prop. 5.14.

The continuity of the map (47) implies that the image of $(\mathcal{H}_{g,n})_n$ is equal to the limit $\lim_{k\to\infty} \mathcal{H}_{g,J(N_0/N_k)}$, therefore is the integral $\mathcal{H}_g$ of $\alpha_{g,0}$. The additional properties for $\mathcal{H}_g$ are inherited from the corresponding properties of the $\mathcal{H}_{g,n}$. $\square$

Under the assumptions of Prop. 6.16, we associate to $(s, M, \mathfrak{C})$, an A-algebra homomorphism

$$\widetilde{\mathrm{Res}} \ : \ \mathcal{A}_{\mathcal{C}\subset G/P} \ \to \ \mathrm{End}_A(M(\mathfrak{C})^P) \ .$$

via propositions 5.14 , 5.12, which extends the A-algebra homomorphism

$$\mathrm{Res} \ : \ C_c^\infty(\mathcal{C}, A)\#P \ \to \ \mathrm{End}_A(M(\mathfrak{C})^P)$$

constructed in proposition 3.17. The homomorphism Res gives rise to a P-equivariant sheaf on $\mathcal{C}$ as described in detail in the theorem 3.23. The homomorphism $\widetilde{\mathrm{Res}}$ defines on the global sections with compact support $M(\mathfrak{C})_c^P$ of the sheaf on $\mathcal{C}$ the structure of a nondegenerate $\mathcal{A}_{\mathcal{C}\subset G/P}$-module. The latter leads, by proposition 5.10, to the unital $C_c^\infty(G/P, A)\#G$-module $\mathcal{Z} \otimes_{\mathcal{A}} M(\mathfrak{C})_c^P$ which corresponds to a G-equivariant sheaf on G/P extending the earlier sheaf on $\mathcal{C}$ (remark 5.11).

7 Classical (φ, Γ)-modules on $\mathcal{O}_{\mathcal{E}}$

7.1 The Fontaine ring $\mathcal{O}_{\mathcal{E}}$

Let $K/\mathbb{Q}_p$ be a finite extension of ring of integers o, of uniformizer p_K and residue field k. By definition the Fontaine ring $\mathcal{O}_{\mathcal{E}}$ over o is the p-adic completion of the localisation of the Iwasawa o-algebra $\Lambda(\mathbb{Z}_p) := o[[\mathbb{Z}_p]]$ with respect to the multiplicative set of elements which are not divisible by p. We choose a generator γ of $\mathbb{Z}_p$ of image $[\gamma]$ in $\mathcal{O}_{\mathcal{E}}$ and we denote $X = [\gamma] - 1 \in \mathcal{O}_{\mathcal{E}}$. The Iwasawa o-algebra $\Lambda(\mathbb{Z}_p)$ is a local noetherian ring of maximal ideal $\mathcal{M}(\mathbb{Z}_p)$ generated by p_K, X. It is a compact ring for the $\mathcal{M}(\mathbb{Z}_p)$-adic topology. The ring $\mathcal{O}_{\mathcal{E}}$ can be viewed as the ring of infinite Laurent series $\sum_{n\in\mathbb{Z}} a_n X^n$ over o in the variable X with $\lim_{n\to-\infty} a_n = 0$, and $\Lambda(\mathbb{Z}_p)$ as the subring $o[[X]]$ of Taylor series. The Fontaine ring $\mathcal{O}_{\mathcal{E}}$ is a local noetherian ring of maximal ideal $p_K\mathcal{O}_{\mathcal{E}}$ and residue field isomorphic to $k((X))$; it is a pseudo-compact ring for the p-adic (= strong) topology and a complete ring (with continuous multiplication) for the weak topology. A fundamental system of open neighborhoods of 0 for the weak topology of $\mathcal{O}_{\mathcal{E}}$ is given by

$$(O_{n,k} = p^n\mathcal{O}_{\mathcal{E}} + \mathcal{M}(\mathbb{Z}_p)^k)_{n,k\in\mathbb{N}}$$

or by

$$(B_{n,k} = p^n\mathcal{O}_{\mathcal{E}} + X^k\Lambda(\mathbb{Z}_p)_{n,k\in\mathbb{N}}$$

Other fundamental systems of neighborhoods of 0 for the weak topology are

$$(O_n := O_{n,n})_{n\geq 1} \quad \text{or} \quad (B_n := B_{n,n})_{n\geq 1} \ .$$

7.2 The group $GL(2, \mathbb{Q}_p)$

We consider the group $G = GL(2, \mathbb{Q}_p)$ and

$$N_0 := \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \ , \ \Gamma := \begin{pmatrix} \mathbb{Z}_p^* & 0 \\ 0 & 1 \end{pmatrix} \ , \ L_0 := \begin{pmatrix} \mathbb{Z}_p^* & 0 \\ 0 & \mathbb{Z}_p^* \end{pmatrix} \ , \ L_* := \begin{pmatrix} \mathbb{Z}_p - \{0\} & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

$$N_k := \begin{pmatrix} 1 & p^k\mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \ , \ L_k := \begin{pmatrix} 1 + p^k\mathbb{Z}_p & 0 \\ 0 & 1 + p^k\mathbb{Z}_p \end{pmatrix} \text{ for } k \geq 1 \ ,$$

$P_k = L_kN_k$ for $k \in \mathbb{N}$, the upper triangular subgroup P, the diagonal subgroup L, the upper unipotent subgroup N, the center Z, the mirabolic monoid $P_* = N_0L_*$, and the monoids $L_+ = L_*Z$, $P_+ = N_0L_+$. The subset of non invertible elements in the monoid L_* is

$$\Gamma s_p^{\mathbb{N}-\{0\}} = \{s_a := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \ \text{ for } \ a \in p\mathbb{Z}_p - \{0\}\} \ .$$

An element $s \in \Gamma s_p^{\mathbb{N}-\{0\}}Z$ is called strictly dominant. In the following we identify the group $\mathbb{Z}_p$ with N_0. The action of P_+ on N_0 induces an étale ring action of P_+ (trivial on Z) on $\Lambda(N_0)$ which respects the ideal generated by p. This action extends first to the localisation and then to the completion to give an étale ring action of P_+ on $\mathcal{O}_{\mathcal{E}}$ determined by its restriction to P_*. For the weak topology (and not for the p-adic topology), the action $P_+ \times \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$ of the monoid P_+ on $\mathcal{O}_{\mathcal{E}}$ is continuous (see Lemma 8.24.i in [12]). For $t \in L_+$ the canonical left inverse ψ_t of the action φ_t of t is continuous (this is proved in a more general setting later in Prop. 8.22).

7.3 Classical étale (φ, Γ)-modules

Let $s \in \Gamma s_p^{\mathbb{N}-\{0\}}Z$. A finitely generated étale φ_s-module D over $\mathcal{O}_{\mathcal{E}}$ is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module with an étale semilinear endomorphism φ_s. These modules form an abelian category $\mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E}}}(\varphi_s)$. We fix such a module D.

In the following, the topology of D is its weak topology. For any surjective $\mathcal{O}_{\mathcal{E}}$-linear map $f : \oplus^d \mathcal{O}_{\mathcal{E}} \to D$, the image in D of a fundamental system of neighborhoods of 0 in $\oplus^d \mathcal{O}_{\mathcal{E}}$ for the weak topology is a fundamental system of neighborhoods of 0 in D. Finitely generated $\Lambda(N_0)$-submodules of D generating the $\mathcal{O}_{\mathcal{E}}$-module D will be called lattices. The map f sends $\oplus^d \Lambda(\mathbb{Z}_p)$ onto a lattice D^0 of D. We note $\mathcal{O}_{n,k} := p^n D + \mathcal{M}(\mathbb{Z}_p)^k D^0$ and $\mathcal{B}_{n,k} := p^n D + X^k D^0$. Writing $\mathcal{O}_n := \mathcal{O}_{n,n}$ and $\mathcal{B}_n := \mathcal{B}_{n,n}$, $(\mathcal{O}_n)_n$ and $(\mathcal{B}_n)_n$ are two fundamental systems of neighborhoods of 0 in D. The topological $\mathcal{O}_{\mathcal{E}}$-module D is Hausdorff and complete.

A treillis D_0 in D is a compact $\Lambda(N_0)$-submodule D_0 such that the image of D_0 in the finite dimensional $k((X))$-vector space $D/p_K D$ is a $k[[X]]$-lattice ([5] Déf. I.1.1). A lattice is a treillis and a treillis contains a lattice.

For $n \geq 1$, the reduction modulo p^n of D is the finitely generated $\mathcal{O}_{\mathcal{E}}$-module $D/p^n D$ with the induced action of φ_s. The action remains étale, because the multiplication by p^n being a morphism in $\mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E}}}(\varphi_s)$ its cokernel belongs to the category. The reduction modulo p^n of ψ_s is the canonical left inverse of the reduction modulo p^n of φ_s. The reduction modulo p^n of a treillis of D is a treillis of $D/p^n D$.

Conversely, if the reduction modulo p^n of a finitely generated φ_s-module D over $\mathcal{O}_{\mathcal{E}}$ is étale for all $n \geq 1$, then D is an étale φ_s-module over $\mathcal{O}_{\mathcal{E}}$ because $D = \varprojlim_n D/p^n D$.

The weak topology of D is the projective limit of the weak topologies of $D/p^n D$.

When D is killed by a power of p and D_0 is a treillis of D, we have :

1. D_0 is open and closed in D.
2. $(\mathcal{M}(\mathbb{Z}_p)^n D_0)_{n \in \mathbb{N}}$ and $(X^n D_0)_{n \in \mathbb{N}}$ form two fundamental systems of open neighborhoods of zero in D.
3. Any treillis of D is contained in $X^{-n} D_0$ for some $n \in \mathbb{N}$.
4. $D = \bigcup_{k \in \mathbb{N}} X^{-k} D_0$.
5. D_0 is a lattice.

The first four properties are easy; a reference is [5] Prop. I.1.2. To show that D_0 is a lattice, we pick some lattice D^0 then D_0 is contained in the lattice $X^{-n} D^0$ for some $n \in \mathbb{N}$ by the property 3. Since the ring $\Lambda(N_0)$ is noetherian the assertion follows.

When D is killed by a power of p, the weak topology of D is locally compact (by properties 2 and 5).

Proposition 7.1. *Let D be a finitely generated étale φ_s-module over $\mathcal{O}_{\mathcal{E}}$. Then φ_s and its canonical inverse ψ_s are continuous.*

Proof. a) The above $\mathcal{O}_{\mathcal{E}}$-linear surjective map $f : \oplus^d \mathcal{O}_{\mathcal{E}} \to D$ sends $(a_i)_i$ to $\sum_i a_i d_i$ for some elements $d_i \in D$. As φ_s is étale, the map $(a_i)_i \mapsto \sum_i a_i \varphi_s(d_i)$ also gives an $\mathcal{O}_{\mathcal{E}}$-linear surjective map $\oplus^d \mathcal{O}_{\mathcal{E}} \to D$. Both surjections are topological quotient maps by the definition of the topology on D, and the morphism φ_s of $\mathcal{O}_{\mathcal{E}}$ is continuous. We deduce that the morphism φ_s of D is continuous.

b) The image of $\oplus^d \Lambda(N_0)$ by f is a lattice D_0 of D. For any $k \in \mathbb{N}$ the $\Lambda(N_0)$-submodule $D_{0,k}$ of D generated by $(\varphi_s(X^k e_i))_{1 \leq i \leq d}$ also is a treillis of D because φ_s is étale. Here $\{e_i \mid 1 \leq i \leq d\}$ is a generating family of D_0. We have $\psi_s(D_{0,k}) = X^k D_0$ (cf. lemma 3.4).

c) When D is killed by a power of p, we deduce that ψ_s is continuous by the properties 1 and 2 of the treillis. When D is not killed by a power of p, we deduce that the reduction modulo p^n of ψ_s is continuous for all n; this implies that ψ_s is continuous because $(A = o, D)$ satisfy the properties a, b, c, d of section 6.5, and $D/p^n D$ is a (finitely generated) étale φ_s-module over $\mathcal{O}_{\mathcal{E}}$. $\square$

We put

$$D^+ := \{x \in D : \text{ the sequence } (\varphi_s^k(x))_{k\in\mathbb{N}} \text{ is bounded in } D\}$$

and

(48) $$D^{++} := \{x \in D \mid \lim_{k\to\infty} \varphi_s^k(x) = 0\} \ .$$

Proposition 7.2. *(i) If D is killed by a power of p, then D^+ and D^{++} are lattices in D.*

(ii) There exists a unique maximal treillis $D^\sharp$ such that $\psi_s(D^\sharp) = D^\sharp$.

(iii) The set of ψ_s-stable treillis in D has a unique minimal element $D^\natural$; it satisfies $\psi_s(D^\natural) = D^\natural$.

(iv) $X^{-k}D^\sharp$ is a treillis stable by ψ_s for all $k \in \mathbb{N}$.

Proof. The references given in the following are stated for étale (φ_{s_p}, Γ)-modules but the proofs never use that there exists an action of Γ and they are valid for étale φ_{s_p}-modules.

(i) For $s = s_p$ this is [5] Prop. II.2.2(iii) and Lemma II.2.3. The properties of s_p which are needed for the argument are still satisfied for general s in the following form:

– $\varphi_s(X) \in \varphi_{s_p}^m(X)\Lambda(\mathbb{Z}_p)^\times$ where $s = s_0 s_p^m z$ with $s_0 \in \Gamma$, $m \geq 1$, and $z \in Z$.

– $(\varphi_s(X)X^{-1})^{p^k} \in p^{k+1}\Lambda(\mathbb{Z}_p) + X^{(p-1)p^k}\Lambda(\mathbb{Z}_p)$ for any $k \in \mathbb{N}$.

(ii) and (iii) For any finitely generated $\mathcal{O}_\mathcal{E}$-torsion module M we denote its Pontrjagin dual of continuous o-linear maps from M to K/o by $M^\vee := \mathrm{Hom}_o^{cont}(M, K/o)$. Obviously, $M^\vee$ again is an $\mathcal{O}_\mathcal{E}$-module by $(\lambda f)(x) := f(\lambda x)$ for $\lambda \in \mathcal{O}_\mathcal{E}$, $f \in M^\vee$, and $x \in M$. It is shown in [5] Lemma I.2.4 that:

– $M^\vee$ is a finitely generated $\mathcal{O}_\mathcal{E}$-torsion module,

– the topology of pointwise convergence on $M^\vee$ coincides with its weak topology as an $\mathcal{O}_\mathcal{E}$-module, and

– $M^{\vee\vee} = M$.

Now let D be as in the assertion but killed by a power of p. One checks that $D^\vee$ also belongs to $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E}}(\varphi_s)$ with respect to the semilinear map $\varphi_s(f) := f \circ \psi_s$ for $f \in D^\vee$; moreover, the canonical left inverse is $\psi_s(f) = f \circ \varphi_s$. Next, [5] Lemma I.2.8 shows that:

– If $D_0 \subset D$ is a lattice then $D_0^\perp := \{d \in D^\vee : f(D_0) = 0\}$ is a lattice in $D^\vee$, and $D_0^{\vee\vee} = D_0$.

We now define $D^\natural := (D^\vee)^+$ and $D^\sharp := (D^\vee)^{++}$. The purely formal arguments in the proofs of [5] Prop. II.6.1, Lemma II.6.2, and Prop. II.6.3 show that $D^\natural$ and $D^\sharp$ have the asserted properties.

For a general D in $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E}}(\varphi_s)$ the (formal) arguments in the proof of [5] Prop. II.6.5 show that $((D/p^nD)^\natural)_{n\in\mathbb{N}}$ and $((D/p^nD)^\sharp)_{n\in\mathbb{N}}$ are well defined projective systems of compact $\Lambda(\mathbb{Z}_p)$-modules (with surjective transition maps). Hence

$$D^\natural := \varprojlim (D/p^nD)^\natural \quad \text{and} \quad D^\sharp := \varprojlim (D/p^nD)^\sharp$$

have the asserted properties.

(iv) $X^{-k}D^\sharp$ is clearly a treillis. As X divides $\varphi_s(X) = (1+X)^a - 1$ in $\Lambda(\mathbb{Z}_p) = o[[X]]$ (when $s \in s_a Z$ for some $a \in p\mathbb{Z}_p \setminus \{0\}$), there exists $f(X) \in o[[X]]$ such that $\varphi_s(X^k) = X^k f(X)^k$. So we have

$$\psi_s(X^{-k}D^\sharp) = \psi_s(\varphi_s(X^{-k})f(X)^k D^\sharp) = X^{-k}\psi_s(f(X)^k D^\sharp) \subset X^{-k}\psi_s(D^\sharp) \subset X^{-k}D^\sharp$$

since $D^\sharp$ is ψ_s-stable by definition. □

Proposition 7.3. *Let D be a finitely generated étale φ_s-module over $\mathcal{O}_\mathcal{E}$ killed by a power of p. For any compact subset $C \subseteq D$, there exists an $r \in \mathbb{N}$ such that*

$$\bigcup_{k \geq 0} \psi_s^k(N_0 C) \subseteq X^{-r} D^{++} \ .$$

Proof. Since N_0C is compact and D^{++} and $D^\sharp$ are treillis, there exists $l \in \mathbf{N}$ such that $N_0 C \subset X^{-l} D^\sharp \subset X^{-2l} D^{++}$. By iv) of prop 7.2 we obtain for all $k \geq 0$

$$\psi_s^k(N_0 C) \subset \psi_s^k(X^{-l} D^\sharp) \subset X^{-l} D^\sharp \subset X^{-2l} D^{++}$$

and we can take $r = 2l$. □

Corollary 7.4. *Let D be a finitely generated étale φ_s-module over $\mathcal{O}_\mathcal{E}$. For any compact subset $C \subseteq D$ and any $n \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that*

$$\bigcup_{k \geq k_0} \psi_s^k(N_0 C) \subseteq D^\sharp + p^n D \ .$$

Proof. We may assume that D is killed by a power of p. In view of (the proof of) prop. 7.3 it suffices to show that for all $l > 0$ there exists a k_0 such that $\psi_s^{k_0}(X^{-l} D^\sharp) = D^\sharp$. By prop. 7.2(ii) and (iv) we have

$$D^\sharp = \psi_s^{k+1}(D^\sharp) \subseteq \psi_s^{k+1}(X^{-l} D^\sharp) \subseteq \psi_s^k(X^{-l} D^\sharp) \subseteq X^{-l} D^\sharp$$

for any $k \geq 0$. Hence $\bigcap_k \psi_s^k(X^{-l} D^\sharp)$ is a treillis in D on which ψ_s is surjective. Therefore it coincides with $D^\sharp$ by the maximality of $D^\sharp$ (Prop. 7.2(ii)). On the other hand, the $\mathbb{Z}_p[[X]]$-module $(X^{-l} D^\sharp)/D^\sharp$ is killed by both X^l and p^n and hence is finite. So there exists a k_0 such that $\psi_s^k(X^{-l} D^\sharp) = D^\sharp$ for all $k \geq k_0$. □

For any submonoid $L' \subset L_+$ containing a strictly dominant element, an étale L'-module over $\mathcal{O}_\mathcal{E}$ is a finitely generated $\mathcal{O}_\mathcal{E}$-module with an étale semilinear action of L'.

A topologically étale L'-module over $\mathcal{O}_\mathcal{E}$ will be an étale L'-module D over $\mathcal{O}_\mathcal{E}$ such that the action $L' \times D \to D$ of L' on D is continuous. This terminology is provisional since we will show later on (Cor. 8.28) in a more general context that any étale L'-module over $\mathcal{O}_\mathcal{E}$ in fact is topologically étale and, in particular, is a complete topologically étale $o[N_0 L']$-module in our previous sense.

Let D be a topologically étale L_+-module over $\mathcal{O}_\mathcal{E}$ and let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$. Denote

$$w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Using the formula

$$g u(r) w_0 = \begin{pmatrix} ar+b & a \\ cr+d & c \end{pmatrix},$$

one checks that the set U_g defined by $g^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 = U_g w_0 P/P$ is

$$U_g = u(X_g) \quad \text{where} \quad X_g = \{r \in \mathbb{Z}_p | cr+d \neq 0, \quad \frac{ar+b}{cr+d} \in \mathbb{Z}_p\}.$$

For each $r \in X_g$ we can write

$$g u(r) w_0 = u(g[r]) t(g,r) w_0 u\left(\frac{c}{cr+d}\right),$$

where

$$g[r]=\frac{ar+b}{cr+d}\in\mathbb{Z}_p,\quad t(g,r)=\begin{pmatrix}\frac{\det g}{cr+d} & 0\\ 0 & cr+d\end{pmatrix}.$$

We deduce that for $u(r)\in U_g$ we have

$$\alpha(g,x_{u(r)})=u(g[r])t(g,r).$$

Let now $s=s_a z\in L_+$ be strictly dominant, with $z\in Z$ and $a\in p\mathbb{Z}_p-\{0\}$. There exists a positive integer $k_{g,s}$ such that for all $k\geq k_{g,s}$ we have $t(g,r)s^k\in L_+$. Note that $N_k=s^kN_0s^{-k}=u(a^k\mathbb{Z}_p)$. We deduce that for $k\geq k_{g,s}$ the operators $\mathcal{H}_{g,J(N_0/N_k)}$ introduced in (28) are equal to the operators

$$\mathcal{H}_{g,s,J(\mathbb{Z}_p/a^k\mathbb{Z}_p)}=\sum_{r\in X_g\cap J(\mathbb{Z}_p/a^k\mathbb{Z}_p)}(1+X)^{g[r]}\varphi_{t(g,r)s^k}\psi_s^k((1+X)^{-r}).$$

Proposition 7.5. *Let D be a topologically étale L_+-module over $\mathcal{O}_{\mathcal{E}}$. For the compact open topology in $\mathrm{End}_o^{cont}(D)$, the maps $\alpha_{g,0}:N_0\longrightarrow\mathrm{End}_o^{cont}(D)$, for $g\in N_0\overline{P}N_0$, are integrable with respect to s and* res*, for all $s\in L_+$ strictly dominant, i.e. $s=s_az$ with $a\in p\mathbb{Z}_p-\{0\}$ and $z\in Z$, their integrals*

$$\mathcal{H}_g=\int_{N_0}\alpha_{g,0}d\,\mathrm{res}=\lim_{k\to\infty}\mathcal{H}_{g,s,J(\mathbb{Z}_p/a^k\mathbb{Z}_p)}$$

for any choices of $J(\mathbb{Z}_p/a^k\mathbb{Z}_p)\subset\mathbb{Z}_p$, do not depend on the choice of s and satisfy the relations H1, H2, H3 of Proposition 5.14.

Proof. By Prop. 6.16, we reduce to the case that D is killed by a power of p and to showing the assumptions of Prop. 6.9 for the family of all compact subsets of D. The axioms $\mathfrak{C}_i$, for $1\leq i\leq 6$, are obviously satisfied by continuity of φ_s,ψ_s, and of the action of $n\in N_0$ on D.

i. We check first the convergence criterion of Proposition 6.8, using the theory of treillis, i.e. of lattices, in D.

Given a lattice $\mathcal{M}\subseteq D$, a compact subset $C\subseteq D$ such that $N_0C\subseteq C$, and a compact subset $C_+\subseteq L_+$, we want to find a compact open subgroup $P_1\subset P_0$ and an integer $k_0\in\mathbb{N}$ such that

$$s^k(1-P_1)C_+\psi_s^k(C)\subseteq\mathcal{M}\tag{49}$$

for all $k\geq k_0$.

We choose $r_0\in\mathbb{N}$ with $\varphi_s^k(D^{++})\subset\mathcal{M}$ for all $k\geq r_0$, as we may by the definition of D^{++}. We choose $r,k_0\in\mathbb{N}$ such that $k_0\geq r_0$ and

$$\bigcup_{k\geq k_0}\psi_s^k(C)\subseteq X^{-r}D^{++}\ ,$$

as we may by Prop. 7.3. Applying C_+ we obtain

$$\bigcup_{k\geq k_0}C_+\psi_s^k(C)\subseteq C_+(X^{-r}D^{++})\ .$$

The continuity of the action of P_+ on D implies that $C_+(X^{-r}D^{++})$ is compact. Hence we can choose $r'\in\mathbb{N}$ such that $C_+(X^{-r}D^{++})\subseteq X^{-r'}D^{++}$ and we obtain

$$\bigcup_{k\geq k_0}C_+\psi_s^k(C)\subseteq X^{-r'}D^{++}\ .$$

As $X^{-r'}D^{++}$ is compact and D^{++} an open neighborhood of 0, the continuity of the action of P_+ on D there exists a compact open subgroup $P_1 \subseteq P_+$ such that

$$(1-P_1)X^{-r'}D^{++} \subseteq D^{++} \ .$$

Hence we have $s^k(1-P_1)C_+\psi_s^k(C) \subset \varphi_s^k(D^{++}) \subset \mathcal{M}$ for all $k \geq k_0$.

ii. To obtain all the assumptions of Prop. 6.9 for the family of all compact subsets of D, it remains to prove that, given $x \in D$ and $g \in N_0\overline{P}N_0$, $s = s_a z$ with $a \in p\mathbb{Z}_p - \{0\}$ and $z \in Z$, and $(J(\mathbb{Z}_p/a^k\mathbb{Z}_p))_k$, there exists a compact $C_{x,g,s} \subset D$ and a positive integer $k_{x,g,s}$ such that $\mathcal{H}_{g,s,J(\mathbb{Z}_p/a^k\mathbb{Z}_p)}(x) \in C_{x,g,s}$ for any $k \geq k_{x,g,s}$. This is clear because D is locally compact (by hypothesis D is killed by a power of p) and the sequence $(\mathcal{H}_{g,s,J(\mathbb{Z}_p/a^k\mathbb{Z}_p)}(x))_k$ converges.

iii. The independence of the choice of $s \in L_+$ strictly dominant results from the fact that, for $z \in Z$, $e \in \mathbb{Z}_p^*$, and a positive integer r, we have $(zs_{p^re})^kN_0(zs_{p^re})^{-k} = s_p^{kr}N_0s_p^{kr}$ and $\varphi^k_{zs_{p^re}}\psi^k_{zs_{p^re}} = \varphi^{rk}_{s_p}\psi^{kr}_{s_p}$ as ψ_{zs_e} is the right and left inverse of φ_{zs_e}. □

Remark 7.6. Let D be a topologically étale L_+-module over $\mathcal{O}_\mathcal{E}$, on which Z acts through a character ω. The pointwise convergence of the integrals $\int_{N_0}\alpha_{g,0}d\mathrm{res}$ is a basic theorem of Colmez, allowing him the construction of the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ that he denotes $D \boxtimes_\omega \mathbb{P}^1$. Our construction coincides with Colmez's construction because our $\mathcal{H}_g \in \mathrm{End}_o^{\mathrm{cont}}(D)$ are the same as the H_g of Colmez given in [6] lemma II.1.2 (ii). Indeed,

$$\alpha(g, x_{u(r)}) = u(g[r])t(g,r) =$$

$$\omega(cr+d)u(g[r])\begin{pmatrix} \frac{\det g}{(cr+d)^2} & 0 \\ 0 & 1 \end{pmatrix} = \omega(cr+d)\begin{pmatrix} g'[r] & g[r] \\ 0 & 1 \end{pmatrix},$$

where $g'[r] = \frac{\det g}{(cr+d)^2}$. This coincides with Colmez's formula.

The major goal of the paper is to generalize Prop. 7.5. See Prop. 9.16.

8 A generalisation of (φ,Γ)-modules

We return to a general group G. We denote $G^{(2)} := GL(2,\mathbb{Q}_p)$ and the objects relative to $G^{(2)}$ will be affected with an upper index $^{(2)}$.

a) We suppose that N_0 has the structure of a p-adic Lie group and that we have a continuous surjective homomorphism

$$\ell : N_0 \to N_0^{(2)} \ .$$

We choose a continuous homomorphism $\iota : N_0^{(2)} \to N_0$ which is a section of ℓ (which is possible because $N_0^{(2)} \simeq \mathbb{Z}_p$).

We have $N_0 = N_\ell\, \iota(N_0^{(2)})$ where N_ℓ is the kernel of ℓ.

We denote by $L_{\ell,+} := \{t \in L \mid tN_\ell t^{-1} \subset N_\ell \ , \ tN_0t^{-1} \subset N_0\}$ the stabilizer of N_ℓ in the L-stabilizer of N_0, and by $L_{\ell,\iota} := \{t \in L \mid tN_\ell t^{-1} \subset N_\ell \ , \ t\iota(N_0^{(2)})t^{-1} \subset \iota(N_0^{(2)})\}$ the stabilizer of N_ℓ in the L-stabilizer of $\iota(N_0^{(2)})$. We have $L_{\ell,\iota} \subset L_{\ell,+}$.

b) We suppose given a submonoid L_* of $L_{\ell,\iota}$ containing s and a continuous homomorphism $\ell : L_* \to L_+^{(2)}$ such that (ℓ,ι) satisfies

$$\ell(tut^{-1}) = \ell(t)\ell(u)\ell(t)^{-1} \ , \quad t\iota(y)t^{-1} = \iota(\ell(t)y\ell(t)^{-1}) \ , \ \text{for } u \in N_0, y \in N_0^{(2)}, t \in L_* \ .$$

The sequence $\ell(s^n N_0 s^{-n}) = \ell(s)^n N_0^{(2)} \ell(s)^{-n}$ in $N^{(2)}$ is decreasing with trivial intersection. The maps ℓ in a) and b) combine to a unique continuous homomorphism

$$\ell \ : \ P_* := N_0 \rtimes L_* \to P_+^{(2)} .$$

8.1 The microlocalized ring $\Lambda_\ell(N_0)$

The ring $\Lambda_\ell(N_0)$, denoted by $\Lambda_{N_\ell}(N_0)$ in [12], is a generalisation of the ring $\mathcal{O}_\mathcal{E}$, which corresponds to $\Lambda_{\mathrm{id}}(N_0^{(2)})$. We refer the reader to [12] for the proofs of some claims in this section.

The maximal ideal $\mathcal{M}(N_\ell)$ of the completed group o-algebra $\Lambda(N_\ell) = o[[N_\ell]]$ is generated by p_K and by the kernel of the augmentation map $o[N_\ell] \to o$.

The ring $\Lambda_\ell(N_0)$ is the $\mathcal{M}(N_\ell)$-adic completion of the localisation of $\Lambda(N_0)$ with respect to the Ore subset $S_\ell(N_0)$ of elements which are not in $\mathcal{M}(N_\ell)\Lambda(N_0)$. The ring $\Lambda(N_0)$ can be viewed as the ring $\Lambda(N_\ell)[[X]]$ of skew Taylor series over $\Lambda(N_\ell)$ in the variable $X = [\gamma] - 1$ where $\gamma \in N_0$ and $\ell(\gamma)$ is a topological generator of $\ell(N_0)$. Then $\Lambda_\ell(N_0)$ is viewed as the ring of infinite skew Laurent series $\sum_{n \in \mathbb{Z}} a_n X^n$ over $\Lambda(N_\ell)$ in the variable X with $\lim_{n \to -\infty} a_n = 0$ for the pseudo-compact topology of $\Lambda(N_\ell)$.

The ring $\Lambda_\ell(N_0)$ is strict-local noetherian of maximal ideal $\mathcal{M}_\ell(N_0)$ generated by $\mathcal{M}(N_\ell)$. It is a pseudocompact ring for the $\mathcal{M}(N_\ell)$-adic topology (called the strong topology). It is a complete Hausdorff ring for the weak topology ([12] Lemma 8.2) with fundamental system of open neighborhoods of 0 given by

$$O_{n,k} := \mathcal{M}_\ell(N_0)^n + \mathcal{M}(N_0)^k \ \text{ for } \ n \in \mathbb{N}, k \in \mathbb{N} \ .$$

In the computations it is sometimes better to use the fundamental systems of open neighborhoods of 0 defined by

$$B_{n,k} := \mathcal{M}_\ell(N_0)^n + X^k \Lambda(N_0) \ \text{ for } \ n \in \mathbb{N}, k \in \mathbb{N} \ ,$$

and

$$C_{n,k} := \mathcal{M}_\ell(N_0)^n + \Lambda(N_0) X^k \ \text{ for } \ n \in \mathbb{N}, k \in \mathbb{N} \ ,$$

which are equivalent due to the two formulae

$$X^k \Lambda(N_0) \subseteq \Lambda(N_0) X^k + \mathcal{M}(N_0)^k \quad \text{and} \quad \Lambda(N_0) X^k \subseteq X^k \Lambda(N_0) + \mathcal{M}(N_0)^k \ ,$$

We write $O_n := O_{n,n}, B_n := B_{n,n}$, and $C_n = C_{n,n}$. Then $(O_n)_n, (B_n)_n$, and $(C_n)_n$ are also fundamental system of open neighborhoods of 0 in $\Lambda_\ell(N_0)$.

The action $(b = ut, n_0) \mapsto b.n_0 = utn_0t^{-1}$ of the monoid $P_{\ell,+} = N_0 \rtimes L_{\ell,+}$ on N_0 induces a ring action $(t, x) \mapsto \varphi_t(x)$ of $L_{\ell,+}$ on the o-algebra $\Lambda(N_0)$ respecting the ideal $\Lambda(N_0)\mathcal{M}(N_\ell)$, and the Ore set $S_\ell(N_0)$ hence defines a ring action of $L_{\ell,+}$ on the o-algebra $\Lambda_\ell(N_0)$. This action respects the maximal ideals $\mathcal{M}(N_0)$ and $\mathcal{M}_\ell(N_0)$ of the rings $\Lambda(N_0)$ and $\Lambda_\ell(N_0)$ and hence the open neighborhoods of zero $O_{n,k}$.

Lemma 8.1. *For $t \in L_{\ell,+}$, a fundamental system of open neighborhoods of 0 in $\Lambda_\ell(N_0)$ is given by*

$$(\varphi_t(O_{n,k})\Lambda(N_0))_{n,k \in \mathbb{N}} \ .$$

Proof. We have just seen $\varphi_t(O_{n,k})\Lambda(N_0) \subset O_{n,k}$. Conversely, given $n, k \in \mathbb{N}$, we have to find $n', k' \in \mathbb{N}$ such that $O_{n',k'} \subset \varphi_t(O_{n,k})\Lambda(N_0)$. This can be deduced from the following fact. Let $H' \subset H$ be an open subgroup. Then given $k' \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that

$$\mathcal{M}(H')^{k'} \Lambda(H) \supset \mathcal{M}(H)^k \ .$$

Indeed by taking a smaller H' we can suppose that $H' \subset H$ is open normal. Then $\mathcal{M}(H')^{k'}\Lambda(H)$ is a two-sided ideal in $\Lambda(H)$ and the factor ring $\Lambda(H)/\mathcal{M}(H')\Lambda(H)$ is an artinian local ring with maximal ideal $\mathcal{M}(H)/\mathcal{M}(H')\Lambda(H)$. It remains to observe that in any artinian local ring the maximal ideal is nilpotent. $\square$

Proposition 8.2. *The action of $L_{\ell,+}$ on $\Lambda_\ell(N_0)$ is étale : for any $t \in L_{\ell,+}$, the map*

$$(\lambda, x) \mapsto \lambda\varphi_t(x) : \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} \Lambda_\ell(N_0) \to \Lambda_\ell(N_0)$$

is bijective.

Proof. We follow ([12] Prop. 9.6, Proof, Step 1).

1) The conjugation by t gives a natural isomorphism

$$\Lambda_\ell(N_0) \to \Lambda_{tN_\ell t^{-1}}(tN_0t^{-1}) \ .$$

2) Obviously $\Lambda_{tN_\ell t^{-1}}(tN_0t^{-1}) = \Lambda(tN_0t^{-1}) \otimes_{\Lambda(tN_0t^{-1})} \Lambda_{tN_\ell t^{-1}}(tN_0t^{-1})$, and the map

$$\Lambda(tN_0t^{-1}) \otimes_{\Lambda(tN_0t^{-1})} \Lambda_{tN_\ell t^{-1}}(tN_0t^{-1}) \to \Lambda(N_0) \otimes_{\Lambda(tN_0t^{-1})} \Lambda_{tN_\ell t^{-1}}(tN_0t^{-1})$$

is injective because $\Lambda_{tN_\ell t^{-1}}(tN_0t^{-1})$ is flat on $\Lambda(tN_0t^{-1})$.

3) The natural map

$$\Lambda(N_0) \otimes_{\Lambda(tN_0t^{-1})} \Lambda_{tN_\ell t^{-1}}(tN_0t^{-1}) \to \Lambda_\ell(N_0)$$

is bijective.

4) The ring action $\varphi_t : \Lambda_\ell(N_0) \to \Lambda_\ell(N_0)$ of t on $\Lambda_\ell(N_0)$ is the composite of the maps of 1), 2), 3), hence is injective.

5) The proposition is equivalent to 3) and φ_t injective. $\square$

Remark 8.3. The proposition is equivalent to : for any $t \in L_{\ell,+}$, the map

$$(u, x) \mapsto u\varphi_t(x) : o[N_0] \otimes_{o[N_0],\varphi_t} \Lambda_\ell(N_0) \to \Lambda_\ell(N_0)$$

is bijective.

8.2 The categories $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ and $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$

By the universal properties of localisation and adic completion the continuous homomorphisms ℓ and ι between N_0 and $N_0^{(2)}$ extend to continuous o-linear homomorphisms of pseudocompact rings,

$$\ell : \Lambda_\ell(N_0) \to \mathcal{O}_\mathcal{E} \ , \ \iota : \mathcal{O}_\mathcal{E} \to \Lambda_\ell(N_0) \quad , \quad \ell \circ \iota = \mathrm{id} \ . \tag{50}$$

If we view the rings as rings of Laurent series, $\ell(X) = X^{(2)}$, $\iota(X^{(2)}) = X$, and ℓ is the augmentation map $\Lambda(N_\ell) \to o$ and ι is the natural injection $o \to \Lambda(N_\ell)$, on the coefficients. We have for $n, k \in \mathbb{N}$,

$$\begin{aligned} \ell(\mathcal{M}_\ell(N_0)) = p_K\mathcal{O}_\mathcal{E} \quad &, \quad \ell(B_{n,k}) = B^{(2)}_{n,k} \ , \\ \iota(p_K\mathcal{O}_\mathcal{E}) = \mathcal{M}_\ell(N_0) \cap \iota(\mathcal{O}_\mathcal{E}) \quad &, \quad \iota(B^{(2)}_{n,k}) = B_{n,k} \cap \iota(\mathcal{O}_\mathcal{E}) \ . \end{aligned} \tag{51}$$

We denote by $J(N_0)$ the kernel of $\ell : \Lambda(N_0) \to \Lambda(N_0^{(2)})$ and by $J_\ell(N_0)$ the kernel of $\ell : \Lambda_\ell(N_0) \to \mathcal{O}_\mathcal{E}$. They are the closed two-sided ideals generated (as left or right ideals) by the kernel of the augmentation map $o[N_\ell] \to o$. We have

$$\begin{aligned} \Lambda_\ell(N_0) = \iota(\mathcal{O}_\mathcal{E}) \oplus J_\ell(N_0) \quad &, \quad \mathcal{M}_\ell(N_0) = p_K\iota(\mathcal{O}_\mathcal{E}) \oplus J_\ell(N_0) \ , \\ X^k\Lambda(N_0) = \iota((X^{(2)})^k\Lambda(N_0^{(2)}) \oplus X^kJ(N_0) \quad &, \quad B_{n,k} = \iota(B^{(2)}_{n,k}) \oplus (J_\ell(N_0) \cap B_{n,k}) \ . \end{aligned} \tag{52}$$

The maps ℓ and ι are L_*-equivariant: for $t \in L_*$,

$$\ell \circ \varphi_t = \varphi_{\ell(t)} \circ \ell \quad , \quad \iota \circ \varphi_{\ell(t)} = \varphi_t \circ \iota \ , \tag{53}$$

thanks to the hypothesis b) made at the beginning of this chapter. The map ι is equivariant for the canonical action of the inverse monoid L_*^{-1}, but not the map ℓ as the following lemma shows.

Lemma 8.4. *For $t \in L_*$, we have $\iota \circ \psi_{\ell(t)} = \psi_t \circ \iota$. We have $\ell \circ \psi_t = \psi_{\ell(t)} \circ \ell$ if and only if $N_\ell = tN_\ell t^{-1}$.*

Proof. Clearly $N_0 = N_\ell \rtimes \iota(N_0^{(2)})$ and $tN_0t^{-1} = tN_\ell t^{-1} \rtimes t\iota(N_0^{(2)})t^{-1}$ for $t \in L$. We choose, as we may, for $t \in L_{\ell,\iota}$, a system $J(N_0/tN_0t^{-1})$ of representatives of N_0/tN_0t^{-1} containing 1 such that

$$J(N_0/tN_0t^{-1}) = \{u\iota(v) \ |u \in J(N_\ell/tN_\ell t^{-1}) \ , \ v \in J(N_0^{(2)}/\ell(t)N_0^{(2)}\ell(t^{-1}))\} \ . \tag{54}$$

We have $\iota \circ \psi_{\ell(t)} = \psi_t \circ \iota$ because, for $\lambda \in \mathcal{O}_{\mathcal{E}}$, we have on one hand (12)

$$\lambda = \sum_{v \in J(N_0^{(2)}/\ell(t)N_0^{(2)}\ell(t)^{-1})} v\varphi_{\ell(t)}(\lambda_{v,\ell(t)}) \quad , \quad \lambda_{v,\ell(t)} = \psi_{\ell(t)}(v^{-1}\lambda) \ ,$$

$$\iota(\lambda) = \sum_{v \in J(N_0^{(2)}/\ell(t)N_0^{(2)}\ell(t)^{-1})} \iota(v)\varphi_t(\iota(\lambda_{v,\ell(t)})) \ ,$$

and on the other hand (12)

$$\iota(\lambda) = \sum_{u \in J(N_\ell/tN_\ell t^{-1}), v \in J(N_0^{(2)}/\ell(t)N_0^{(2)})\ell(t)^{-1})} u\iota(v)\varphi_t(\iota(\lambda)_{u\iota(v),t}) \ ,$$

where $\iota(\lambda)_{u\iota(v),t} = \psi_t(\iota(v)^{-1}u^{-1}\iota(\lambda))$. By the uniqueness of the decomposition,

$$\iota(\lambda)_{\iota(v),t} = \iota(\lambda_{v,\ell(t)}) \ , \ \iota(\lambda)_{u\iota(v),t} = 0 \ \text{ if } u \neq 1 \ .$$

Taking $u = 1, v = 1$, we get $\psi_t(\iota(\lambda)) = \iota(\psi_{\ell(t)}(\lambda))$.

A similar argument shows that $\ell \circ \psi_t = \psi_{\ell(t)} \circ \ell$ if and only if $N_\ell = tN_\ell t^{-1}$. For $\lambda \in \Lambda_\ell(N_0)$,

$$\lambda = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(\lambda_{u,t}) \quad , \quad \lambda_{u,t} = \psi_t(u^{-1}\lambda) \ ,$$

$$\ell(\lambda) = \sum_{u \in J(N_0/tN_0t^{-1})} \ell(u)\varphi_{\ell(t)}(\ell(\lambda_{u,t})) = \sum_{v \in J(N_0^{(2)}/\ell(t)N_0^{(2)})} v\varphi_{\ell(t)}(\ell(\lambda)_{v,\ell(t)})$$

By the uniqueness of the decomposition,

$$\ell(\lambda)_{v,\ell(t)} = \sum_{u \in J(N_\ell/tN_\ell t^{-1})} \ell(\lambda_{u\iota(v),t}) \ .$$

We deduce that $\ell \circ \psi_t = \psi_{\ell(t)} \circ \ell$ if and only if $N_\ell = tN_\ell t^{-1}$. □

Remark 8.5. $\ell \circ \psi_s \neq \psi_{\ell(s)} \circ \ell$, except in the trivial case where $\ell : N_0 \to N_0^{(2)}$ is an isomorphism, because $sN_\ell s^{-1} \neq N_\ell$ as the intersection of the decreasing sequence $s^kN_\ell s^{-k}$ for $k \in \mathbb{N}$ is trivial.

For future use, we note:

Lemma 8.6. *The left or right $o[N_0]$-submodule generated by $\iota(\mathcal{O}_{\mathcal{E}})$ in $\Lambda_\ell(N_0)$ is dense.*

Proof. As $o[N_0]$ is dense in $\Lambda(N_0)$ it suffices to show that the left or right $\Lambda(N_0)$-submodule generated by $\iota(\mathcal{O}_{\mathcal{E}})$ in $\Lambda_\ell(N_0)$ is dense. This will be shown even with respect to the $\mathcal{M}_\ell(N_0)$-adic topology.

View $\lambda \in \Lambda_\ell(N_0)$ as an infinite Laurent series $\lambda = \sum_{n\in\mathbb{Z}} \lambda_n X^n$ with $\lambda_n \in \Lambda(N_\ell)$ and $\lim_{n\to-\infty} \lambda_n = 0$ in the $\mathcal{M}(N_\ell)$-adic topology of $\Lambda(N_\ell)$. Further, note that the left, resp. right, $\Lambda(N_0)$-submodule of $\Lambda_\ell(N_0)$ generated by $\iota(\mathcal{O}_{\mathcal{E}})$ contains $\Lambda(N_0)X^{-m}$, resp. $X^{-m}\Lambda(N_0)$, for any positive integer m. Finally, for each $n \in \mathbb{N}$ there exists μ_n in $\Lambda(N_0)X^{-m}$, resp. $X^{-m}\Lambda(N_0)$, for some large m, such that $\lambda - \mu_n \in \mathcal{M}_\ell(N_0)^n$. $\square$

Let M be a finitely generated $\Lambda_\ell(N_0)$-module and let $f : \oplus_{i=1}^n \Lambda_\ell(N_0) \to M$ be a $\Lambda_\ell(N_0)$-linear surjective map. We put on M the quotient topology of the weak topology on $\oplus_{i=1}^n \Lambda_\ell(N_0)$; this is independent of the choice of f. Then M is a Hausdorff and complete topological $\Lambda_\ell(N_0)$-module and every submodule is closed ([12] Lemma 8.22). In the same way we can equip M with the pseudocompact topology. Again M is Hausdorff and complete and every submodule is closed in the pseudocompact topology, because $\Lambda_\ell(N_0)$ is noetherian. The weak topology on M is weaker than the pseudocompact topology which is weaker than the p-adic topology. In particular the intersection of the submodules $p^n M$ for $n \in \mathbb{N}$ is 0. By [9] IV.3.Prop. 10, M is p-adically complete, i.e., the natural map $M \to \varprojlim_n M/p^n M$ is bijective.

Unless otherwise indicated, M is always understood to carry the weak topology.

Lemma 8.7. *The properties a,b,c,d of section 6.5 are satisfied by (o, M) and M is complete.*

Definition 8.8. *A finitely generated module M over $\Lambda_\ell(N_0)$ with an étale semilinear action of a submonoid L' of $L_{\ell,+}$ is called an étale L'-module over $\Lambda_\ell(N_0)$.*

We denote by $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L')$ the category of étale L'-modules on $\Lambda_\ell(N_0)$.

Lemma 8.9. *The category $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L')$ is abelian.*

Proof. As in the proof of Proposition 3.8 and using that the ring $\Lambda_\ell(N_0)$ is noetherian. $\square$

The continuous homomorphism $\ell : L_* \to L_+^{(2)}$ defines an étale semilinear action of L_* on the ring $\Lambda_{id}(N_0^{(2)})$ isomorphic to $\mathcal{O}_{\mathcal{E}}$.

Definition 8.10. *A finitely generated module D over $\mathcal{O}_{\mathcal{E}}$ with an étale semilinear action of L_* is called an étale L_*-module over $\mathcal{O}_{\mathcal{E}}$.*

An element $t \in L_*$ in the kernel $L_*^{\ell=1}$ of ℓ acts trivially on $\mathcal{O}_{\mathcal{E}}$ hence bijectively on an étale L_*-module over $\mathcal{O}_{\mathcal{E}}$.

Remark 8.11. The action of $L_*^{\ell=1}$ on D extends to an action of the subgroup of L generated by $L_*^{\ell=1}$ if $L_*^{\ell=1}$ is commutative or if we assume that for each $t \in L_*^{\ell=1}$ there exists an integer $k > 0$ such that $s^k t^{-1} \in L_*$. The assumption is trivially satisfied whenever $L_* = H \cap L_+$ for some subgroup $H \subset L$.

Indeed, the subgroup generated by $L_*^{\ell=1}$ is the set of words of the form $x_1^{\pm1} \dots x_n^{\pm1}$ with $x_i \in L_*^{\ell=1}$ for $i = 1, \dots, n$. So if we have an action of all the elements and all the inverses, then we can take the products of these, as well. We need to show that this action is well defined, i.e., whenever we have a relation

$$x_1^{\pm1} \dots x_n^{\pm1} = y_1^{\pm1} \dots y_r^{\pm1} \tag{55}$$

in the group then the action we just defined is the same using the x's or the y's. If $L_*^{\ell=1}$ is commutative, this is easily checked. In the second case, we can choose a big enough

$k = \sum_{i=1}^n k_i + \sum_{j=1}^r k_j$ such that $s^{k_i} x_i^{-1} \in L_*$ and $s^{k_j} y_j^{-1} \in L_*$. Then multiplying the relation (55) by s^k we obtain a relation in L_* so the two sides will define the same action on D. This shows that the actions defined using the two sides of (55) are equal on $\varphi_s^k(D) \subset D$. However, they are also equal on group elements $u \in N_0^{(2)}$ hence on the whole $D = \bigoplus_{u \in J(N_0^{(2)}/\varphi_s^k(N_0^{(2)}))} u\varphi_s^k(D)$.

We denote by $\mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E},\ell}}(L_*)$ the category of étale L_*-modules on $\mathcal{O}_{\mathcal{E}}$.

Lemma 8.12. *The category $\mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E},\ell}}(L_*)$ is abelian.*

Proof. As in the proof of Proposition 3.8 and using that the ring $\mathcal{O}_{\mathcal{E}}$ is noetherian. $\square$

We will prove later that the categories $\mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E},\ell}}(L_*)$ and $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ are equivalent.

8.3 Base change functors

We recall a general argument of semilinear algebra (see [12]). Let A be a ring with a ring endomorphism φ_A, let B be another ring with a ring endomorphism φ_B, and let $f : A \to B$ be a ring homomorphism such that $f \circ \varphi_A = \varphi_B \circ f$. When M is an A-module with a semilinear endomorphism φ_M, its image by base change is the B-module $B \otimes_{A,f} M$ with the semilinear endomorphism $\varphi_B \otimes \varphi_M$. The endomorphism φ_M of M is called étale if the natural map

$$a \otimes m \mapsto a\varphi_M(m) : A \otimes_{A,\varphi_A} M \to M$$

is bijective.

Lemma 8.13. *When φ_M is étale, then $\varphi_B \otimes \varphi_M$ is étale.*

Proof. We have

$$B \otimes_{B,\varphi_B} (B \otimes_{A,f} M) = B \otimes_{A,\varphi_B \circ f} M = B \otimes_{f \circ \varphi_A} M = B \otimes_{A,f} (A \otimes_{A,\varphi_A} M) \cong B \otimes_{A,f} M.$$

$\square$

Applying these general considerations to the L_*-equivariant maps $\ell : \Lambda_\ell(N_0) \to \mathcal{O}_{\mathcal{E}}$ and $\iota : \mathcal{O}_{\mathcal{E}} \to \Lambda_\ell(N_0)$ satisfying $\ell \circ \iota = \mathrm{id}$ (see (50), (53)), we have the base change functors

$$M \mapsto \mathbb{D}(M) := \mathcal{O}_{\mathcal{E}} \otimes_{\Lambda_\ell(N_0),\ell} M$$

from the category of $\Lambda_\ell(N_0)$-modules to the category of $\mathcal{O}_{\mathcal{E}}$-modules, and

$$D \mapsto \mathbb{M}(D) := \Lambda_\ell(N_0) \otimes_{\mathcal{O}_{\mathcal{E}},\iota} D$$

in the opposite direction. Obviously these base change functors respect the property of being finitely generated. By the general lemma we obtain:

Proposition 8.14. *The above functors restrict to functors*

$$\mathbb{D}\colon \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*) \to \mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E},\ell}}(L_*) \quad \textit{and} \quad \mathbb{M}\colon \mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E},\ell}}(L_*) \to \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*) \ .$$

When $M \in \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$, the diagonal action of L_* on $\mathbb{D}(M)$ is:

$$\varphi_t(\mu \otimes m) = \varphi_{\ell(t)}(\mu) \otimes \varphi_t(m) \ \text{ for } t \in L_*, \mu \in \mathcal{O}_{\mathcal{E}}, m \in M \ . \tag{56}$$

When $D \in \mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E},\ell}}(L_*)$, the diagonal action of L_* on $\mathbb{M}(D)$ is:

$$\varphi_t(\lambda \otimes d) = \varphi_t(\lambda) \otimes \varphi_t(d) \ \text{ for } t \in L_*, \lambda \in \Lambda_\ell(N_0), d \in D \ . \tag{57}$$

The natural map

$$\ell_M : M \to \mathbb{D}(M) \quad , \quad \ell_M(m) = 1 \otimes m$$

is surjective, L_*-equivariant, with a P_*-stable kernel $M_\ell := J_\ell(N_0)M$. The injective L_*-equivariant map

$$\iota_D : D \to \mathbb{M}(D) \quad , \quad \iota_D(d) = 1 \otimes d$$

is ψ_t-equivariant for $t \in L_*$ (same proof as Lemma 8.4).

For future use we note the following property.

Lemma 8.15. *Let $d \in D$ and $t \in L_*$. We have*

$$\psi_t(u^{-1}\iota_D(d)) = \begin{cases} \iota_D(\psi_t(v^{-1}d)) & \text{if } u = \iota(v) \text{ with } v \in N_0^{(2)}, \\ 0 & \text{if } u \in N_0 \setminus \iota(N_0^{(2)})tN_0t^{-1}. \end{cases}$$

Proof. We choose a set $J \subset N_0^{(2)}$ of representatives for the cosets in $N_0^{(2)}/\ell(t)N_0^{(2)}\ell(t)^{-1}$. The semilinear endomorphism φ_t of D is étale hence

$$d = \sum_{v \in J} v\varphi_t(d_{v,t}) \quad \text{where } d_{v,t} = \psi_t(v^{-1}d) \ .$$

Applying ι_D we obtain

$$\iota_D(d) = \sum_v \iota(v)\iota_D(\varphi_t(d_{v,t})) = \sum_v \iota(v)\varphi_t(\iota_D(d_{v,t})) = \sum_v \iota(v)\varphi_t(\psi_t(\iota_D(v^{-1}d))) \ .$$

The map ι induces an injective map from J into N_0/tN_0t^{-1} with image included in a set $J(N_0/tN_0t^{-1}) \subset N_0$ of representatives for the cosets in N_0/tN_0t^{-1}. As the action φ_t of t in $\mathbb{M}(D)$ is étale, we have (12)

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} u\varphi_t(m_{u,t})) \quad \text{where } m_{u,t} = \psi_t(u^{-1}m)$$

for any $m \in \mathbb{M}(D)$. We deduce that $\psi_t(\iota(v^{-1})\iota_D(d)) = \iota_D(d_{v,t})$ when $v \in J$ and $\psi_t(u^{-1}\iota_D(d)) = 0$ when $u \in J(N_0/tN_0t^{-1}) \setminus \iota(J)$. As any element of $N_0^{(2)}$ can belong to a set of representatives of $N_0^{(2)}/\ell(t)N_0^{(2)}\ell(t)^{-1}$, we deduce that $\psi_t(\iota(v^{-1})\iota_D(d)) = \iota_D(d_{v,t})$ for any $v \in N_0^{(2)}$. For the same reason $\psi_t(\iota(u^{-1})\iota_D(d)) = 0$ for any $u \in N_0$ which does not belong to $\iota(N_0^{(2)})tN_0t^{-1}$. $\square$

8.4 Equivalence of categories

Let $D \in \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$. By definition $\mathbb{D}(\mathbb{M}(D)) = \mathcal{O}_\mathcal{E} \otimes_{\Lambda_\ell(N_0),\ell} (\Lambda_\ell(N_0) \otimes_{\mathcal{O}_\mathcal{E},\iota} D)$, and we have a natural map

$$\mu \otimes (\lambda \otimes d) \mapsto \mu\ell(\lambda)d : \mathcal{O}_\mathcal{E} \otimes_{\Lambda_\ell(N_0),\ell} (\Lambda_\ell(N_0) \otimes_{\mathcal{O}_\mathcal{E},\iota} D) \to D \ .$$

Proposition 8.16. *The natural map $\mathbb{D}(\mathbb{M}(D)) \to D$ is an isomorphism in $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$.*

Proof. The natural map is bijective because $\ell \circ \iota = \mathrm{id} : \mathcal{O}_\mathcal{E} \to \Lambda_\ell(N_0) \to \mathcal{O}_\mathcal{E}$, and L_*-equivariant because the action of $t \in L_*$ satisfies

$$\varphi_t(\mu \otimes (\lambda \otimes d)) = \varphi_{\ell(t)}(\mu) \otimes \varphi_t(\lambda \otimes d) = \varphi_{\ell(t)}(\mu) \otimes (\varphi_t(\lambda) \otimes \varphi_t(d)) \ ,$$

$$\varphi_t(\mu\ell(\lambda)d) = \varphi_{\ell(t)}(\mu(\ell(\lambda))\varphi_t(d) = \varphi_{\ell(t)}(\mu)\ell(\varphi_t(\lambda))\varphi_t(d) \ ,$$

by (56), (57). $\square$

The kernel N_ℓ of $\ell : N_0 \to \mathbb{Z}_p$ being a closed subgroup of N_0 is also a p-adic Lie group, hence contains an open pro-p-subgroup H with the following property ([11] Remark 26.9 and Thm. 27.1):

For any integer $n \geq 1$, the map $h \mapsto h^{p^n}$ is an homeomorphism of H onto an open subgroup $H_n \subseteq H$, and $(H_n)_{n\geq 1}$ is a fundamental system of open neighborhoods of 1 in H.

The groups $s^k N_\ell s^{-k}$ for $k \geq 1$ are open and form a fundamental system of neighborhoods of 1 in N_ℓ. For any integer $n \geq 1$ there exists a positive integer k such that any element in $s^k N_\ell s^{-k}$ is contained in H_n, hence is a p^n-th power of some element in N_ℓ. We denote by k_n the smallest positive integer such that any element in $s^{k_n} N_\ell s^{-k_n}$ is a p^n-th power of some element in N_ℓ.

Lemma 8.17. *For any positive integers n and $k \geq k_n$, we have*

$$\varphi^k(J_\ell(N_0)) \subset \mathcal{M}_\ell(N_0)^{n+1} \ .$$

Proof. For $u \in N_\ell$, and $j \in \mathbb{N}$, the value at u of the p^j-th cyclotomic polynomial $\Phi_{p^j}(u)$ lies in $\mathcal{M}_\ell(N_0)$ and

$$u^{p^n} - 1 = \prod_{j=0}^{n} \Phi_{p^j}(u)$$

lies in $\mathcal{M}_\ell(N_0)^{n+1}$. An element $v \in s^k N_\ell s^{-k}$ is a p^n-th power of some element in N_ℓ hence $v-1$ lies in $\mathcal{M}_\ell(N_0)^{n+1}$. The ideal $J_\ell(N_0)$ of $\Lambda_\ell(N_0)$ is generated by $u-1$ for $u \in N_\ell$ and $\varphi^k(J_\ell(N_0))$ is contained in the ideal generated by $v-1$ for $v \in s^k N_\ell s^{-k}$. As $\mathcal{M}_\ell(N_0)$ is an ideal of $\Lambda_\ell(N_0)$ we deduce that $\varphi^k(J_\ell(N_0)) \subset \mathcal{M}_\ell(N_0)^{n+1}$. $\square$

Lemma 8.18. *i. The functor $\mathbb{D}$ is faithful.*

ii. The functor $\mathbb{M}$ is fully faithful.

Proof. Obviously ii. follows from i. by proposition 8.16. To prove i. let $f : M_1 \to M_2$ be a morphism in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ such that $\mathbb{D}(f) = 0$, i. e., such that $f(M_1) \subseteq J_\ell(N_0)M_2$. Since M_1 is étale we deduce that $f(M_1) \subseteq \bigcap_k \varphi^k(J_\ell(N_0))M_2$ and hence, by lemma 8.17, in $\bigcap_n \mathcal{M}_\ell(N_0)^n M_2$. Since the pseudocompact topology on M_2 is Hausdorff we have $\bigcap_n \mathcal{M}_\ell(N_0)^n M_2 = 0$. It follows that $f = 0$. $\square$

Let $M \in \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$. By definition,

$$\mathbb{M}\mathbb{D}(M) = \Lambda_\ell(N_0) \otimes_{\mathcal{O}_\mathcal{E},\iota} (\mathcal{O}_\mathcal{E} \otimes_{\Lambda_\ell(N_0),\ell} M) = \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\iota\circ\ell} M \ .$$

In the particular case where $L_* = s^{\mathbb{N}}$ is the monoid generated by s, we denote the category $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ (resp. $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$), by $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$ (resp. $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(\varphi)$). The category $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ (resp. $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$) is a subcategory of $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$ (resp. $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(\varphi)$).

Proposition 8.19. *For any $M \in \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$ there is a unique morphism*

$$\Theta_M : M \to \mathbb{M}\mathbb{D}(M) \qquad \textit{in } \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$$

such that the composed map $\mathbb{D}'(\Theta_M) : \mathbb{D}(M) \xrightarrow{\mathbb{D}(\Theta_M)} \mathbb{D}\mathbb{M}\mathbb{D}(M) \cong \mathbb{D}(M)$ is the identity. The morphism Θ_M, in fact, is an isomorphism.

Proof. The uniqueness follows immediately from Lemma 8.18.i. The construction of such an isomorphism Θ_M will be done in three steps.

Step 1: We assume that M is free over $\Lambda_\ell(N_0)$, and we start with an arbitrary finite $\Lambda_\ell(N_0)$-basis $(\epsilon_i)_{i\in I}$ of M. By (52), we have

$$M = (\oplus_{i\in I}\iota(\mathcal{O}_\mathcal{E})\epsilon_i) \oplus (\oplus_{i\in I} J_\ell(N_0)\epsilon_i) \ .$$

The $\Lambda_\ell(N_0)$-linear map from M to $\mathbb{M}\mathbb{D}(M)$ sending ϵ_i to $1 \otimes (1 \otimes \epsilon_i)$ is bijective. If $\oplus_{i\in I}\iota(\mathcal{O}_{\mathcal{E}})\epsilon_i$ is φ-stable, the map is also φ-equivariant and is an isomorphism in the category $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$. We will construct a $\Lambda_\ell(N_0)$-basis $(\eta_i)_{i\in I}$ of M such that $\oplus_{i\in I}\iota(\mathcal{O}_{\mathcal{E}})\eta_i$ is φ-stable.

We have

$$\varphi(\epsilon_i) = \sum_{j\in I}(a_{i,j} + b_{i,j})\epsilon_j \quad \text{where } a_{i,j} \in \iota(\mathcal{O}_{\mathcal{E}}) \ , \ b_{i,j} \in J_\ell(N_0) \ .$$

If the $b_{i,j}$ are not all 0, we will show that there exist elements $x_{i,j} \in J_\ell(N_0)$ such that $(\eta_i)_{i\in I}$ defined by

$$\eta_i := \epsilon_i + \sum_{j\in I} x_{i,j}\epsilon_j \ ,$$

satisfies $\varphi(\eta_i) = \sum_{j\in I} a_{i,j}\eta_j$ for $i \in I$. By the Nakayama lemma ([1] II §3.2 Prop. 5), the set $(\eta_i)_{i\in I}$ is a $\Lambda_\ell(N_0)$-basis of M, and we obtain an isomorphism in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$,

$$\Theta_M \ : \ M \to \mathbb{M}\mathbb{D}(M) \quad , \quad \Theta(\eta_i) = 1 \otimes (1 \otimes \eta_i) \ \text{ for } i \in I \ ,$$

such that $\mathbb{D}'(\Theta_M)$ is the identity morphism of $\mathbb{D}(M)$.

The conditions on the matrix $X := (x_{i,j})_{i,j\in I}$ are :

$$\varphi(\mathrm{Id} + X)(A + B) = A(\mathrm{Id} + X)$$

for the matrices $A := (a_{i,j})_{i,j\in I}$, $B := (b_{i,j})_{i,j\in I}$. The coefficients of A belong to the commutative ring $\iota(\mathcal{O}_{\mathcal{E}})$. The matrix A is invertible because the $\Lambda_\ell(N_0)$-endomorphism f of M defined by

$$f(\epsilon_i) = \varphi(\epsilon_i) \ \text{ for } i \in I$$

is an automorphism of M as φ is étale. We have to solve the equation

$$A^{-1}B + A^{-1}\varphi(X)(A + B) = X \ .$$

For any $k \geq 0$ define

$$U_k = A^{-1}\varphi(A^{-1})\ldots\varphi^{k-1}(A^{-1})\,\varphi^k(A^{-1}B)\,\varphi^{k-1}(A + B)\ldots\varphi(A + B)(A + B) \ .$$

We have

$$A^{-1}\varphi(U_k)(A + B) = U_{k+1} \ .$$

Hence $X := \sum_{k\geq 0} U_k$ is a solution of our equation provided this series converges with respect to the pseudocompact topology of $\Lambda_\ell(N_0)$. The coefficients of $A^{-1}B$ belong to the two-sided ideal $J_\ell(N_0)$ of $\Lambda_\ell(N_0)$. Therefore the coefficients of U_k belong to the two-sided ideal generated by $\varphi^k(J_\ell(N_0))$. Hence the series converges (Lemma 8.17). The coefficients of every term in the series belong to $J_\ell(N_0)$ and $J_\ell(N_0)$ is closed in $\Lambda_\ell(N_0)$, hence $x_{i,j} \in J_\ell(N_0)$ for $i, j \in I$.

Step 2: We show that any module M in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$ is the quotient of another module M_1 in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$ which is free over $\Lambda_\ell(N_0)$.

Let $(m_i)_{i\in I}$ be a minimal finite system of generators of the $\Lambda_\ell(N_0)$-module M. As φ is étale, $(\varphi(m_i))_{i\in I}$ is also a minimal system of generators. We denote by $(e_i)_{i\in I}$ the canonical $\Lambda_\ell(N_0)$-basis of $\oplus_{i\in I}\Lambda_\ell(N_0)$, and we consider the two surjective $\Lambda_\ell(N_0)$-linear maps

$$f, g : \oplus_{i\in I}\Lambda_\ell(N_0) \to M \quad , \quad f(e_i) = m_i \ , \ g(e_i) = \varphi(m_i) \ .$$

In particular, we find elements $m'_i \in M$, for $i \in I$, such that $g(m'_i) = \varphi(m_i)$. By the Nakayama lemma ([1] II §3.2 Prop. 5) the $(m'_i)_{i\in I}$ form another $\Lambda_\ell(N_0)$-basis of $\oplus_{i\in I}\Lambda_\ell(N_0)$. The φ-linear map

$$\oplus_{i\in I}\Lambda_\ell(N_0) \to \oplus_{i\in I}\Lambda_\ell(N_0) \quad , \quad \varphi(\sum_{i\in I}\lambda_i e_i) := \sum_{i\in I}\varphi(\lambda_i)m'_i$$

therefore is étale. With this map, $M_1 := \oplus_{i\in I}\Lambda_\ell(N_0)$ is a module in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$ which is free over $\Lambda_\ell(N_0)$, and the surjective map f is a morphism in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$.

Step 3: As $\Lambda_\ell(N_0)$ is noetherian, we deduce from Step 2 that for any module M in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$ we have an exact sequence

$$M_2 \xrightarrow{f} M_1 \xrightarrow{f'} M \to 0$$

in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(\varphi)$ such that M_1 and M_2 are free over $\Lambda_\ell(N_0)$. We now consider the diagram

$$\begin{tikzcd} \mathbb{M}\mathbb{D}(M_2) \arrow[r, "\mathbb{M}\mathbb{D}(f)"] & \mathbb{M}\mathbb{D}(M_1) \arrow[r, "\mathbb{M}\mathbb{D}(f')"] & \mathbb{M}\mathbb{D}(M) \arrow[r] & 0 . \\ M_2 \arrow[r, "f"] \arrow[u, "\Theta_{M_2}", "\cong"'] & M_1 \arrow[r, "f'"] \arrow[u, "\Theta_{M_1}", "\cong"'] & M \arrow[r] \arrow[u, dashed, "\Theta_M"] & 0 \end{tikzcd}$$

Since the functors $\mathbb{M}$ and $\mathbb{D}$ are right exact both rows of the diagram are exact. By Step 1 the left two vertical maps exist and are isomorphisms. Since

$$\mathbb{D}(\mathbb{M}\mathbb{D}(f) \circ \Theta_{M_2} - \Theta_{M_1} \circ f) = \mathbb{D}(f) \circ \mathbb{D}'(\Theta_{M_2}) - \mathbb{D}'(\Theta_{M_1}) \circ \mathbb{D}(f) = 0$$

it follows from lemma 8.18.i that the left square of the diagram commutes. Hence we obtain an induced isomorphism Θ_M as indicated, which moreover by construction satisfies $\mathbb{D}'(\Theta_M) = \mathrm{id}_{\mathbb{D}(M)}$. □

Theorem 8.20. *The functors*

$$\mathbb{M} : \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*) \to \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*) \quad , \quad \mathbb{D} : \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*) \to \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*) \ ,$$

are quasi-inverse equivalences of categories.

Proof. By proposition 8.16 and lemma 8.18.ii it remains to show that the functor $\mathbb{M}$ is essentially surjective. Let $M \in \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$. We have to find a $D \in \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$ together with an isomorphism $M \cong \mathbb{M}(D)$ in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$. It suffices to show that the morphism Θ_M in proposition 8.19 is L_*-equivariant.

We want to prove that $(\Theta_M \circ \varphi_t - \varphi_t \circ \Theta_M)(m) = 0$ for any $m \in M$ and $t \in L_*$. Since $\mathbb{D}'(\Theta_M) = \mathrm{id}_{\mathbb{D}(M)}$ we certainly have $(\Theta \circ \varphi_t - \varphi_t \circ \Theta)(m) \in J_\ell(N_0)\mathbb{M}\mathbb{D}(M)$ for any $m \in M$ and $t \in L_*$. We choose for any positive integer r a set $J(N_0/N_r) \subseteq N_0$ of representatives for the cosets in N_0/N_r. Writing (12)

$$m = \sum_{u \in J(N_0/N_r)} u\varphi^r(m_{u,s^r}) \ , \ m_{u,s^r} = \psi^r(u^{-1}m)$$

and using that $st = ts$ we see that

$$(\Theta_M \circ \varphi_t - \varphi_t \circ \Theta_M)(m) = \sum_{u \in J(N_0/N_r)} \varphi_t(u)\varphi^r((\Theta_M \circ \varphi_t - \varphi_t \circ \Theta_M)(m_{u,s^r}))$$

lies, for any r, in the $\Lambda_\ell(N_0)$-submodule of $\mathbb{M}\mathbb{D}(M)$ generated by $\varphi^r(J_\ell(N_0))\mathbb{M}\mathbb{D}(M)$. As in the proof of lemma 8.18.ii we obtain $\bigcap_{r>0} \varphi^r(J_\ell(N_0))\mathbb{M}\mathbb{D}(M) = 0$. □

Since the functors $\mathbb{M}$ and $\mathbb{D}$ are right exact they commute with the reduction modulo p^n, for any integer $n \geq 1$.

8.5 Continuity

In this section we assume that L_* contains a subgroup L_1 which is open in L_* and is a topologically finitely generated pro-p-group.

We will show that the L_*-action on any étale L_*-module over $\Lambda_\ell(N_0)$ is automatically continuous. Our proof is highly indirect so that we temporarily will have to make some definitions. But first a few partial results can be established directly.

Let M be a finitely generated $\Lambda_\ell(N_0)$-module.

Definition 8.21. *A lattice in M is a $\Lambda(N_0)$-submodule of M generated by a finite system of generators of the $\Lambda_\ell(N_0)$-module M.*

The lattices of M are of the form $M^0 = \sum_{i=1}^r \Lambda(N_0) m_i$ for a set $(m_i)_{1\leq i\leq r}$ of generators of the $\Lambda_\ell(N_0)$-module M.

We have the three fundamental systems of neighborhoods of 0 in M :

$$(\sum_{i=1}^r O_{n,k} m_i = \mathcal{M}_\ell(N_0)^n M + \mathcal{M}(N_0)^k M^0)_{n,k\in\mathbb{N}} \ , \tag{58}$$

$$(\sum_{i=1}^r B_{n,k} m_i = \mathcal{M}_\ell(N_0)^n M + X^k M^0)_{n,k\in\mathbb{N}} \ , \tag{59}$$

$$(\sum_{i=1}^r C_{n,k} m_i = \mathcal{M}_\ell(N_0)^n M + M^0_k)_{n,k\in\mathbb{N}} \ , \tag{60}$$

where M_k^0 is the lattice $\sum_{i=1}^r \Lambda(N_0) X^k m_i$, and is different from the set $X^k M_0$ when N_0 is not commutative.

If M is an étale L_*-module over $\Lambda_\ell(N_0)$, for any fixed $t \in L_{\ell,+}$ we have a fourth fundamental system of neighborhoods of 0 in M :

$$(\sum_{i=1}^r \varphi_t(O_{n,k})\Lambda(N_0)\varphi_t(m_i))_{n,k\in\mathbb{N}} \ ,$$

given by Lemma 8.1, because $(\varphi_t(m_i)_{1\leq i\leq r}$ is also a system of generators of the $\Lambda_\ell(N_0)$-module M.

Proposition 8.22. *Let L' be a submonoid of $L_{\ell,+}$. Let M be an étale L'-module over $\Lambda_\ell(N_0)$. Then the maps φ_t and ψ_t, for any $t\in L'$, are continuous on M.*

Proof. The ring endomorphisms φ_t of $\Lambda_\ell(N_0)$ are continuous since they preserve $\mathcal{M}(N_0)$ and $\mathcal{M}(N_\ell)$. The continuity of the φ_t on M follows as in part a) of the proof of proposition 7.1. The continuity of the ψ_t follows from

$$\psi_t(\sum_{i=1}^r \varphi_t(O_{n,k})\Lambda(N_0)\varphi_t(m_i)) = \sum_{i=1}^r O_{n,k}\psi_t(\Lambda(N_0)) m_i = \sum_{i=1}^r O_{n,k} m_i \ .$$

□

The same proof shows that, for any $D \in \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$, the maps φ_t and ψ_t, for any $t \in L_*$, are continuous on D.

Proposition 8.23. *The L_*-action $L_* \times D \to D$ on an étale L_*-module D over $\mathcal{O}_\mathcal{E}$ is continuous.*

Proof. Let D be in $\mathfrak{M}^{et}_{\mathcal{O}_{\mathcal{E}},\ell}(L_*)$. Since we already know from Prop. 8.22 that each individual φ_t, for $t \in L_*$, is a continuous map on D and since L_1 is open in L_* it suffices to show that the action $L_1 \times D \to D$ of L_1 on D is continuous. As D is p-adically complete with its weak topology being the projective limit of the weak topologies on the $D/p^n D$ we may further assume that D is killed by a power of p. In this situation the weak topology on D is locally compact. By Ellis' theorem ([8] Thm. 1) we therefore are reduced to showing that the map $L_1 \times D \to D$ is separately continuous. Because of Prop. 8.22 it, in fact, remains to prove that, for any $d \in D$, the map

$$L_1 \longrightarrow D \ , \ g \longmapsto gd$$

is continuous at $1 \in L_1$. This amounts to finding, for any $d \in D$ and any lattice $D_0 \subset D$, an open subgroup $H \subset L_1$ such that $(H-1)d \subset D_0$. We observe that $(X^m D_{++})_{m\in\mathbb{Z}}$ is a fundamental system of L_1-stable open neighbourhoods of zero in D such that $\bigcup_m X^m D_{++} = D$. We now choose an $m \geq 0$ large enough such that $d \in X^{-m} D_{++}$ and $X^m D_{++} \subset D_0$. The L_1 action on D induces an L_1-action on $X^{-m} D_{++}/X^m D_{++}$ which is o-linear hence given by a group homomorphism $L_1 \to \mathrm{Aut}_o(X^{-m} D_{++}/X^m D_{++})$. Since D_{++} is a finitely generated $o[[X]]$-module which is killed by a power of p we see that $X^{-m} D_{++}/X^m D_{++}$ is finite. It follows that the kernel H of the above homomorphism is of finite index in L_1. Our assumption that L_1 is a topologically finitely generated pro-p-group finally implies, by a theorem of Serre ([7] Thm. 1.17), that H is open in L_1. We obtain

$$(H-1)d \subset (H-1)X^{-m} D_{++} \subset X^m D_{++} \subset D_0 \ .$$

$\square$

In the special case of classical (φ, Γ)-modules on $\mathcal{O}_{\mathcal{E}}$ the proposition is stated as Exercise 2.4.6 in [10] (with the indication of a totally different proof).

Proposition 8.24. *Let L' be a submonoid of $L_{\ell,+}$ containing an open subgroup L_2 which is a topologically finitely generated pro-p-group. Then the L'-action $L' \times \Lambda_\ell(N_0) \to \Lambda_\ell(N_0)$ on $\Lambda_\ell(N_0)$ is continuous.*

Proof. Since we know already from Prop. 8.2 and 8.22 that each individual φ_t, for $t \in L'$, is a continuous map on $\Lambda_\ell(N_0)$ and since L_2 is open in L' it suffices to show that the action $L_2 \times \Lambda_\ell(N_0) \to \Lambda_\ell(N_0)$ of L_2 on $\Lambda_\ell(N_0)$ is continuous. The ring $\Lambda_\ell(N_0)$ is $\mathcal{M}_\ell(N_0)$-adically complete with its weak topology being the projective limit of the weak topologies on the $\Lambda_\ell(N_0)/\mathcal{M}_\ell(N_0)^n \Lambda_\ell(N_0)$. It suffices to prove that the induced action of L_2 on $\Lambda' = \Lambda_\ell(N_0)/\mathcal{M}_\ell(N_0)^n$ is continuous. The weak topology on Λ' is locally compact since $(B'_k = (X^k \Lambda(N_0) + \mathcal{M}_\ell(N_0)^n)/\mathcal{M}_\ell(N_0)^n)_{k\in\mathbb{Z}}$ forms a fundamental system of compact neighborhoods of 0. By Ellis' theorem ([8] Thm. 1) we therefore are reduced to showing that the map $L_2 \times \Lambda' \to \Lambda'$ is separately continuous. Because of Prop. 8.22 it, in fact, remains to prove that, for any $x \in \Lambda'$, the map

$$L_2 \longrightarrow \Lambda' \ , \ g \longmapsto gx$$

is continuous at $1 \in L_2$. This amounts to finding, for any $x \in \Lambda'$ and any large $k \geq 1$, an open subgroup $H \subset L_2$ such that $(H-1)x \subset B'_k$. We observe that the B'_k, for $k \in \mathbb{Z}$, are L_2-stable of union Λ'. We now choose an $m \geq k$ large enough such that $x \in B'_{-m}$. The L_2-action on Λ' induces an L_2-action on B'_{-m}/B'_m which is o-linear hence given by a group homomorphism $L_2 \to \mathrm{Aut}_o(B'_{-m}/B'_m)$. Since B'_0 is isomorphic to $o[[X]] \otimes_o \Lambda(N_\ell)/\mathcal{M}(N_\ell)^n$ as an $o[[X]]$-module, and $\Lambda(N_\ell)/\mathcal{M}(N_\ell)^n$ is finite, we see that B'_{-m}/B'_m is finite. It follows that the kernel H of the above homomorphism is of finite index in L_2. Our assumption that L_2 is a topologically finitely generated pro-p-group finally implies, by a theorem of Serre ([7] Thm. 1.17), that H is open in L_2. We obtain

$$(H-1)x \subset (H-1)B'_{-m} \subset B'_m \subset B'_k \ .$$

$\square$

Lemma 8.25. *i. For any $M \in \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ the weak topology on $\mathbb{D}(M)$ is the quotient topology, via the surjection $\ell_M : M \to \mathbb{D}(M)$, of the weak topology on M.*

ii. For any $D \in \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$ the weak topology on $\mathbb{M}(D)$ induces, via the injection $\iota_D : D \to \mathbb{M}(D)$, the weak topology on D.

Proof. i. If we write M as a quotient of a finitely generated free $\Lambda_\ell(N_0)$-module then we obtain an exact commutative diagram of surjective maps of the form

$$\begin{tikzcd} \oplus_{i=1}^n \Lambda_\ell(N_0) \arrow[r, two heads] \arrow[d, two heads, "\oplus_i \ell"'] & M \arrow[d, two heads, "\ell_M"] \\ \oplus_{i=1}^n \mathcal{O}_\mathcal{E} \arrow[r, two heads] & \mathbb{D}(M) \end{tikzcd} .$$

The horizontal maps are continuous and open by the definition of the weak topology. The left vertical map is continuous and open by direct inspection of the open zero neighbourhoods $B_{n,k}$ (see (51)). Hence the right vertical map ℓ_M is continuous and open.

ii. An analogous argument as for i. shows that ι_D is continuous. Moreover ι_D has the continuous left inverse $\ell_{\mathbb{M}(D)}$. Any continuous map with a continuous left inverse is a topological inclusion. $\square$

An étale L_*-module M over $\Lambda_\ell(N_0)$, resp. over $\mathcal{O}_\mathcal{E}$, will be called topologically étale if the L_*-action $L_* \times M \to M$ is continuous. Let $\mathfrak{M}^{et,c}_{\Lambda_\ell(N_0)}(L_*)$ and $\mathfrak{M}^{et,c}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$ denote the corresponding full subcategories of $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ and $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$, respectively. Note that, by construction, all morphisms in $\mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ and in $\mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$ are automatically continuous. Also note that by proposition 8.22 any object in these categories is a complete topologically étale $o[N_0L_*]$-module in our earlier sense.

Proposition 8.26. *The functors $\mathbb{M}$ and $\mathbb{D}$ restrict to quasi-inverse equivalences of categories*

$$\mathbb{M} : \mathfrak{M}^{et,c}_{\mathcal{O}_\mathcal{E},\ell}(L_*) \to \mathfrak{M}^{et,c}_{\Lambda_\ell(N_0)}(L_*) \quad , \quad \mathbb{D} : \mathfrak{M}^{et,c}_{\Lambda_\ell(N_0)}(L_*) \to \mathfrak{M}^{et,c}_{\mathcal{O}_\mathcal{E},\ell}(L_*) \ .$$

Proof. It is immediate from lemma 8.25.i that if L_* acts continuously on $M \in \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$ then it also acts continuously on $\mathbb{D}(M)$.

On the other hand, let $D \in \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$ such that the action of L_* on D is continuous. We choose a lattice D_0 in D with a finite system (d_i) of generators. Given $t \in L_*$ we introduce $D_t := \sum_i \Lambda(N_0^{(2)})t.d_i$ which is a lattice in D since the action of t on D is étale. Also $D_0 + D_t$ is a lattice in D. The $\Lambda_\ell(N_0)$-module $\mathbb{M}(D)$ is generated by $\iota_D(D_0)$ as well as by $\iota_D(D_0 + D_t)$ and both

$$(C_n\iota_D(D_0))_{n\in\mathbb{N}} \quad \text{and} \quad (C_n\iota_D(D_0 + D_t))_{n\in\mathbb{N}}$$

are fundamental systems of neighbourhoods of 0 in $\mathbb{M}(D)$ for the weak topology. To show that the action of L_* on $\mathbb{M}(D)$ is continuous, it suffices to find for any $t \in L_*, \lambda_0 \in \Lambda_\ell(N_0), d_0 \in D_0, n \in \mathbb{N}$ a neighborhood $L_t \subset L_*$ of t and $n' \in \mathbb{N}$ such that

$$L_t.(\lambda_0\iota_D(d_0) + C_{n'}\iota_D(D_0)) \subset t.\lambda_0\iota_D(d_0) + C_n\iota_D(D_0 + D_t) \ . \tag{61}$$

The three maps

$$\begin{aligned} \lambda &\mapsto \lambda\iota_D(d_0) : \Lambda_\ell(N_0) \to \mathbb{M}(D) \\ d &\mapsto \lambda_0\iota_D(d) : D \to \mathbb{M}(D) \\ (\lambda, d) &\mapsto \lambda\iota_D(d) : \Lambda_\ell(N_0 \times D) \to \mathbb{M}(D) \end{aligned}$$

are continuous because ι_D is continuous. The action of L_* on D and on $\Lambda_\ell(N_0)$ is continuous (Prop. 8.24). Altogether this implies that we can find a small L_t such that

$$L_t.\lambda_0\iota_D(d_0) \subset t.\lambda_0\iota_D(d_0) + C_n\iota_D(D_0 + D_t) \ .$$

Since ι_D is L_*-equivariant we have, for any $n' \in \mathbb{N}$,

$$L_t.C_{n'}\iota_D(D_0) = (L_t.C_{n'})\,\iota_D(L_t.D_0) \ .$$

The continuity of the action of L_* on $\Lambda_\ell(N_0)$ shows that $L_t.C_{n'} \subset C_n$ when L_t is small enough and n' is large enough.

For $d \in D_0$ we have $L_t.\Lambda(N_0^{(2)})d \subset \Lambda(N_0^{(2)})(L_t.d)$. The action of L_* on D is continuous hence, for any n', we can choose a small L_t such that $L_t.d \subset t.d + C_{n'}^{(2)}D_0$. We can choose the same L_t for each d_i and we obtain

$$L_t.D_0 \subset \sum_i \Lambda(N_0^{(2)})t.d_i + C_{n'}^{(2)}D_0 \ .$$

Applying ι_D, we obtain

$$\iota_D(L_t.D_0) \subset \iota_D(D_t) + C_{n'}\iota_D(D_0)$$

and then

$$(L_t.C_{n'})\,\iota_D(L_t.D_0) \subset C_n\iota_D(D_t) + C_nC_{n'}\iota_D(D_0) \ .$$

We check that $C_nC_{n'} \subset C_{n,n+n'} \subset C_n$ when $n' \geq n$. Hence when n' is large enough,

$$L_t.(C_{n'}\iota_D(D_0)) \subset C_n\iota_D(D_t + D_0) \ .$$

This ends the proof of (61). □

Proposition 8.27. *We have $\mathfrak{M}^{et,c}_{\mathcal{O}_\mathcal{E},\ell}(L_*) = \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_*)$ and $\mathfrak{M}^{et,c}_{\Lambda_\ell(N_0)}(L_*) = \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_*)$.*

Proof. The first identity was shown in proposition 8.23 and is equivalent to the second identity by theorem 8.20 and proposition 8.26. □

Corollary 8.28. *Any étale L_*-module over $\Lambda_\ell(N_0)$, resp. over $\mathcal{O}_\mathcal{E}$, is a complete topologically étale $o[N_0L_*]$-module in our sense.*

Proof. Use propositions 8.22 and 8.27. □

9 Convergence in L_+-modules on $\Lambda_\ell(N_0)$

In this section, we use the notations of section 8 where we assume that N is a p-adic Lie group. We assume that ℓ and ι are continuous group homomorphisms

$$\ell : P \to P^{(2)} \ , \ \iota : N^{(2)} \to N \ , \ \ell \circ \iota = \mathrm{id} \ ,$$

such that $\ell(L_+) \subset L_+^{(2)}$, $\ell(N) = N^{(2)}$, $(\iota \circ \ell)(N_0) \subset N_0$, and

$$t\iota(y)t^{-1} = \iota(\ell(t)y\ell(t)^{-1}) \quad \text{for } y \in N^{(2)}, t \in L \ . \tag{62}$$

The assumptions of Chapter 8 are naturally satisfied with $L_* = L_+$. Indeed, the compact open subgroup N_0 of N is a compact p-adic Lie group, the group $\ell(N_0)$ is a compact non-trivial subgroup $N_0^{(2)}$ of $N^{(2)} \simeq \mathbb{Q}_p$ hence $N_0^{(2)}$ is isomorphic to $\mathbb{Z}_p$ and is open in $N^{(2)}$, the kernel of $\ell|_{N_0}$ is normalized by $L_{\ell,+}$. Note that L_+ normalizes $\iota(N_0^{(2)})$ since $\ell(L_+)$ normalises $N_0^{(2)}$ and (62).

Let $M \in \mathfrak{M}^{et}_{\Lambda_\ell(N_0)}(L_+)$ and $D \in \mathfrak{M}^{et}_{\mathcal{O}_\mathcal{E},\ell}(L_+)$ be related by the equivalence of categories (Thm. 8.20),

$$M = \Lambda_\ell(N_0) \otimes_{\mathcal{O}_\mathcal{E},\iota} D = \Lambda_\ell(N_0)\iota_D(D) \ .$$

We will exhibit in this chapter a special family $\mathfrak{C}_s$ of compact subsets in M such that $M(\mathfrak{C}_s)$ is a dense o-submodule of M, and such that the P-equivariant sheaf on $\mathcal{C}$ associated to the étale $o[P_+]$-module $M(\mathfrak{C}_s)$ by the theorem 3.23 extends to a G-equivariant sheaf on G/P. We will follow the method explained in subsection 6.5 which reduces the most technical part to the easier case where M is killed by a power of p.

9.1 Bounded sets

Definition 9.1. *A subset A of M is called bounded if for any open neighborhood $\mathcal{B}$ of 0 in M there exists an open neighborhood B of 0 in $\Lambda_\ell(N_0)$ such that*

$$BA \subset \mathcal{B} \ .$$

Compare with [12] Def. 8.5. The properties satisfied by bounded subsets of M can be proved directly or deduced from the properties of bounded subsets of $\Lambda_\ell(N_0)$ ([15] §12). Using the fundamental system (59) of neighborhoods of 0, the set A is bounded if and only if for any large n there exists $n' > n$ such that

$$(\mathcal{M}_\ell(N_0)^{n'} + X^{n'}\Lambda(N_0))A \subset \mathcal{M}_\ell(N_0)^n M + X^n M^0 \ ,$$

equivalently $X^{n'-n}A \subset \mathcal{M}_\ell(N_0)^n M + M^0$. We obtain (compare with [12] Lemma 8.8):

Lemma 9.2. *A subset A of M is bounded if and only if for any large positive n there exists a positive integer n' such that*

$$A \subset \mathcal{M}_\ell(N_0)^n M + X^{-n'} M^0 \ .$$

The following properties of bounded subsets will be used in the construction of a special family $\mathfrak{C}_s$ in the next subsection.

- Let $f : \oplus_{i=1}^r \Lambda_\ell(N_0) \to M$ be a surjective homomorphism of $\Lambda_\ell(N_0)$-modules. The image by f of a bounded subset of $\oplus_{i=1}^r \Lambda_\ell(N_0)$ is a bounded subset of M. For $1 \leq i \leq r$, the i-th projections $A_i \subset \Lambda_\ell(N_0)$ of a subset A of $\oplus_{i=1}^r \Lambda_\ell(N_0)$ are all bounded if and only if A is bounded.
- A compact subset is bounded.
- The $\Lambda(N_0)$-module generated by a bounded subset is bounded.
- The closure of a bounded subset is bounded.
- Given a compact subset C in $\Lambda_\ell(N_0)$ and a bounded subset A of M, the subset CA of M is bounded.
- The image of a bounded subset by $f \in \mathrm{End}_o^{cont}(M)$ is bounded. The image by ℓ_M of a bounded subset in M is bounded in D.
- A subset A of D is bounded if and only if the image A_n of A in D/p^nD is bounded for all large n.
- When D is killed by a power of p, a subset A of D is bounded if and only if A is contained in a lattice, i.e. if A is contained in a compact subset (by the properties of lattices given in Section 7.3).

Lemma 9.3. *The image by ι_D of a bounded subset in D is bounded in M.*

Proof. Let $A \subset D$ be a bounded subset and let D^0 be a fixed lattice in D. For all $n \in \mathbb{N}$ there exists $n' \in \mathbb{N}$ such that $A \subset p^n D + (X^{(2)})^{-n'} D^0$ by Lemma 9.2. Applying ι_D we obtain

$$\iota_D(A) \subset p^n \iota_D(D) + X^{-n'} \iota_D(D^0) \subset \mathcal{M}_\ell(N_0)^n M + X^{-n'} M^0$$

where $M^0 = \Lambda(N_0)\iota_D(D^0)$ is a lattice in M. By the same lemma, this means that $\iota_D(A)$ is bounded in M. $\square$

9.2 The module M_s^{bd}

Definition 9.4. *M_s^{bd} is the set of $m \in M$ such that the set of $\ell_M(\psi^k(u^{-1}m))$ for $k \in \mathbb{N}, u \in N_0$ is bounded in D.*

The definition of M_s^{bd} depends on s because ψ is the canonical left inverse of the action φ of s on M. We recognize $m_{u,s^k} = \psi^k(u^{-1}m)$ appearing in the expansion (12).

Proposition 9.5. *M_s^{bd} is an étale $o[P_+]$-submodule of M.*

Proof. a) We check first that M_s^{bd} is P_+-stable. As M_s^{bd} is N_0-stable and $P_+ = N_0 L_+$, it suffices to show that $tm = \varphi_t(m) \in M_s^{bd}$ when $t \in L_+$ and $m \in M_s^{bd}$. Using the expansion (12) of m and $st = ts$, for $k \in \mathbb{N}$ and $n_0 \in N_0$, we write $\psi^k(n_0^{-1}tm)$ as the sum over $u \in J(N_0/N_k)$ of

$$\psi^k(n_0^{-1}tu\varphi^k(m_{u,s^k})) = \psi^k(n_0^{-1}tut^{-1}\varphi^k(\varphi_t(m_{u,s^k}))) = \psi^k(n_0^{-1}tut^{-1})\varphi_t(m_{u,s^k}) \ ,$$

and $\ell_M(\psi^k(n_0^{-1}\varphi_t(m)))$ as the sum over $u \in J(N_0/N_k)$ of

$$\ell_M(\psi^k(n_0^{-1}tut^{-1})\varphi_t(m_{u,s^k})) = v_{k,n_0}\ell_M(\varphi_t(m_{u,s^k})) = v_{k,n_0}\varphi_t(\ell_M(m_{u,s^k})) \ ,$$

where $v_{k,n_0} := \ell(\psi^k(n_0^{-1}tut^{-1}))$ belongs to $N_0^{(2)}$ or is 0. As $m \in M_s^{bd}$, the set of $\ell_M(m_{u,s^k})$ for $k \in \mathbb{N}$ and $u \in N_0$ is bounded in D. Its image by the continuous map φ_t is bounded and generates a bounded $o[N_0^{(2)}]$-submodule of D. Hence $\varphi_t(m) \in M_s^{bd}$.

b) The $o[P_+]$-module M_s^{bd} is ψ-stable (hence M_s^{bd} is étale by Corollary 3.30) because we have, for $m \in M_s^{bd}, u \in N_0, k \in \mathbb{N}$,

$$\psi^k(u^{-1}\psi(m)) = \psi^{k+1}(\varphi(u^{-1})m) \ . \tag{63}$$

$\square$

The goal of this section is to show that the P-equivariant sheaf on $\mathcal{C}$ associated to the étale $o[P_+]$-module M_s^{bd} extends to a G-equivariant sheaf on G/P. We will follow the method explained in subsection 6.5.

Let $p_n : M \to M/p^n M$ be the reduction modulo p^n for a positive integer n. Recall that M is p-adically complete.

Lemma 9.6. *The o-submodule $M_s^{bd} \subset M$ is closed for the p-adic topology, in particular*

$$M_s^{bd} = \varprojlim_n (M_s^{bd}/p^n M_s^{bd}) \ .$$

Moreover M_s^{bd} is the set of $m \in M$ such that $p_n(m)$ belongs to $(M/p^n M)_s^{bd}$ for all $n \in \mathbb{N}$, and we have

$$M_s^{bd} = \varprojlim_n (M/p^n M)_s^{bd} \ .$$

Proof. a) Let m be an element in the closure of M_s^{bd} in M for the p-adic topology. For any $r \in \mathbb{N}$, we choose $m'_r \in M_s^{bd}$ with $m - m'_r \in p^r M$. For each r, we choose $r' \geq 1$ such that $\ell_M(\psi^k(u^{-1}m'_r)) \in p^r D + X^{-r'} D^0$ for all $k \in \mathbb{N}, u \in N_0$, applying Lemma 9.2. We have

$$\ell_M(\psi^k(u^{-1}m)) \in \ell_M(\psi^k(u^{-1}m'_r) + p^r M) = \ell_M(\psi^k(u^{-1}m'_r)) + p^r D \subset p^r D + X^{-r'} D^0 \ .$$

By the same lemma, $m \in M_s^{bd}$. This proves that M_s^{bd} is closed in M hence p-adically complete.

b) The reduction modulo p^n commutes with ℓ_M, ψ, and the action of N_0. The following properties are equivalent :

$m \in M_s^{bd}$,

$\{\ell_M(\psi^k(u^{-1}m))$ for $k \in \mathbb{N}, u \in N_0\} \subset D$ is bounded,

$\{\ell_{M/p^nM}(\psi^k(u^{-1}p_n(m)))$ for $k \in \mathbb{N}, u \in N_0\} \subset D/p^nD$ is bounded for all positive integers n,

$p_n(m) \in (M/p^nM)_s^{bd}$ for all positive integers n.

We deduce that $m \mapsto (p_n(m))_n : M_s^{bd} \to \varprojlim_n (M/p^nM)_s^{bd}$ is an isomorphism. □

Proposition 9.7. *$D = D_s^{bd}$ and M_s^{bd} contains $\iota_D(D)$.*

Proof. i) We show that $D = D_s^{bd}$. By Lemma 9.6, we can suppose that D is killed by a power of p. Let $d \in D$. By Cor. 7.4, for $n \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $\psi^k(v^{-1}d) \in D^\sharp$ for $k \geq k_0, v \in N_0^{(2)}$. As $D^\sharp \subset D$ is bounded, and as the set of $\psi^k(v^{-1}d)$ for all $0 \leq k < k_0, v \in N_0^{(2)}$, is also bounded because the set of $v^{-1}d$ for $v \in N_0^{(2)}$ is bounded and ψ^k is continuous, we deduce that $d \in D_s^{bd}$.

ii) We show that M_s^{bd} contains $\iota_D(D)$ by showing

$$\{\ell_M(\psi^k(u^{-1}\iota_D(d))) \text{ for } k \in \mathbb{N}, u \in N_0\} = \{\psi^k(v^{-1}d) \text{ for } k \in \mathbb{N}, v \in N_0^{(2)}\}$$

when $d \in D$ (the right hand side is bounded in D by i)). We write an element of N_0 as $\iota(v)u$ for u in N_ℓ and $v \in N_0^{(2)}$. By Lemma 8.15,

$$\psi^k(u^{-1}\iota(v)^{-1}\iota_D(d)) = \psi^k(u^{-1}\iota_D(v^{-1}d)) = s^{-k}u^{-1}s^k\psi^k(\iota_D(v^{-1}d))$$

when $u \in s^k N_\ell s^{-k}$ and is 0 when u is not in $s^k N_\ell s^{-k}$. When $u \in s^k N_\ell s^{-k}$ we have $\ell_M(s^{-k}u^{-1}s^k\psi^k(\iota_D(v^{-1}d))) = \psi^k(v^{-1}d)$ as ι_D is ψ-equivariant. □

Proposition 9.8. *M_s^{bd} is dense in M*

Proof. $M_s^{bd} \subset M$ is an $o[N_0]$-submodule, which by Proposition 9.7 contains $\iota_D(D)$. The $o[N_0]$-submodule of M generated by $\iota_D(D)$ is dense by Lemma 8.6. □

We summarize: we proved that $M_s^{bd} \subset M$ is a dense $o[N_0]$-submodule, stable by L_+, and the action of L_+ on M_s^{bd} is étale.

Remark 9.9. It follows from Lemma 9.6 and the subsequent proposition 9.10 that M_s^{bd} is a $\Lambda(N_0)$-submodule of M.

9.3 The special family $\mathfrak{C}_s$ when M is killed by a power of p

We suppose that M is killed by a power of p.

Proposition 9.10. *1. For any lattice D_0 in D, the o-submodule*

$$M_s^{bd}(D_0) := \{m \in M \mid \ell_M(\psi^k(u^{-1}m)) \in D_0 \text{ for all } u \in N_0 \text{ and } k \in \mathbb{N}\}.$$

of M is compact, and is a ψ-stable $\Lambda(N_0)$-submodule.

2. The family $\mathfrak{C}_s$ of compact subsets of M contained in $M_s^{bd}(D_0)$ for some lattice D_0 of D, is special (Def. 6.1), satisfies $\mathfrak{C}(5)$ (Prop. 6.8) and $\mathfrak{C}(6)$ (Prop. 6.9), and $M(\mathfrak{C}_s) = M_s^{bd}$ is a $\Lambda(N_0)$-submodule of M.

Proof. 1. a) As ℓ and ψ are continuous (Proposition 8.22) and $D_0 \subset D$ is closed, it follows that $M_s^{bd}(D_0)$ is an intersection of closed subsets in M, hence $M_s^{bd}(D_0)$ is closed in M. As $M_s^{bd}(D_0)$ is an $o[N_0]$-submodule of M and $o[N_0]$ is dense in $\Lambda(N_0)$ we deduce that $M_s^{bd}(D_0)$ is a $\Lambda(N_0)$-submodule. It is ψ-stable by (63). The weak topology on M is the projective limit of the weak topologies on $M/\mathcal{M}_\ell(N_0)^n M$, and we have ([2] I.29 Corollary)

$$M_s^{bd}(D_0) = \varprojlim_{n \geq 1} (M_s^{bd}(D_0) + \mathcal{M}_\ell(N_0)^n M)/\mathcal{M}_\ell(N_0)^n M \ .$$

Therefore it suffices to show that

$$(M_s^{bd}(D_0) + \mathcal{M}_\ell(N_0)^n M)/\mathcal{M}_\ell(N_0)^n M$$

is compact for each large n. We will show the stronger property that it is a finitely generated $\Lambda(N_0)$-module.

b) We prove first that $M_s^{bd}(D_0)$ is the intersection of the $\Lambda(N_0)$-modules generated by the image by φ^k of the inverse image $\ell_M^{-1}(D_0)$ of D_0 in M, for $k \in \mathbb{N}$,

$$M_s^{bd}(D_0) = \bigcap_{k \in \mathbb{N}} \Lambda(N_0)\varphi^k(\ell_M^{-1}(D_0)) \ . \tag{64}$$

The inclusion from left to right follows from the expansion (12), as $m \in M_s^{bd}(D_0)$ is equivalent to $m_{u,s^k} = \psi^k(u^{-1}m) \in \ell_M^{-1}(D_0)$ for all $u \in N_0$ and $k \in \mathbb{N}$. The inclusion from right to left follows from

$$\ell_M \psi^k u^{-1}(\Lambda(N_0)\varphi^k(\ell_M^{-1}(D_0))) = D_0 \ .$$

c) We pick a lattice M_0 of M such that $\ell_M^{-1}(D_0) = M_0 + J_\ell(N_0)M$, as $J_\ell(N_0)M$ is the kernel of ℓ_M. By Lemma 8.17 we can choose for each $n \in \mathbb{N}$ a large integer r such that $\varphi^r(J_\ell(N_0)M) \subseteq \mathcal{M}_\ell(N_0)^n M$. Therefore we have

$$M_s^{bd}(D_0) \subseteq \Lambda(N_0)\varphi^r(M_0 + J_\ell(N_0)M) \subseteq \Lambda(N_0)\varphi^r(M_0) + \mathcal{M}_\ell(N_0)^n M \ .$$

We deduce

$$(M_s^{bd}(D_0) + \mathcal{M}_\ell(N_0)^n M)/\mathcal{M}_\ell(N_0)^n M \subseteq (\Lambda(N_0)\varphi^r(M_0) + \mathcal{M}_\ell(N_0)^n M)/\mathcal{M}_\ell(N_0)^n M \ .$$

The right term is a finitely generated $\Lambda(N_0)$-module hence the left term is finitely generated as a $\Lambda(N_0)$-module since $\Lambda(N_0)$ is noetherian.

2. The family is stable by finite union because a finite sum of lattices is a lattice. If $C \in \mathfrak{C}_s$ then $N_0 C \in \mathfrak{C}_s$ because $M_s^{bd}(D_0)$ is a $\Lambda(N_0)$-module. We have

$$M(\mathfrak{C}_s) = \cup_{D_0} M_s^{bd}(D_0) = M_s^{bd} \ ,$$

when D_0 runs over the lattices of D, the last follows from the fact that a bounded subset of D is contained in a lattice (this is the only part in the proof where the assumption that M is killed by a power of p is used). Apply Prop. 9.5.

Property $\mathfrak{C}(5)$ is immediate because $M_s^{bd}(D_0)$ is ψ-stable. Property $\mathfrak{C}(6)$ follows from $\varphi(M_s^{bd}(D_0)) \subset M_s^{bd}(D_s)$ where D_s is the lattice of D generated by $\varphi(D_0)$ (this uses the part a) of the proof of Prop. 9.5). $\square$

Consider the lattice $M^{++} = \Lambda(N_0)i_D(D^{++})$ of M. Since $\Lambda(N_0)$ and D^{++} are φ-stable and since φ and ι_D commute, M^{++} is φ-stable and $\ell_M(M^{++}) = D^{++}$. Hence for a subset $S \subset M$ we have

$$S \subset X^r M^{++} + J_\ell(N_0)M \Leftrightarrow \ell_M(S) \subset (X^{(2)})^r D^{++} \ . \tag{65}$$

Proposition 9.11. *Let $r \in \mathbf{N}$, $C_+ \subset L_+$ a compact subset and let D_0 be a lattice in D. There is a compact open subgroup $P_1 \subset P_0$ and $k_0 \geq 0$ such that for all $k \geq k_0$*

$$s^k(1 - P_1)C_+ M_s^{bd}(D_0) \subset X^r M^{++} + \mathcal{M}_\ell(N_0)^r M \ .$$

Proof. Denote for simplicity $S = M_s^{bd}(D_0)$. By definition $\ell_M(S) \subset D_0$. Since $P_+^{(2)}$ acts continuously on D, $\ell(C_+)D_0$ is compact and $(X^{(2)})^r D^{++}$ is open, there is a compact open subgroup $P_1^{(2)} \subset P_+^{(2)}$ such that $(1 - P_1^{(2)})\ell(C_+)D_0 \subset (X^{(2)})^r D^{++}$. We may choose a compact open subgroup P_1 of P_0 such that $\ell(P_1) \subset P_1^{(2)}$, hence $\ell_M((1 - P_1)C_+S) \subset (X^{(2)})^r D^{++}$. Relation (65) yields

$$(1 - P_1)C_+S \subset J_\ell(N_0)M + X^r M^{++} \ .$$

Choosing k_0 such that $\varphi^k(J_\ell(N_0)) \subset \mathcal{M}_\ell(N_0)^r$ for $k \geq k_0$ (as we may by Lemma 8.17), the result follows from the φ-stability of $X^r M^{++}$(which follows from the fact that $\varphi(X^r) \in X^r\Lambda(N_0)$ and $\varphi(M^{++}) \subset M^{++}$). $\square$

Corollary 9.12. *Property $\mathfrak{T}(1)$ in Prop. 6.8 is satisfied.*

Proof. Let $C, C_+, \mathcal{M}$ as in Prop. 6.8 and choose r such that $\mathcal{M}_\ell(N_0)^r M + X^r M^{++} \subset \mathcal{M}$. Choose a lattice D_0 such that $C \subset M_s^{bd}(D_0)$. As $M_s^{bd}(D_0)$ is ψ-stable (Prop. 9.10), we can choose the subgroup P_1 and $k(C, \mathcal{M}, C_+) = k_0$ given by Proposition 9.11. $\square$

Recall that we defined (42) operators $s_g^{(k)} = \mathcal{H}_g^{(k+1)} - \mathcal{H}_g^{(k)}$ on $M_s^{bd} = M(\mathfrak{C}_s)$. From now on we fix a lattice D_0 in D and $g \in N_0\overline{P}N_0$. We denote $S = M_s^{bd}(D_0)$.

Corollary 9.13. *There is $k_g \geq 0$ such that for all $x \in \mathbf{N}$ and $k \geq x + k_g$*

$$\psi^x \circ N_0 \circ s_g^{(k)}(S) \subset \ell_M^{-1}(D^{++}) \ .$$

Proof. Let $r = 1$, $C_+ = \Lambda_g s$ and choose P_1 and k_0 as in Proposition 9.11, so that $s^k(1-P_1)C_+S \subset \ell_M^{-1}(D^{++})$ for $k \geq k_0$. Since S is $\Lambda(N_0)[\psi]$-stable and $\ell_M^{-1}(D^{++})$ is $o[N_0]$-stable, relation (43) and the inequality $k_g^{(2)}(P_1) \geq k_g^{(1)}$ yield $s_g^{(k)}(S) \subset N_0\varphi^x\ell_M^{-1}(D^{++})$ for $k \geq x+k_0+k_g^{(2)}(P_1)$. Applying $\psi^x \circ N_0$ yields the desired result, with $k_g = k_0+k_g^{(2)}(P_1)$. $\square$

Lemma 9.14. *There is a lattice D_1 in D such that for all $u \in U_g$, $k \geq k_g \geq k_g^{(1)}$ and $x \geq k - k_g$*

$$\psi^x \circ N_0 \circ \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k})(S) \subset \ell_M^{-1}(D_1) \ . \tag{66}$$

Proof. Let $C'_+ = s^{k_g}t(g, U_g)$. This is a compact subset of L_+, since $k_g \geq k_g^{(1)}$. Since S is $\Lambda(N_0)[\psi]$-stable, we have $\mathrm{Res}(1_{uN_k})(S) \subset u \circ \varphi^k(S)$, hence

$$\alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k})(S) \subset N_0 \circ t(g, u) \circ \varphi^k(S) \subset N_0\varphi^{k-k_g}(C'_+S) \ .$$

Hence the left hand-side of (66) is contained in $\psi^{x-k+k_g}(\Lambda(N_0)C'_+S)$, which is a subset of $\Lambda(N_0)C'_+\ell_M^{-1}(D_0)$, because $S \subset \Lambda(N_0)\varphi^{x-k+k_g}(\ell_M^{-1}(D_0))$ and $C'_+\Lambda(N_0) \subset \Lambda(N_0)C'_+$. Thus

$$\ell_M(\psi^x \circ N_0 \circ \alpha(g, x_u) \circ \mathrm{Res}(1_{uN_k})(S)) \subset \Lambda(N_0^{(2)})\ell(C'_+)(D_0)$$

and the last subset of D is compact, hence contained in some lattice D_1. $\square$

Corollary 9.15. *For all $k \geq k_g$ we have $\mathcal{H}_g^{(k)}(S) \subset M_s^{bd}(D_1+D^{++})$. Moreover, $\mathcal{H}_g(S) \subset M_s^{bd}(D_1 + D^{++})$.*

Proof. The second assertion follows from the first by letting $k \to \infty$, since $M_s^{bd}(D_1+D^{++})$ is closed in M. For the first assertion, we need to prove that $\psi^x(N_0\mathcal{H}_g^{(k)}(S)) \subset \ell_M^{-1}(D_1 + D^{++})$ for all $x \geq 0$ and $k \geq k_g$. Fix $x \geq 0$. If $k \leq k_g + x$, simply add all relations (66) for $u \in J(U_g/N_k)$. If $k > k_g + x$, the equation $\mathcal{H}_g^{(k)} = \mathcal{H}_g^{(x+k_g)} + \sum_{j=x+k_g}^{k-1} s_g^{(j)}$ and corollary 9.13 show that

$$\psi^x(N_0\mathcal{H}_g^{(k)}(S)) \subset \psi^x(N_0\mathcal{H}_g^{(x+k_g)}(S)) + \ell_M^{-1}(D^{++}) \ .$$

But we have already seen that $\psi^x(N_0\mathcal{H}_g^{(x+k_g)}(S)) \subset \ell_M^{-1}(D_1 + D^{++})$. $\square$

Proposition 9.16. *All the assumptions of Prop. 6.9 are satisfied.*

Proof. Property $\mathfrak{T}(1)$ was checked in corollary 9.12. Property $\mathfrak{T}(2)$ and the fact that $\mathcal{H}_g$ preserves M_s^{bd} (for $g \in N_0\overline{P}N_0$) follow from Corollary 9.15 and the fact that any $m \in M_s^{bd}$ is in $S = M_s^{bd}(D_0)$ for some lattice D_0 in D. $\square$

9.4 Functoriality and dependence on s

Let $Z(L)_{\dagger\dagger} \subset Z(L)$ be the subset of elements s such that $L = L_- s^{\mathbb{N}}$ and $(s^k N_0 s^{-k})_{k\in\mathbb{Z}}$ and $(s^{-k}w_0N_0w_0^{-1}s^k)_{k\in\mathbb{Z}}$ are decreasing sequences of trivial intersection and union N and $w_0Nw_0^{-1}$, respectively (see section 6).

Let M be a topologically etale L_+-module over $\Lambda_\ell(N_0)$ and let $D := \mathbb{D}(M)$. We have $D/p^nD = \mathbb{D}(M/p^nM)$ for $n \geq 1$. By Lemma 8.7, M satisfies the properties a,b,c,d of subsection 6.5 and is complete (the same is true for M/p^nM). The image $D_{0,n}$ in D/p^nD of any lattice $D_{0,n+1}$ in $D/p^{n+1}D$ is a lattice and the maps ℓ and ψ commute with the reduction modulo p^n, hence $(M/p^{n+1}M)_s^{bd}(D_{0,n+1})$ maps into $(M/p^nM)_s^{bd}(D_{0,n})$. Therefore the special family $\mathfrak{C}_{s,n+1}$ in $M/p^{n+1}M$ maps to the special family $\mathfrak{C}_{s,n}$ in M/p^nM. As in Lemma 6.14 we define the special family $\mathfrak{C}_s$ in M to consist of all compact subsets $C \subset M$ such that $p_n(C) \in \mathfrak{C}_{s,n}$ for all $n \geq 1$. By Prop. 9.10 and Lemma 9.6 we have

$$M(\mathfrak{C}_s) = M_s^{bd} \ .$$

Theorem 9.17. *Let $s \in Z(L)_{\dagger\dagger}$ and $M \in \mathcal{M}^{et}_{\Lambda_\ell(N_0)}(L_+)$.*

(i) The $(s, \mathrm{res}, \mathfrak{C}_s)$-integrals $\mathcal{H}_{g,s}$ of the functions $\alpha_{g,0}|_{M_s^{bd}}$ for $g \in N_0\overline{P}N_0$ exist, lie in $\mathrm{End}_o(M_s^{bd})$, and satisfy the relations H1, H2, H3 of Prop. 5.14.

(ii) The map $M \mapsto (M_s^{bd}, (\mathcal{H}_{g,s})_{g\in N_0\overline{P}N_0})$ is functorial.

Proof. (i) By Prop. 9.16 the assumptions of Prop. 6.16 are satisfied.

(ii) Let $f : M \to M'$ be a morphism in $\mathcal{M}^{et}_{\Lambda_\ell(N_0)}(L_+)$. For $m \in M$ we denote $E_s(m) = \{\ell_M(\psi_s^k u^{-1}m) \text{ for } u \in N_0, k \in \mathbb{N}\}$. We have

$$\mathbb{D}(f)(E_s(m)) = E_s(f(m)) \text{ when } m \in M \ , \tag{67}$$

because the maps $\ell_M : M \to D$ and $\ell_{M'} : M' \to D'$ sending x to $1 \otimes x$ for $x \in M$ or $x \in M'$ satisfy $\ell_{M'} \circ f = \mathbb{D}(f) \circ \ell_M$, and f is P^--equivariant by Lemma 3.7. Any morphism between finitely generated modules on $\mathcal{O}_{\mathcal{E}}$ is continuous for the weak topology (cf. [12] Lemma 8.22). The image of a bounded subset by a continuous map is bounded. We deduce from (67) that $E_s(m)$ bounded implies $E_s(f(m))$ bounded, equivalently $m \in M_s^{bd}$ implies $f(m) \in {M'}_s^{bd}$. For $m \in M_s^{bd}$ we have $f(\mathcal{H}_{g,s}(m)) = \mathcal{H}_{g,s}(f(m))$ where

$$\mathcal{H}_{g,s}(.) = \lim_{k\to\infty} \sum_{u\in J(N_0/s^kN_0s^{-k})} n(g,u)\varphi_{t(g,u)s^k}\psi_s^k u^{-1}(.) \ ,$$

because f is P_+ and P_--equivariant by Lemma 3.7. $\square$

We investigate now the dependence on $s \in Z(L)_{\dagger\dagger}$ of the dense subset $M_s^{bd} \subseteq M$ and of the $(s, \mathrm{res}, \mathfrak{C}_s)$-integrals $\mathcal{H}_{g,s}$.

Lemma 9.18. *$Z(L)_{\dagger\dagger}$ is stable by product.*

Proof. Let $s, s' \in Z(L)_{\dagger\dagger}$. Clearly $L_- s'^n = L_- s^{-n} s^n s'^n \subset L_-(ss')^n$ because L_- is a monoid and $s^{-1} \in Z(L)_- = Z(L) \cap L_-$. Therefore $L = L_-(ss')^{\mathbb{N}}$. The sequence $((ss')^k N_0 (ss')^{-k})_{k \in \mathbb{Z}}$ is decreasing because

$$s'^{k+1} s^{k+1} N_0 s^{-k-1} s'^{-k-1} \subset s'^k s^{k+1} N_0 s^{-k-1} s'^{-k} \subset s'^k s^k N_0 s^{-k} s'^{-k} \ .$$

The intersection is trivial and the union is N because $s'^k s^k N_0 s^{-k} s'^{-k} \subset s^k N_0 s^{-k}$ when $k \in \mathbb{N}$ and $s'^k s^k N_0 s^{-k} s'^{-k} \supset s^k N_0 s^{-k}$ when $-k \in \mathbb{N}$. One makes the same argument with $w_0 N_0 w_0^{-1}$. $\square$

Lemma 9.19. *(i) The action of $t_0 \in \ell^{-1}(L_0^{(2)}) \cap L_+$ on D is invertible.*

(ii) There exists a treillis D_0 in D which is stable by $\ell^{-1}(L_0^{(2)}) \cap L_+$.

Proof. (i) is true because the action of t_0 on D is étale and $N_0^{(2)} = \ell(t_0) N_0^{(2)} \ell(t_0)^{-1}$.

(ii) Let $s \in Z(L)_{\dagger\dagger}$ and let ψ_s be the canonical inverse of the étale action φ_s of s on D. We show that the minimal ψ_s-stable treillis $D^\natural$ of D (Prop. 7.2(iii)) is stable by $\ell^{-1}(L_0^{(2)}) \cap L_+$.

For $t_0 \in \ell^{-1}(L_0^{(2)}) \cap L_+$ we claim that $\varphi_{t_0}(D^\natural)$ is also a ψ_s-stable treillis in D. We have $\psi_s \psi_{t_0} = \psi_{t_0} \psi_s$ as $t_0 \in Z(L)$. Multiplying by φ_{t_0} on both sides, one gets $\varphi_{t_0} \psi_s \psi_{t_0} \varphi_{t_0} = \varphi_{t_0} \psi_{t_0} \psi_s \varphi_{t_0}$. Since ψ_{t_0} is the two-sided inverse of φ_{t_0} by (i) we get that φ_{t_0} and ψ_s commute. Hence $\varphi_{t_0}(D^\natural)$ is a compact o-module which is ψ_s-stable. It is a $\Lambda(N_0^{(2)})$-module because any $\lambda \in \Lambda(N_0^{(2)})$ is of the form $\lambda = \varphi_{\ell(t_0)}(\mu)$ for some $\mu \in \Lambda(N_0^{(2)})$ and $\lambda \varphi_{t_0}(d) = \varphi_{t_0}(\mu d)$ for all $d \in D$. As $D^\natural$ contains a lattice and φ_{t_0} is étale, we deduce that $\varphi_{t_0}(D^\natural)$ contains a lattice and therefore is a treillis. By the minimality of $D^\natural$ we must have

$$D^\natural \subset \varphi_{t_0}(D^\natural) \ .$$

Similarly one checks that $\psi_{t_0}(D^\natural)$ is a treillis. It is ψ_s-stable because ψ_s and ψ_{t_0} commute. Hence

$$D^\natural \subset \psi_{t_0}(D^\natural) \ .$$

Applying φ_{t_0} which is the two-sided inverse of ψ_{t_0} we obtain $\varphi_{t_0}(D^\natural) \subset D^\natural$ hence $D^\natural = \varphi_{t_0}(D^\natural)$. $\square$

We denote by $Z(L)_\dagger \subset Z(L)$ the monoid of $z \in Z(L)_+ = Z(L) \cap L_+$ such that $z^{-1} w_0 N_0 w_0^{-1} z \subset w_0 N_0 w_0^{-1}$. We have $Z(L)_{\dagger\dagger} Z(L)_\dagger \subset Z(L)_{\dagger\dagger}$.

Note that $L_0^{(2)}$ contains the center of $GL(2, \mathbb{Q}_p)$ and that $Z(L^{(2)})_\dagger = L_+^{(2)}$.

For $m \in M, t \in L_+, u \in U$, and a system of representatives $J(N_0/tN_0t^{-1}) \subset N_0$ for the cosets in N_0/tN_0t^{-1} we have (12)

$$m = \sum_{u \in J(N_0/tN_0t^{-1})} u \mu_{t,u} \quad , \quad \mu_{t,u} := \varphi_t \psi_t(u^{-1} m) \ . \tag{68}$$

For $g \in N_0 \overline{P} N_0$ and $s \in Z(L)_{\dagger\dagger}$, we have the smallest positive integer $k_{g,s}^{(0)}$ as in (27). For $k \geq k_{g,s}^{(0)}$, we have $\mathcal{H}_{g,s,J(N_0/N_k)} \in \mathrm{End}_o^{cont}(M)$ where (compare with (28))

$$\mathcal{H}_{g,s,J(N_0/N_k)}(m) = \sum_{u \in J(U_g/N_k)} n(g,u) t(g,u) \mu_{s^k,u} \ . \tag{69}$$

When $m \in M_s^{bd}$, the integral $\mathcal{H}_{g,s}(m)$ is the limit of $\mathcal{H}_{g,s,J(N_0/N_k)}(m)$ by Theorem 9.17 and (29).

Proposition 9.20. *Let $s \in Z(L)_{\dagger\dagger}$, $t_0 \in \ell^{-1}(L_0^{(2)}) \cap Z(L)_\dagger$ and r a positive integer.*
(i) We have $M_{st_0}^{bd} \subseteq M_s^{bd} = M_{s^r}^{bd}$.
(ii) For $g \in N_0\overline{P}N_0$ we have $\mathcal{H}_{g,s} = \mathcal{H}_{g,st_0}$ on $M_{st_0}^{bd}$ and $\mathcal{H}_{g,s} = \mathcal{H}_{g,s^r}$ on M_s^{bd}.

Proof. a) Note that st_0 and s^r in proposition belong also to $Z(L)_{\dagger\dagger}$.

For a treillis D_0 in D which is stable by $\ell^{-1}(L_0^{(2)}) \cap L_+$ (Lemma 9.19), $(X^{(2)})^{-r}D_0$ is a treillis in D; it is also stable by $t_0 \in \ell^{-1}(L_0^{(2)}) \cap L_+$ because

$$\varphi_{\ell(t_0)}((X^{(2)})^{-r}\Lambda(N_0^{(2)})) = \varphi_{\ell(t_0)}((X^{(2)})^{-r})\varphi_{\ell(t_0)}(\Lambda(N_0^{(2)})) = (X^{(2)})^{-r}\Lambda(N_0^{(2)}) \ .$$

When M is killed by a power of p, this implies with Prop. 9.10 that M_s^{bd} is the union of $M_s^{bd}(D_0)$ when D_0 runs over the lattices of D which are stable by $\ell^{-1}(L_0^{(2)}) \cap L_+$.

b) We suppose from now on, as we can by Lemma 9.6, that M is killed by a power of p to prove $M_{st_0}^{bd} \subset M_s^{bd} = M_{s^r}^{bd}$. Let $m \in M_{st_0}^{bd}(D_0)$ where D_0 is a $\ell^{-1}(L_0^{(2)}) \cap L_+$-stable lattice of D. For $u \in N_0$ and $k \in \mathbb{N}$, using (12) for $t = t_0^k$ we obtain that

$$\begin{aligned}\ell_M(\psi_s^k(u^{-1}m)) = \ell_M(&\sum_{v \in J(N_0/t_0^k N_0 t_0^{-k})} v \circ \varphi_{t_0}^k \circ \psi_{t_0}^k \circ v^{-1} \circ \psi_s^k(u^{-1}m)) = \\ &= \sum_{v \in J(N_0/t_0^k N_0 t_0^{-k})} \ell(v)\varphi_{t_0}^k(\ell_M(\psi_{st_0}^k(\varphi_s^k(v^{-1})u^{-1}m)))\end{aligned}$$

lies in D_0, since D_0 is both $N_0^{(2)}$- and φ_{t_0}-invariant and $\ell_M(\psi_{st_0}^k(u'm)) \in D_0$ for $u' \in N_0$. Therefore $M_{st_0}^{bd}(D_0) \subset M_s^{bd}(D_0)$ and by a) we deduce $M_{st_0}^{bd} \subset M_s^{bd}$.

For any $m \in M$ we observe that

$$\{\ell_M(\psi_{s^r}^k(u^{-1}m)) \text{ for } k \in \mathbb{N}, u \in N_0\} \subset \{\ell_M(\psi_s^k(u^{-1}m)) \text{ for } k \in \mathbb{N}, u \in N_0\} \ ,$$

as $\psi_{s^r}^k = \psi_s^{rk}$. We deduce that $M_s^{bd}(D_0) \subset M_{s^r}^{bd}(D_0)$ for any lattice D_0 of D hence $M_s^{bd} \subset M_{s^r}^{bd}$. Conversely, for $k_1 \in \mathbb{N}$ we write $k_1 = rk - k_2$ with $k \in \mathbb{N}$ and $0 \le k_2 < r$ and we observe that

$$\begin{aligned}\ell_M(\psi_s^{k_1}(u^{-1}m)) = \ell_M(&\sum_{v \in J(N_0/s^{k_2}N_0 s^{-k_2})} v \circ \varphi_s^{k_2} \circ \psi_s^{rk}(\varphi_s^{k_1}(v^{-1})u^{-1}m)) \\ &= \sum_{v \in J(N_0/s^{k_2}N_0 s^{-k_2})} \ell(v)\varphi_s^{k_2}(\ell_M(\psi_{s^r}^k(\varphi_s^{k_1}(v^{-1})u^{-1}m))) \ .\end{aligned}$$

The $\Lambda(N_0^{(2)})$-submodule D_r generated by $\sum_{i=1}^{r-1} \varphi_s^i(D_0)$ is a lattice because the action φ_s of s on D is étale. We deduce that $M_{s^r}^{bd}(D_0) \subset M_s^{bd}(D_r)$ since $\ell_M(\psi_{s^r}^k(u'm)) \in D_0$ for $u' \in N_0, m \in M_{s^r}^{bd}(D_0)$. Therefore $M_{s^r}^{bd}(D_0) \subset M_s^{bd}(D_r)$ hence $M_{s^r}^{bd} \subset M_s^{bd}$. It is obvious that $\mathcal{H}_{g,s} = \mathcal{H}_{g,s^r}$ on M_s^{bd}.

c) Let $g \in N_0\overline{P}N_0, k \ge k_{g,s}^{(0)}, t_0 \in \ell^{-1}(L_0^{(2)}) \cap Z(L)_\dagger$ and $r \ge 1$. We have

$$k_{g,st_0}^{(0)} \le k_{g,s}^{(0)} \quad , \quad k_{g,s^r}^{(0)} \le k_{g,s}^{(0)}$$

because $(st_0)^k N_0 (st_0)^{-k} \subset N_k$ and $(s^r)^k N_0 (s^r)^{-k} = N_{kr} \subset N_k$.

Let d in D and $v \in N_0$. By (12) we have

$$\begin{aligned}d &= \sum_{u \in J(N_0^{(2)}/\ell(st_0)^k N_0^{(2)} \ell(st_0)^{-k})} u\varphi_{st_0}^k \circ \psi_{st_0}^k(u^{-1}d) \\ &= \sum_{u \in J(N_0^{(2)}/\ell(s)^k N_0^{(2)} \ell(s)^{-k})} u\varphi_s^k \circ \psi_s^k(u^{-1}d) \ ,\end{aligned}$$

with the second equality holding true summand per summand, because ψ_{t_0} is the left and right inverse of φ_{t_0} on D (Lemma 9.19 (i)) and $\ell(t_0)N_0^{(2)}\ell(t_0)^{-1} = N_0^{(2)}$. Since ι_D commutes with φ_t and ψ_t for $t \in L_+$, this implies

$$\begin{aligned} v\iota_D(d) &= \sum_{u \in J(N_0^{(2)}/\ell(st_0)^k N_0^{(2)} \ell(st_0)^{-k})} v\iota(u)\varphi_{st_0}^k \circ \psi_{st_0}^k(\iota(u)^{-1}\iota_D(d)) \\ &= \sum_{u \in J(N_0^{(2)}/\ell(s)^k N_0^{(2)} \ell(s)^{-k})} v\iota(u)\varphi_s^k \circ \psi_s^k(\iota(u)^{-1}\iota_D(d)) \ , \end{aligned}$$

again with the second equality holding true summand per summand. We choose, as we can, systems of representatives $J(N_0/(st_0)^k N_0 (st_0)^{-k})$ and $J(N_0/s^k N_0 s^{-k})$ containing $\iota(J(N_0^{(2)}/\ell(s)^k N_0^{(2)}\ell(s)^{-k}))$. For $k \geq k_{g,s}^{(0)} \geq k_{g,st_0}^{(0)}$, we obtain

$$\mathcal{H}_{g,st_0,vJ(N_0/(st_0)^k N_0(st_0)^{-k})}(v\iota_D(d)) = \mathcal{H}_{g,s,vJ(N_0/s^k N_0 s^{-k})}(v\iota_D(d)) \ .$$

Passing to the limit when k goes to infinity, and using linearity we deduce that $\mathcal{H}_{g,st_0} = \mathcal{H}_{g,s}$ on the $o[N_0]$-submodule $< N_0\iota_D(D) >_o$ generated by $\iota_D(D)$ in $M_{st_0}^{bd}$.

d) Let $m \in M_s^{bd}(D_1)$ with $D_1 \subset D$ a ψ_s-stable lattice (Prop. 7.2 (iv)). For a positive integer k, and a set of representatives $J(N_0/s^k N_0 s^{-k})$, we write m in the form (12)

$$m = \sum_{u \in J(N_0/s^k N_0 s^{-k})} u\varphi_s^k(\iota_D(d(s,u)) + m(s,u))$$

with $m(s,u)$ in $J_\ell(N_0)M$ and $d(s,u) = \ell_M(\psi_s^k(u^{-1}m))$ in D_1. Then

$$m(s) := \sum_{u \in J(N_0/s^k N_0 s^{-k})} u\varphi_s^k(\iota_D(d(s,u))) \ \text{ lies in } < N_0\iota_D(D) >_o$$

because ι_D is L_+-equivariant. Moreover $m - m(s)$ is contained in the $o[N_0]$-submodule $N_0\varphi_s^k(J_\ell(N_0)M)$ generated by $\varphi_s^k(J_\ell(N_0)M)$. We show that

$$m(s) \in M_s^{bd}(D_1) \ . \tag{70}$$

For $v \in N_0$ and $r \leq k$ we have

$$\begin{aligned} \psi_s^r(v^{-1}(m - m(s))) &= \psi_s^r(v^{-1} \sum_{u \in J(N_0/s^k N_0 s^{-k})} u\varphi_s^k(m(s,u))) \\ &= \sum_{u \in J(N_0/s^k N_0 s^{-k})} \psi_s^r(v^{-1}u)\varphi_s^{k-r}(m(s,u)) \end{aligned}$$

which lies in $J_\ell(N_0)M$ since $m(s,u)$ is in $J_\ell(N_0)M$ and $J_\ell(N_0)M$ is N_0 and φ_s-stable. This shows that $\ell_M(\psi_s^r(v^{-1}m(s))) = \ell_M(\psi_s^r(v^{-1}m))$ lies in D_1. On the other hand, for $r > k$ we have

$$\begin{aligned} \ell_M(\psi_s^r(v^{-1}m(s))) &= \ell_M(\psi_s^r(v^{-1} \sum_{u \in J(N_0/s^k N_0 s^{-k})} u\varphi_s^k(\iota_D(d(s,u))))) \\ &= \sum_{u \in J(N_0/s^k N_0 s^{-k})} \ell_M(\psi_s^{r-k}(\psi_s^k(v^{-1}u)\iota_D(d(s,u)))) \end{aligned}$$

which lies in D_1. Indeed, since D_1 is ψ_s-stable the formula in part ii) of the proof of Prop. 9.7 implies that $\iota_D(D_1) \subseteq M_s^{bd}(D_1)$; hence the $\iota_D(d(s,u))$ lie in the ψ_s- and N_0-invariant subspace $M_s^{bd}(D_1)$. We conclude that $m(s) \in M_s^{bd}(D_1)$.

Therefore, for any ψ_{st_0}-stable lattice $D_1 \subset D$, any $k \geq 1$, and any set of representatives $J(N_0/(st_0)^k N_0 (st_0)^{-k})$, we have defined an o-linear homomorphism

$$m \mapsto m(st_0) \quad M^{bd}_{st_0}(D_1) \to M^{bd}_{st_0}(D_1) \cap \ < N_0 \iota_D(D) >_o$$

such that

$$m - m(st_0) \in M^{bd}_{st_0}(D_1) \cap \varphi^k_{st_0}(J_\ell(N_0)M). \tag{71}$$

By c) we have $\mathcal{H}_{g,st_0}(m(st_0)) = \mathcal{H}_{g,s}(m(st_0))$ for $m \in M^{bd}_{st_0}(D_1)$.

e) To end the proof that $\mathcal{H}_{g,st_0} = \mathcal{H}_{g,s}$ on $M^{bd}_{st_0}(D_1)$ we use the $\mathfrak{C}_s$-uniform convergence of $(\mathcal{H}_{g,s,J(N_0/s^k N s^{-k})})_k$. We fix systems of representatives $J(N_0/(st_0)^k N_0(st_0)^{-k})$ and $J(N_0/s^k N s^{-k})$, for any $k \geq 1$. We also choose a lattice $D_0 \subset D$ which is stable by $\ell^{-1}(L_0^{(2)}) \cap L_+$ and such that $D_1 \subset D_0$. We recall that $M^{bd}_{st_0}(D_1)$ is compact (Prop. 9.10 i)) and that $M^{bd}_{st_0}(D_1) \subset M^{bd}_{st_0}(D_0) \subset M^{bd}_s(D_0)$ by b). For any open $\Lambda(N_0)$-submodule in the weak topology $M_0 \subset M$, there exists a common constant $k_0 \geq k^{(0)}_{g,s} \geq k^{(0)}_{g,st_0}$ (by c)) such that for $k \geq k_0$,

$$\mathcal{H}_{g,st_0,J(N_0/(st_0)^k N (st_0)^{-k})} \in \mathcal{H}_{g,st_0} + E(M^{bd}_{st_0}(D_1), M_0) \tag{72}$$

$$\mathcal{H}_{g,s,J(N_0/s^k N s^{-k})} \in \mathcal{H}_{g,s} + E(M^{bd}_{st_0}(D_1), M_0)\ . \tag{73}$$

On the left hand side of (72), (73), we have continuous endomorphisms of M. By Lemma 8.17, there exists an integer $k_1 \geq k_0$ such that they send $N_0 \varphi^{k_1}_{st_0}(J_\ell(N_0)M)$ into M_0. Therefore, for $m \in M^{bd}_{st_0}(D_1)$, they send the element $m - m(st_0)$ associated to k_1 and $J(N_0/((st_0)^{k_1} N_0 (st_0)^{-k_1})$ as in d) (71) into M_0 hence

$$\mathcal{H}_{g,st_0}(m - m(st_0)) \text{ and } \mathcal{H}_{g,s}(m - m(st_0)) \text{ lie in } M_0.$$

By d) we obtain that $H_{g,st_0}(m) - H_{g,s_2}(m)$ lies in M_0 for $m \in M^{bd}_{st_0}(D_1)$. The statement follows since we chose M_0 to be an arbitrary open neighborhood of zero in the weak topology of M. $\square$

Definition 9.21. *We define the transitive relation $s_1 \leq s_2$ on $Z(L)_{\dagger\dagger}$ generated by*

$$s_1 = s_2 t_0 \ \textit{for}\ t_0 \in \ell^{-1}(L_0^{(2)}) \cap Z(L)_\dagger \quad \textit{or} \quad s_1^{r_1} = s_2^{r_2} \ \textit{for positive integers}\ r_1, r_2.$$

Proposition 9.20 admits the following corollary.

Corollary 9.22. *Let $s_1, s_2 \in Z(L)_{\dagger\dagger}$.*

i) When $s_1 \leq s_2$ we have $M^{bd}_{s_1} \subseteq M^{bd}_{s_2}$ and $\mathcal{H}_{g,s_1} = \mathcal{H}_{g,s_2}$ on $M^{bd}_{s_1}$.

ii) When the relation $\leq$ on $Z(L)_{\dagger\dagger}$ is right filtered, we have $\mathcal{H}_{g,s_1} = \mathcal{H}_{g,s_2}$ on $M^{bd}_{s_1} \cap M^{bd}_{s_2}$.

Proof. i) If $s_1 \leq s_2$ then there exists, by definition, a sequence $s_1 = s'_1 \leq s'_2 \leq \ldots \leq s'_m = s_2$ in $Z(L)_{\dagger\dagger}$ such that each pair s'_i, s'_{i+1} satisfies one of the two conditions in Def. 9.21. Hence we may assume, by induction, that the pair s_1, s_2 satisfies one of these conditions, and we apply Prop. 9.20.

ii) When there exists $s_3 \in Z(L)_{\dagger\dagger}$ such that $s_1 \leq s_3$ and $s_2 \leq s_3$, by i) $M^{bd}_{s_1}$ and $M^{bd}_{s_2}$ are contained in $M^{bd}_{s_3}$ and $\mathcal{H}_{g,s_1} = \mathcal{H}_{g,s_2} = \mathcal{H}_{g,s_3}$ on $M^{bd}_{s_1} \cap M^{bd}_{s_2}$. $\square$

Proposition 9.23. *We assume that the relation $\leq$ on $Z(L)_{\dagger\dagger}$ is right filtered. Then, the intersection and the union*

$$M^{bd}_\cap := \bigcap_{s \in Z(L)_{\dagger\dagger}} M^{bd}_s \ \subset \ M^{bd}_\cup := \bigcup_{s \in Z(L)_{\dagger\dagger}} M^{bd}_s$$

are dense étale L_+-submodules of M over $\Lambda(N_0)$.

For $g \in N_0 \overline{P} N_0$ the endomorphisms $\mathcal{H}_g \in \mathrm{End}_o(M^{bd}_\cup)$ equal to $\mathcal{H}_{g,s}$ on M^{bd}_s for each $s \in Z(L)_{\dagger\dagger}$, are well defined, stabilize $M^{bd}_\cap$ and satisfy the relations H1, H2, H3 of Prop. 5.14.

Proof. $M_{\cap}^{bd}$ is an L_+-submodule of M over $\Lambda(N_0)$ by Prop. 9.5 and Remark 9.9. It is dense in M by Prop. 9.7 and Lemma 8.6. The action of L_+ on $M_{\cap}^{bd}$ is étale because $M_{\cap}^{bd}$ is L_--stable. When $\leq$ is right filtered, $M_{\cup}^{bd}$ is a $\Lambda_\ell(N_0)$-module by Cor. 9.22 i). For the same reasons as for $M_{\cap}^{bd}$, it is an étale L_+-submodule of M over $\Lambda(N_0)$.

By Cor. 9.22 the $\mathcal{H}_g$ are well defined and stabilize $M_{\cap}^{bd}$. They satisfy the relations H1, H2, H3 of Prop. 5.14 because the $\mathcal{H}_{g,s}$ satisfy them (Theorem 9.17). □

We summarize our results and give our main theorem.

Theorem 9.24. *For any $s \in Z(L)_{\dagger\dagger}$, we have a faithful functor*

$$\mathbb{Y}_s : \mathcal{M}^{et}_{\Lambda_\ell(N_0)}(L_+) \quad \to \quad G\text{-equivariant sheaves on } G/P \ ,$$

which associates to $M \in \mathcal{M}^{et}_{\Lambda_\ell(N_0)}(L_+)$ the G-equivariant sheaf $\mathfrak{Y}_s$ on G/P such that $\mathfrak{Y}_s(\mathcal{C}_0) = M_s^{bd}$.

When the relation $\leq$ on $Z(L)_{\dagger\dagger}$ is right filtered, we have faithful functors

$$\mathbb{Y}_\cap, \mathbb{Y}_\cup : \mathcal{M}^{et}_{\Lambda_\ell(N_0)}(L_+) \quad \to \quad G\text{-equivariant sheaves on } G/P \ ,$$

which associate to $M \in \mathcal{M}^{et}_{\Lambda_\ell(N_0)}(L_+)$ the G-equivariant sheaves $\mathfrak{Y}_\cap$ and $\mathfrak{Y}_\cup$ on G/P with sections on $\mathcal{C}_0$ equal to $\mathfrak{Y}_\cap(\mathcal{C}_0) = M_{\cap}^{bd}$ and $\mathfrak{Y}_\cup(\mathcal{C}_0) = M_{\cup}^{bd}$.

Proof. The existence of the functors results from Prop. 9.23, Theorem 9.17, Prop. 5.14, and Remark 5.11.

We show the faithfulness of the functors. For a non zero morphism $f : M \to M'$ in $\mathcal{M}^{et}_{\Lambda_\ell(N_0)}(L_+)$, we have $f(M_{\cap}^{bd}) \neq 0$ because f is continuous ([12] Lemma 8.22) and $M_{\cap}^{bd}$ containing $\Lambda(N_0)\iota_D(D)$ is dense (Prop. 9.23). We deduce $\mathbb{Y}_\cap(f) \neq 0$ since it is nonzero on sections on $\mathcal{C}_0$. A fortiori $\mathbb{Y}_s(f) \neq 0$, and $\mathbb{Y}_\cup(f) \neq 0$. □

10 Connected reductive split group

We explain how our results apply to connected reductive groups.

a) Let F be a locally compact non archimedean field of ring of integers o_F and uniformizer p_F. Let G be a connected reductive F-group, let S be a maximal F-split subtorus of G and let P be a parabolic F-subgroup of G with Levi component L containing S and unipotent radical N. Let $X^*(S)$ be the group of characters of S, let Φ_L, resp. Φ, be the subset of roots of S in L, resp. G, and let $\Phi_{+,N}$ be the subset of roots of S in N (we suppress the index N if P is a minimal parabolic F-subgroup of G).

Let s be any element of $S(F)$ such that $\alpha(s) = 1$ for $\alpha \in \Phi_L$ and the p-valuation of $\alpha(s) \in F^*$ is positive for all roots $\alpha \in \Phi_{+,N}$. *For any compact open subgroup N_0 of $N(F)$, the data $(P(F), L(F), N(F), N_0, s)$ satisfy all the conditions introduced in the section on étale P_+-modules (3), (3.2), the assumptions introduced in the section 6, and in the section 9.*

b) We suppose that P is a minimal parabolic F-subgroup. Let $W \subset N_G(L)$ be a system of representatives of the Weyl group $N_G(L)/L$ and let w_0 be the longest element of the Weyl group. *The data $(G(F), P(F), W)$ satisfy the assumptions of the section 5 on G-equivariant sheaves on G/P.*

c) We suppose until the end of this article that

$$F = \mathbb{Q}_p,\ G \text{ is } \mathbb{Q}_p\text{-split and } P \text{ is a Borel } \mathbb{Q}_p\text{-subgroup.}$$

The Levi subgroup $L = T$ of P is a split $\mathbb{Q}_p$-torus. The monoid of dominant elements and the submonoid without unit of strictly dominant elements are

$$T(\mathbb{Q}_p)_+ = \{t \in T(\mathbb{Q}_p),\ \alpha(t) \in \mathbb{Z}_p \text{ for all } \alpha \in \Delta\}\ ,$$
$$T(\mathbb{Q}_p)_{++} = \{t \in T(\mathbb{Q}_p),\ \alpha(t) \in p\mathbb{Z}_p - \{0\} \text{ for all } \alpha \in \Delta\}\ .$$

With our former notation $Z(L) = T(\mathbb{Q}_p), Z(L)_{\dagger\dagger} = T(\mathbb{Q}_p)_{++}$. For each root $\alpha \in \Phi$, let

$$(74) \qquad u_\alpha : \mathbb{Q}_p \to N_\alpha(\mathbb{Q}_p) \quad , \quad tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x) \ \text{ for } \ x \in \mathbb{Q}_p, t \in T(\mathbb{Q}_p)\ ,$$

be a continuous isomorphism from $\mathbb{Q}_p$ onto the root subgroup $N_\alpha(\mathbb{Q}_p)$ of $N(\mathbb{Q}_p)$ normalized by $T(\mathbb{Q}_p)$. We can write an element $u \in N(\mathbb{Q}_p)$ in the form

$$u = \prod_{\alpha \in \Phi_+} u_\alpha(x_\alpha)$$

for any ordering of Φ_+. The coordinates $x_\alpha = x_\alpha(u) \in \mathbb{Q}_p$ of u are determined by the ordering of the roots, but for a simple root α, the coordinate

$$(75) \qquad x_\alpha : N(\mathbb{Q}_p) \to \mathbb{Q}_p$$

is independent of the choice of the ordering, and satisfies $u_\alpha \circ x_\alpha = 1$. We suppose, as we can, that the u_α have be chosen such that the product

$$N_0 = \prod_{\alpha \in \Phi_+} u_\alpha(\mathbb{Z}_p)$$

is a group for some ordering of Φ_+. Then N_0 is the product of the $u_\alpha(\mathbb{Z}_p) = N_\alpha(\mathbb{Z}_p)$ for any ordering of Φ_+.

We choose a simple root α. We consider the continuous homomorphisms

$$\ell_\alpha \ : \ P(\mathbb{Q}_p) \to P^{(2)}(\mathbb{Q}_p) \quad , \quad \iota_\alpha : \ N(\mathbb{Q}_p)^{(2)} \to N(\mathbb{Q}_p) \quad , \quad \ell_\alpha \circ \iota_\alpha = 1\ ,$$

defined by

$$\ell_\alpha(ut) := \begin{pmatrix} \alpha(t) & x_\alpha(u) \\ 0 & 1 \end{pmatrix} \quad , \quad \iota_\alpha(u^{(2)}(x)) := u_\alpha(x) \ \text{ for } \ u^{(2)}(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\ ,$$

for $t \in T(\mathbb{Q}_p), u \in N(\mathbb{Q}_p), x \in \mathbb{Q}_p$. They satisfy the functional equation

$$t\iota_\alpha(y)t^{-1} = \iota_\alpha(\ell_\alpha(t)y\ell_\alpha(t)^{-1})$$

for $y \in N(\mathbb{Q}_p)^{(2)}$ and $t \in T(\mathbb{Q}_p)$. *The data $(N_0, \ell_\alpha, \iota_\alpha)$ satisfies the assumptions introduced in the 8 and in the section 9.*

We consider the binary relation $s_1 \leq s_2$ on $T(\mathbb{Q}_p)_{++}$ generated by

$$s_1 = s_2 s_0 \ \text{ with } s_0 \in T(\mathbb{Q}_p)_+, \alpha(s_0) \in \mathbb{Z}_p^* \quad , \ \text{ or } s_1^n = s_2^m \text{ with } n, m \geq 1.$$

Lemma 10.1. *The relation $s_1 \leq s_2$ on $T(\mathbb{Q}_p)_{++}$ is right filtered.*

Proof. Let $\Delta = \{\alpha = \alpha_1, \ldots, \alpha_n\}$. The image of $T(\mathbb{Q}_p)_{++}$ by $A = (\mathrm{val}_p(\alpha_i(.))_{\alpha_i \in \Delta}$ is contained in $(\mathbb{N} - \{0\})^n$ and $s_1 \leq s_2$ depends only on the cosets $s_1 T(\mathbb{Q}_p)_0$ and $s_1 T(\mathbb{Q}_p)_0$, where

$$T(\mathbb{Q}_p)_0 = \{t \in T(\mathbb{Q}_p),\ \alpha(t) \in \mathbb{Z}_p^* \text{ for all } \alpha \in \Delta\}\ .$$

a) First we assume that, for any positive integer k, there exists $s_{[k]} \in T(\mathbb{Q}_p)$ such $A(s_{[k]}) = (k, 1, \ldots, 1)$. Then we have $s_{[k]} \leq s_{[k+1]}$, and $s \leq s_{[k(s)]}$ for $s \in T(\mathbb{Q}_p)_{++}$

with $k(s) = \mathrm{val}_p(\alpha(s))$. For any s_1, s_2 in $T(\mathbb{Q}_p)_{++}$ we deduce that $s_1 \leq s_{[k(s_1)+k(s_2)]}$ and $s_2 \leq s_{[k(s_1)+k(s_2)]}$. Hence the relation $\leq$ on $T(\mathbb{Q}_p)_{++}$ is right filtered.

b) When G is semi-simple and adjoint the dominant coweights $\omega_{\alpha_1}, \ldots, \omega_{\alpha_n}$ for $\Delta = \{\alpha = \alpha_1, \ldots, \alpha_n\}$ form a basis of $Y = \mathrm{Hom}(\mathbb{G}_m, T)$, and $A(T(\mathbb{Q}_p)_{++}) = (\mathbb{N} - \{0\})^n$. Hence $s_{[k]}$ exists for any $k \geq 1$.

c) When G is semi-simple we consider the isogeny $\pi : G \to G_{ad}$ from G onto the adjoint group G_{ad} ([13] 16.3.5). The image T_{ad} of T is a maximal split $\mathbb{Q}_p$-torus in G_{ad}. The isogeny gives an homomorphism $T(\mathbb{Q}_p) \to T_{ad}(\mathbb{Q}_p)$, inducing an injective map between the cosets

$$T(\mathbb{Q}_p)_{++}/T(\mathbb{Q}_p)_0 \ \to \ T_{ad}(\mathbb{Q}_p)_{++}/T_{ad}(\mathbb{Q}_p)_0$$

respecting $\leq$, and such that for any $t_{ad} \in T_{ad}(\mathbb{Q}_p)$ there exists an integer $n \geq 1$ such that $t_{ad}^n \in \pi(T(\mathbb{Q}_p))$. Given $s_1, s_2 \in T(\mathbb{Q}_p)_{++}$ there exists $s_{ad} \in T_{ad}(\mathbb{Q}_p)_{++}$ such that $\pi(s_1), \pi(s_2) \leq s_{ad}$ by b) and a). Let $n \geq 1$ such that $s_{ad}^n = \pi(s_3)$ for $s_3 \in T(\mathbb{Q}_p)$. We have $s_{ad} \leq s_{ad}^n$ hence $\pi(s_1), \pi(s_2) \leq \pi(s_3)$. This is equivalent to $s_1, s_2 \leq s_3$.

d) When G is reductive let $\pi : G \to G' = G/Z^0$ be the natural $\mathbb{Q}_p$-homomorphism from G to the quotient of G by its maximal split central torus Z^0. The group G' is semi-simple, $\pi(T) = T'$ is a maximal split $\mathbb{Q}_p$-torus in G', $\pi|_T$ gives an exact sequence

$$1 \to Z_0(\mathbb{Q}_p) \to T(\mathbb{Q}_p) \to T'(\mathbb{Q}_p) \to 1 \ ,$$

inducing a bijective map between the cosets

$$T(\mathbb{Q}_p)_{++}/T(\mathbb{Q}_p)_0 \ \to \ T'(\mathbb{Q}_p)_{++}/T'(\mathbb{Q}_p)_0$$

respecting $\leq$. By c), $\leq$ is right filtered on $T'(\mathbb{Q}_p)_{++}$. We deduce that $\leq$ is right filtered on $T(\mathbb{Q}_p)_{++}$. $\square$

By Theorem 8.20 and Theorem 9.24, we can associate functorially to an étale T_+-module D over $\mathcal{O}_{\mathcal{E},\alpha}$ different sheaves :

- For any $s \in T_{++}$, a $G(\mathbb{Q}_p)$-equivariant sheaf $\mathfrak{Y}_s$ on $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ with sections on $\mathcal{C}_0$ equal to $\mathbb{M}(D)_s^{bd}$
- The $G(\mathbb{Q}_p)$-equivariant sheaves $\mathfrak{Y}_\cap$ and $\mathfrak{Y}_\cup$ on $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ with sections on $\mathcal{C}_0$ equal to $\cap_{s \in T_{++}} \mathbb{M}(D)_s^{bd}$ and $\cup_{s \in T_{++}} \mathbb{M}(D)_s^{bd}$.

In general $\mathbb{M}(D)$ is different from $\cup_{s \in T_{++}} \mathbb{M}(D)_s^{bd}$, by the following proposition.

Proposition 10.2. *Let M be an étale T_+-module M over $\Lambda_{\ell_\alpha}(N_0)$. When the root system of G is irreducible of positive rank $rk(G)$, we have:*

(i) If $rk(G) = 1$, the $G(\mathbb{Q}_p)$-equivariant sheaf on $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ with sections M_s^{bd} over $\mathcal{C}_0$ does not depend on the choice of $s \in T_{++}$, and $M = M_s^{bd}$.

(ii) If $rk(G) > 1$, a $G(\mathbb{Q}_p)$-equivariant sheaf of o-modules $\mathfrak{Y}$ on $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ such that $\mathfrak{Y}(\mathcal{C}_0) \subset M$ and $(u_\alpha(1) - 1)$ is bijective on $\mathfrak{Y}(\mathcal{C}_0)$, is zero.

Proof. We prove (i). If $rk(G) = 1$, then $\mathcal{O}_{\mathcal{E}} = \Lambda_{\ell_\alpha}(N_0)$ and $M = D$ is an étale T_+-module over $\mathcal{O}_{\mathcal{E}}$. With the same proof as in Prop. 7.5, we have $M_s^{bd} = M$ for any $s \in T_{++}$ and the integrals $\mathcal{H}_g$ for $g \in N_0 \overline{P} N_0$ do not depend on the choice of s.

(ii) is equivalent to the property: an étale $o[P_+]$-submodule M' of M which is also a $R = o[N_0][(u_\alpha(1) - 1)^{-1}]$-submodule of M, and is endowed with endomorphisms $\mathcal{H}_g \in \mathrm{End}_o(M)$, for all $g \in N_0 \overline{P}(F) N_0$, satisfying the relations H1, H2, H3 (Prop. 5.14), is 0.

a) Preliminaries. As $rk(G) \geq 2$ and the root system is irreducible, there exists a simple root β such that $\alpha + \beta$ is a root. The elements $n_\alpha := u_\alpha(1)$ and $n_\beta := u_\beta(1)$ do not commute. By the commutation formulas, $n_\alpha n_\beta = n_\beta n_\alpha h$ for some $h \neq 1$ in the group

$H = \prod_\gamma N_\gamma(\mathbb{Z}_p)$ for all positive roots of the form $\gamma = i\alpha + j\beta \in \Phi_+$ with $i, j > 0$. Note that H is normalized by $N_\alpha(\mathbb{Z}_p)$. Let $s \in T_{++}$. We have the expansion (12)

$$(n_\alpha h - 1)^{-k} = \sum_{u \in J(N_\alpha(\mathbb{Z}_p)H/sN_\alpha(\mathbb{Z}_p)Hs^{-1})} u\varphi_s(\psi_s(u^{-1}(n_\alpha h - 1)^{-k})) \tag{76}$$

in R. We choose, as we can, a lift w_β of s_β in the normalizer of $T(\mathbb{Q}_p)$ such that

- $w_\beta n_\beta \in n_\beta \overline{P}(\mathbb{Q}_p)$
- w_β normalizes the group $N_{\Phi_+ - \beta}(\mathbb{Z}_p) = \prod_\gamma N_\gamma(\mathbb{Z}_p)$ for all positive roots $\gamma \neq \beta$.

The subset $N'_\beta(\mathbb{Z}_p) \subset N_\beta(\mathbb{Z}_p)$ of $u_\beta(b)$ such that $w_\beta u_\beta(b) \in u_\beta(\mathbb{Z}_p)\overline{P}(\mathbb{Q}_p)$, contains n_β but does not contain 1. The subset $U_{w_\beta} \subset N_0$ of u such that $w_\beta u \in N_0 \overline{P}(\mathbb{Q}_p)$ is equal to

$$U_{w_\beta} = N'_\beta(\mathbb{Z}_p) N_{\Phi_+ - \beta}(\mathbb{Z}_p) = N_{\Phi_+ - \beta}(\mathbb{Z}_p) N'_\beta(\mathbb{Z}_p) \ .$$

Hence $U_{w_\beta} = uU_{w_\beta}$, i.e. $w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 = uw_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0$, for any $u \in N_{\Phi_+ - \beta}(\mathbb{Z}_p)$.

b) Let M' be an $R = o[N_0][(n_\alpha - 1)^{-1}]$-module of M, which is also an étale $o[P_+]$-submodule, and is endowed with endomorphisms $\mathcal{H}_g \in \mathrm{End}_o(M)$, for all $g \in N_0 \overline{P}(F) N_0$, satisfying the relations H1, H2, H3 (Prop. 5.14), and let $m \in M'$ be an arbitrary element. We want to prove that $m = 0$.

The idea of the proof is that, for $s \in T_{++}$, we have $m = 0$ if $\mathcal{H}_{w_\beta}(n_\beta \varphi_s(m)) = 0$ and that $\mathcal{H}_{w_\beta}(n_\beta \varphi_s(m)) = 0$ because it is infinitely divisible by $n_\gamma - 1$, where $\gamma = s_\beta(\alpha)$. An element in M with this property is 0 because $n_\gamma - 1$ lies in the maximal ideal of $\Lambda_{\ell_\alpha}(N_0)$.

Let $a \in \mathbb{Z}_p$. The product formula in Prop. 6.9ii implies

$$\mathcal{H}_{w_\beta} \circ \mathcal{H}_{n_\alpha^a} \circ \mathrm{res}(1_{w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0}) = \mathcal{H}_{w_\beta n_\alpha^a} \circ \mathrm{res}(1_{w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0}) =$$
$$\mathcal{H}_{n_\gamma^a w_\beta} \circ \mathrm{res}(1_{w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0}) = \mathcal{H}_{n_\gamma^a} \circ \mathcal{H}_{w_\beta} \circ \mathrm{res}(1_{w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0})$$

since $n_\alpha^{-a} w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 = w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0 = w_\beta^{-1} n_\gamma^{-a}\mathcal{C}_0 \cap \mathcal{C}_0$. For all $k \in \mathbb{N}$, the elements

$$m_k := (n_\alpha - 1)^{-k} n_\beta \varphi_s(m) = n_\beta (n_\alpha h - 1)^{-k} \varphi_s(m) \tag{77}$$

lie in the image of the idempotent $\mathrm{res}(1_{w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0}) \in \mathrm{End}_o(M)$, because

$$m_k = \sum_{u \in J(N_\alpha(\mathbb{Z}_p)H/sN_\alpha(\mathbb{Z}_p)Hs^{-1})} n_\beta u \varphi_s(\psi_s(u^{-1}(n_\alpha h - 1)^{-k} m)) \tag{78}$$

by (76), (77). Therefore the product relations between $\mathcal{H}_{w_\beta}, \mathcal{H}_{n_\alpha^a}$ and $\mathcal{H}_{n_\gamma^a}$ imply

$$\begin{aligned}
\mathcal{H}_{w_\beta}(n_\beta \varphi_s(m)) = \mathcal{H}_{w_\beta}((n_\alpha - 1)^k m_k) &= \sum_{a=0}^{k} (-1)^{k-a} \binom{k}{a} \mathcal{H}_{w_\beta} \circ \mathcal{H}_{n_\alpha^a}(m_k) \\
&= \sum_{a=0}^{k} (-1)^{k-a} \binom{k}{a} \mathcal{H}_{w_\beta} \circ \mathcal{H}_{n_\alpha^a} \circ \mathrm{res}(1_{w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0})(m_k) \\
&= \sum_{a=0}^{k} (-1)^{k-a} \binom{k}{a} \mathcal{H}_{n_\gamma^a} \circ \mathcal{H}_{w_\beta} \circ \mathrm{res}(1_{w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0})(m_k) \\
&= (n_\gamma - 1)^k \mathcal{H}_{w_\beta}(m_k) \ ,
\end{aligned}$$

Hence $\mathcal{H}_{w_\beta}(n_\beta \varphi_s(m)) = 0$ since it is infinitely divisible by $n_\gamma - 1$ which lies in the maximal ideal of $\Lambda_{\ell_\alpha}(N_0)$. We also have

$$n_\beta \varphi_s(m) = \mathcal{H}_1 \circ \mathrm{res}(1_{w_\beta^{-1}\mathcal{C}_0 \cap \mathcal{C}_0})(n_\beta \varphi_s(m)) = \mathcal{H}_{w_\beta} \circ \mathcal{H}_{w_\beta}(n_\beta \varphi_s(m)) = 0 \ .$$

As $n_\beta \circ \varphi_s \in \mathrm{End}_o(M')$ is injective, we deduce $m = 0$. $\square$

Corollary 10.3. *There exists a $G(\mathbb{Q}_p)$-equivariant sheaf on $G(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ with sections M on $\mathcal{C}_0$ if and only if* $\mathrm{rk}(G) = 1$.

References

[1] Bourbaki N., Algèbre commutative. Ch. 1 à 4. Masson 1985.

[2] Bourbaki N., Topologie générale. Ch. 1 à 4. Hermann 1971.

[3] Bourbaki N., Topologie générale. Ch. 5 à 10. Hermann 1974.

[4] Bosch S., Güntzer U., Remmert R., Non-Archimedean Analysis. Springer 1984.

[5] Colmez P., (φ, Γ)-modules et représentations du mirabolique de $GL_2(\mathbb{Q}_p)$. Astérisque 330, 2010, p.61-153.

[6] Colmez P., Représentations de $GL_2(\mathbb{Q}_p)$ et (φ, Γ)-modules. Astérisque 330, 2010, p.281-509.

[7] Dixon J. D., du Sautoy M. P. F., Mann A., Segal D., Analytic pro-p groups. Second edition. Cambridge Studies in Advanced Mathematics, 61. Cambridge University Press: Cambridge, 1999.

[8] Ellis R., Locally compact transformation groups. Duke Math. J. 24, 119-125 (1957).

[9] Gabriel P., Des catégories abéliennes. Bull. Soc. math. France, 90, 1962, p. 323-448.

[10] Kedlaya K., New methods for (φ, Γ)-modules, preprint (2011), http://math.mit.edu/kedlaya/papers/new-phigamma.pdf

[11] Schneider P., p-Adic Lie Groups. Springer Grundlehren Vol. 344, 2011.

[12] Schneider P., Vigneras M.-F., A functor from smooth o-torsion representations to (φ, Γ)-modules. Volume in honour of F. Shahidi. Clay Mathematics Proceedings Volume 13, 525–601, 2011.

[13] Springer T.A., Linear Algebraic Groups. 2nd Ed. Birkhäuser 2009.

[14] Vigneras M.-F., Représentations ℓ-modulaires d'un groupe réductif p-adique avec $\ell \neq p$. Birkhäuser, PM 137, 1996.

[15] Warner S.: Topological Rings. Elsevier: Amsterdam 1993.

[16] Zábrádi G., Exactness of the reduction of étale modules. J. Algebra 331 (2011) 400-415.