Data Assimilation for Navier-Stokes

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LMS-EPSRC Data Assimilation Meeting
Durham University, August 3rd 2011

Collaboration with Kody Law (Warwick).

Funded by EPSRC, ERC and ONR

Outline

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Navier Stokes Equations and Data

Write NSE as ODE in $H = \left\{u \in L^2(\mathbb{T}^2) \middle| \nabla \cdot u = 0, \int_{\mathbb{T}^2} u\, dx = 0\right\}$:

$$\frac{dv}{dt} + \nu A v + B(v,v) = f, \qquad v(0) = u$$
$$v(t) = \Psi(u;t) \quad \Psi^{(j)}(u) := \Psi(u; jh)$$

Find u given noisy observations y_j:

$$y_j = \Psi^{(j)}(u) + \eta_j,$$
$$\eta_j \sim \mathcal{N}(0, \Gamma).$$

$Y_j = \{y_k\}_{k=1}^{j}$.

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Bayesian Formulation

- Prior $\mathbb{P}(u)$ on u:

$$\mathbb{P}(u) \sim \mathcal{N}(\hat{m}_0, \hat{C}_0).$$

- Bayes formula:

$$\frac{\mathbb{P}(u|Y_j)}{\mathbb{P}(u)} \propto \mathbb{P}(Y_j|u).$$

- Here

$$\mathbb{P}(Y_J|u) \propto \exp\left(-\Phi(u; Y_J)\right)$$

$$\Phi(u; Y_J) = \frac{1}{2}\sum_{j=1}^{J}\left\|\Gamma^{-\frac{1}{2}}\left(y_j - \Psi^{(j)}(u)\right)\right\|^2$$

- Posterior is $\mathbb{P}(u|Y_J)$.

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Outline

Posterior Distribution

- MCMC - Gold standard, state of the art (accurate) :

Propose $u^* = \hat{m}_0 + (1-2\beta)^{\frac{1}{2}}(u^{(n-1)} - \hat{m}_0) + \sqrt{2\beta}\mathcal{N}(0, \hat{\mathcal{C}}_0)$ and let

$$u^{(n)} = \left\{ \begin{array}{cc} u^* & \text{with probability } 1 \wedge \exp\{\Phi(u^{(n-1)}) - \Phi(u^*)\} \\ u^{(n-1)} & else. \end{array} \right\}$$

- 4DVAR - MAP Estimator (approximate) :

$$u \approx \mathcal{N}(m_0', \mathcal{C}_0'), \qquad \mathcal{C}_0' = \left(D^2\Phi(m_0') + \hat{\mathcal{C}}_0^{-1}\right)^{-1}$$

$$m_0' = \mathrm{argmin_u}\left(\Phi(u) + \frac{1}{2}\|\hat{\mathcal{C}}_0^{-\frac{1}{2}}(u - \hat{m}_0)\|^2\right).$$

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Approximate Gaussian Filters I

- Make the Gaussian approximation:

$$\Psi^{(j)} \star \mathbb{P}(u|Y_j) = \mathbb{P}(v_j|Y_j) \approx \mathcal{N}(\hat{m}_j, \hat{C}_j).$$

- Find update rule:

$$(\hat{m}_j, \hat{C}_j) \mapsto (\hat{m}_{j+1}, \hat{C}_{j+1}).$$

- Throughout:

$$\hat{m}_{j+1} = B_j \Psi(\hat{m}_j) + (I - B_j) y_j.$$

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Approximate Gaussian Filters II

In all cases

$$\hat{\mathcal{C}}_{j+1}^{-1} = \mathcal{C}_{j+1}^{-1} + \Gamma^{-1}, \quad B_j = \hat{\mathcal{C}}_{j+1}\mathcal{C}_{j+1}^{-1}.$$

- **3DVAR**

$$\mathcal{C}_j \equiv \mathcal{C}_0 \quad \text{where} \quad \mathcal{C}_0^{-1} = \hat{\mathcal{C}}_0^{-1} - \Gamma^{-1}.$$

- **FDF** For $\mathcal{C}$ chosen from Gaussian SPDE parameter fit:

$$\mathcal{C}_j \equiv \mathcal{C}.$$

- **Kalman Filter** For $\Psi(u) = Lu$:

$$\mathcal{C}_{j+1} = L\hat{\mathcal{C}}_j L^*.$$

- **(LR)ExKF** (Low rank approximation of):

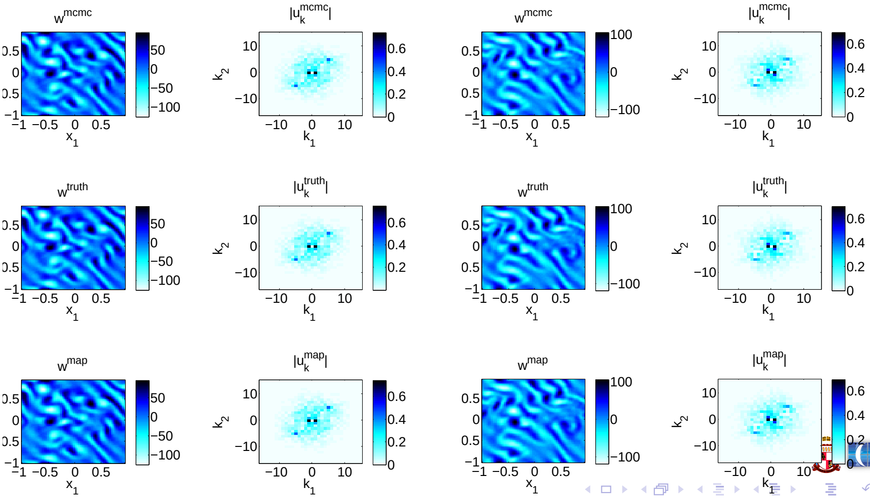
$$\mathcal{C}_{j+1} = D\Psi(\hat{m}_j)\hat{\mathcal{C}}_j D\Psi(\hat{m}_j)^*.$$

- **EnKF** Particle approximations for $\hat{m}_j$ and $\hat{\mathcal{C}}_j$.

Outline

MCMC & MAP

$t = 0$ $t = T$



Relative Error in Mean c/w Posterior

Mean

method	e_{mean}
4DVAR($t = 0$)	0.000731491
4DVAR($t = T$)	0.00130112
3DVAR	0.0634553
FDF	0.165732
LRExKF	0.00614573
EnKF	0.0596825

Relative Error in Variance c/w Posterior

Variance

method	$e_{variance}$
4DVAR($t = 0$)	0.0932748
4DVAR($t = T$)	0.220154
3DVAR	6.34057
FDF	28.9155
LRExKF	0.195101
EnKF	0.516939

Outline

Error Propagation

Theorem

Let $C_0 = \delta^2 A^{-\gamma}$, $\Gamma = \epsilon^2 A^{-\beta}$, $r^2 = \epsilon^2/\delta^2$ *and* $\alpha = \gamma - \beta$. *Assume* $\sup_{j\geq 0} \|\eta_j\|_V = \omega$. *Let* $\hat{m}_0 \in B_{\mathrm{V}}(0, R)$. *Then there exists* $r_c = r_c(R, \alpha)$ *and* $\lambda \in (0, 1)$ *such that, for all* $r < r_c$,

$$\|\hat{m}_j - v_j\| \leq \lambda^n \|\hat{m}_0 - v_0\| + c\omega.$$

Proof:

$$\begin{aligned} v_{j+1} &= B_j\Psi(v_j) + (I - B_j)\Psi(v_j), \\ \hat{m}_{j+1} &= B_j\Psi(\hat{m}_j) + (I - B_j)\Psi(v_j) + (I - B_j)\eta_j \end{aligned}$$

Error $e_j = v_j - \hat{m}_j$ propagates according to

$$e_{j+1} \approx B_j D\Psi(v_j)e_j + (I - B_j)\eta_j$$

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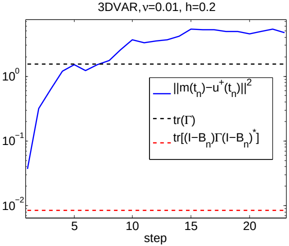
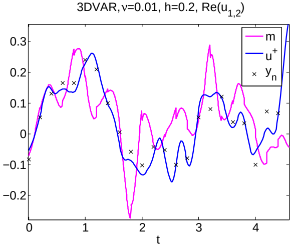
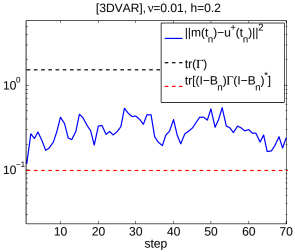
Unstable


3DVAR, ν=0.01, h=0.2
10^0
10^−1
10^−2
5
10
15
20
step
||m(t_n)−u^+(t_n)||^2
tr(Γ)
tr[(I−B_n)Γ(I−B_n)^*]


3DVAR, ν=0.01, h=0.2, Re(u_1,2)
0.3
0.2
0.1
0
−0.1
−0.2
0
1
2
3
4
t
m
u^+
y_n

Stabilized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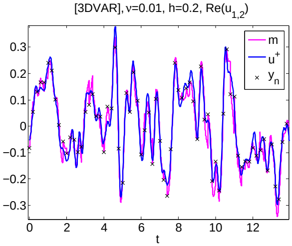
tr(Γ)
tr[(I−B_n)Γ(I−B_n)^*]
10^0
10^−1
10
20
30
40
50
60
70
step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m
u^+
y_n
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
0
2
4
6
8
10
12
t

Relative Error in Mean c/w Posterior: Stabilized

method	e_{mean}	$e_{variance}$
3DVAR	0.458527	1.8214
[3DVAR]	0.27185	6.62328
LRExKF	0.632448	0.4042
[LRExKF]	0.201327	11.2449
EnKF	0.450555	0.583623
[EnKF]	0.279007	6.67466
FDF	0.189832	11.4573

Outline

Conclusions

Approximate filters:

- reproduce the posterior mean accurately;
- fail to reproduce covariance accurately;
- can exhibit instability on longer time-intervals.

This instability:

- can cause loss of accuracy in even mean prediction;

Filter stabilization, via variance inflation can be used:

- ameliorates instability but can reduce mean accuracy on short time intervals;
- makes it impossible to predict the covariance.

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References 1

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