I. Vernacular of the S-matrix: Colour & Kinematics

Jacob Bourjaily, [arXiv:1212.5605, 0705.4305], Friday 05/07/2013, $9^{30} - 10^{30}$

Complete reformulation of perturbative QFT:

i) new perturbation expansion makes all symmetries manifest term-by-term

ii) way more efficient than classical Feynman diagram approach

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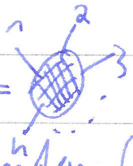
We consider an amplitude $A_n =$  for a theory of massless particles, each of which is specified by momentum (p_a^μ), ($a \in \{1,\dots,n\}$ particle number), helicity $h_a = \pm \sigma_a$ (spin) and colour c_a. For gauge fields we need to introduce polarisation tensors $\varepsilon_a^{\pm\mu}$ s.t. $\varepsilon_a^{\pm\mu} \cdot p_a = 0$ that carry a gauge redundancy $\varepsilon_a \to \varepsilon_a + \alpha(p_a) \cdot p_a$. Not only does this discription enclose fake degrees of freedom, but more severely we can not find a global choice of basis $\varepsilon_a^{\pm\mu}(p)$ without singularities!

Instead, we introduce Spinor-Helicity-variables

$$(p^\mu) \mapsto (p^{\alpha\dot\alpha}) := p^\mu \sigma_\mu^{\alpha\dot\alpha} = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix}.$$

On-shell-ness implies $0 = p^\mu p_\mu = \det(p^{\alpha\dot\alpha})$, such that we can write $p_a^{\alpha\dot\alpha} = \lambda_a^\alpha \tilde\lambda_a^{\dot\alpha} = a\rangle[a$ with

$$\langle a b\rangle := \det\{\lambda_a, \lambda_b\} \qquad [ab] := \det\{\tilde\lambda_a, \tilde\lambda_b\} \qquad (p_a+p_b)^2 = \langle ab\rangle[ab].$$

Little Group

The choice is not unique as $\lambda_a \mapsto t\cdot\lambda_a$, $\tilde\lambda_a \mapsto t^{-1}\cdot\tilde\lambda_a$ gives the same $\lambda_a^\alpha \tilde\lambda_a^{\dot\alpha}$ (so we have only 3 degrees of freedom as we must have for an on-shell momentum $p_a^2=0$). We can define polarization vectors

$$\varepsilon_a^+ := \frac{p\rangle[a}{\langle pa\rangle} \qquad \varepsilon_a^- := \frac{a\rangle[p}{[ap]}.$$

Under the action of the little group, these rescale to give

$$A_n(\ldots, t_a\lambda_a, t_a^{-1}\tilde{\lambda}_a, h_a, \ldots) = t^{-2h_a} A_n(\ldots, \lambda_a, \tilde{\lambda}_a, h_a, \ldots).$$

We organize the kinematics in the matrix

$$\lambda := \begin{pmatrix} \lambda_1^1 & \cdots & \lambda_n^1 \\ \lambda_1^2 & \cdots & \lambda_n^2 \end{pmatrix} = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} = (\lambda_1, \ldots, \lambda_n).$$

By little-group transformations, we can rescale the columns individually; so the Lorentz-invariant content is the span

$$\mathrm{lin}\{\lambda^1, \lambda^2\} \in Gr(2,n)$$

in the Grassmannian. Momentum-conservation translates into

$$\delta^4\Big(\sum_a p_a\Big) = \delta^{2\times 2}(\lambda\cdot\tilde{\lambda}) := \delta^{2\times 2}\Big(\sum_a \lambda_a\cdot\tilde{\lambda}_a\Big).$$

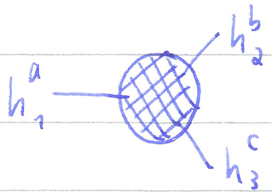
Note that $\lambda\cdot\tilde{\lambda}=0$ means the orthogonality of the 2-planes λ and $\tilde{\lambda}$:

$$\lambda \subseteq \tilde{\lambda}^\perp \in Gr(n-2,2) \qquad \tilde{\lambda} \subseteq \lambda^\perp \in Gr(n-2,2).$$

⇒ The three-particle S-matrix

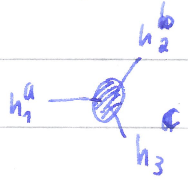
For generic $\lambda \in Gr(2,3)$, momentum-conservation fixes $\tilde{\lambda} \subseteq \lambda^\perp = \mathrm{lin}\,(\langle 23\rangle, \langle 31\rangle, \langle 12\rangle)$. one vector!

Otherwise, for $\tilde\lambda \in Gr(2,3)$ generic we find $\lambda \propto ([23],[31],[12])$. Little group gives

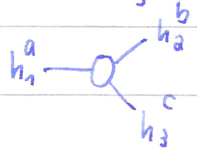


$$\propto \underset{\uparrow \text{ some factor}}{f^{abc}} \begin{cases} \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1} \xrightarrow[\langle ab\rangle\to\varepsilon]{} \mathcal{O}(\varepsilon^{-h}) \\ [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2} \xrightarrow[{[ab]}\to\varepsilon]{} \mathcal{O}(\varepsilon^{h}) \end{cases}$$

for the total helicity $h = h_1 + h_2 + h_3$. The limits correspond to real momenta p_a (which means that $\tilde\lambda$ and λ are complex conjugate). As we need finite real on-shell limits, the amplitude breaks up into two cases:

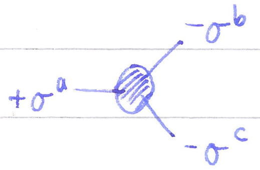
$h<0$: 

$$= f^{abc}\langle 12\rangle^{h_3-h_1-h_2}$$

$h>0$: 

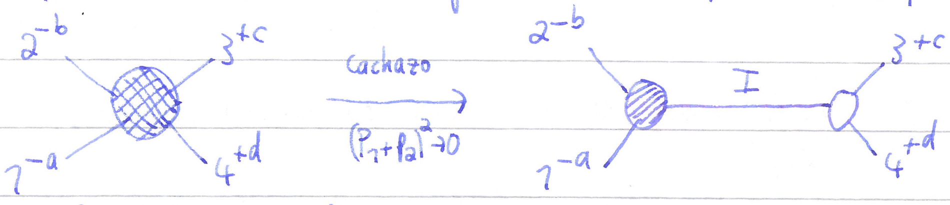
$$= f^{abc}[12]^{h_1+h_2-h_3}$$

▶ <u>Quantum mechanical consequences.</u> Consider bosons (spins $\sigma \in \mathbb{Z}$), then



$$= f^{abc}\left(\frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle}\right)^{\sigma},$$

so for σ odd, f is anti-symmetric and for σ even, f is symmetric. Consider



$$\propto f^{abI} f^{Icd}$$

as the factorization of the S-matrix dictated by unitarity. Remarks:

- Residue (at $(p_1+p_2)^2\to 0$) is non-zero only if $\sigma \leq 2$ ($\to$ Weinberg result included (no long-range forces of spin >2)!)
- $\sigma = 1 \Rightarrow$ f's satisfy a Jacobi-identity
- $\sigma = 2 \Rightarrow$ equivalence principle
- dimensional analysis shows $[f] = \text{mass}^{1-|h|}$

⇒ The Simplest Quantum Field Theory: $\mathcal{N}=4$

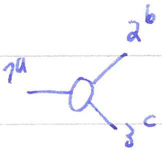
We consider coherent states given collectively in the notation

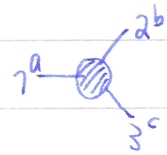
helicities

$$|a\rangle = e^{Q\tilde{\eta}_a}|a^+\rangle = |a^+\rangle + \tilde{\eta}_a^I |a^{+1/2}\rangle_I + \ldots + (\tilde{\eta}_a^1 \cdots \tilde{\eta}_a^4)|a^-\rangle$$

spin + gluon   spin + fermion   spin − gluon

and have two three-valent interaction vertices

$$\mathcal{A}_3^{(1)} \underset{h>0}{:=}$$  $$:= f^{abc} \frac{\delta^{1\times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times 2}(\lambda\cdot\tilde{\lambda}) = (\tilde{\eta}_1^1 \ldots \tilde{\eta}_1^4)\mathcal{A}(a^-, b^+, c^+) + \ldots$$

$$\mathcal{A}_3^{(2)} \underset{h<0}{:=}$$  $$:= f^{abc} \frac{\delta^{2\times 4}(\lambda\cdot\tilde{\eta})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \delta^{2\times 2}(\lambda\cdot\tilde{\lambda}) = (\tilde{\eta}_1^1 \ldots \tilde{\eta}_1^4)\mathcal{A}(a^-, b^-, c^+) + \ldots$$

Here, for a $(k\times n)$-matrix C define

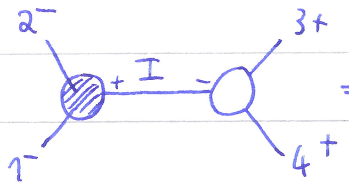
$$\delta^{k\times 4}(C\cdot\tilde{\eta}) := \prod_{I=1}^{4} \left(\bigoplus_{a_1 < \ldots < a_k} (a_1 \ldots a_k)\, \tilde{\eta}_{a_1}^I \cdots \tilde{\eta}_{a_k}^I \right)$$

$I \in \{1,\ldots,4\}$ is supersymmetric species ($\mathcal{N}=4$!)

where $(a_1, \ldots, a_k) := \det(C_{a_1}, \ldots, C_{a_k})$.

$14^{30} - 15^{45}$

We encountered the on-shell diagram as a factorization channel $(p_1+p_2)^2 \to 0$ of the



$$= \sum_{h_I \text{ internal state}}^{\text{“}\int d^4\tilde{\eta}_I\text{''}} \int \frac{d^2\lambda_I d^2\tilde{\lambda}_I}{\mathrm{vol}(GL(1))} = \frac{\langle 12\rangle^3}{\langle I1\rangle\langle 2I\rangle} \cdot \frac{[34]^3}{[I3][4I]} = \frac{\langle 12\rangle^2 [23]^2}{\langle 13\rangle [13]}$$

mod out little-group redundancy in the description of on-shell physical states encoded in $(\lambda, \tilde{\lambda})$.

4-point function.

Note that this function is only supported on $(p_1+p_2)^2 = 0$! For a general diagram we integrate over the phase-space of all (physical) states of the internal lines.

By 3-regularity, the numbers of vertices n_v, internal edges n_I and externals n relate

$$n = 3n_v - 2n_I.$$

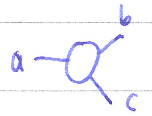
We further define n_B and n_W as the numbers of black and white vertices to set

$$k := 2n_B + n_W - n_I\,,$$

which counts morally the number of negative-helicity gluons. We now like to

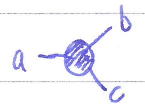
i) linearize the quadratic constraints $\delta^{2\times 2}(\lambda\cdot\tilde\lambda)$

ii) remove the ambiguity of $\delta^{2\times 2}(\lambda\cdot\tilde\lambda)$ as (depending on vertex colour) it is supposed to either pick the solution of generic λ or generic $\tilde\lambda$



$$= \int \frac{d^{1\times 3}W}{\mathrm{vol}(GL(1))} \frac{\delta^{1\times 4}(W\cdot\tilde\eta)}{(a)\,(b)\,(c)} \delta^{1\times 2}(W\cdot\tilde\lambda)\,\delta^{2\times 2}(\lambda\cdot W^\perp)$$

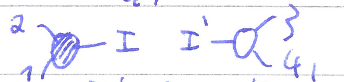
$(w_a \; w_b \; w_c)$



$$= \int \frac{d^{2\times 3}B}{\mathrm{vol}(GL(2))} \frac{\delta^{2\times 4}(B\cdot\tilde\eta)}{(ab)(bc)(ca)} \delta^{2\times 2}(B\cdot\tilde\lambda)\,\delta^{2\times 1}(\lambda\cdot B^\perp)$$

$$= \int \frac{db_c^1\, db_c^2}{b_c^1\; b_c^2}\,\delta\ldots$$

Gauge-fixing $B = \begin{pmatrix} 1 & 0 & b_c^1 \\ 0 & 1 & b_c^2 \end{pmatrix}$

To compute  gauge-fix

$$\begin{array}{cccccc} 1 & 2 & I & I' & 3 & 4 \\ \end{array}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & b_2^1 & 0 \\ 0 & 1 & b_I^2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 1 & w_3 & w_4 \end{pmatrix} \end{pmatrix} \xrightarrow[\downarrow]{\delta's} \begin{pmatrix} 1 & b_2^1 & 0 & 0 \\ 0 & 1 & w_3 & w_4 \end{pmatrix}.$$

A general on-shell diagram thus can be written in the form

$(k\times n)$-matrix

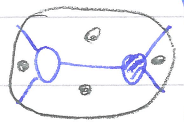
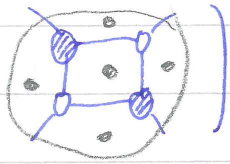
$$f = \int \text{"}dC\text{"}\, \delta^{k\times 4}(C\cdot\tilde\eta)\,\delta^{k\times 2}(C\cdot\tilde\lambda)\,\delta^{2\times(n-k)}(\lambda\cdot C^\perp).$$

Define d as the number of auxiliary (internal) degrees of freedom in this setup:

$$d = 2n_v - n_I = n - n_v - n_I \underset{\uparrow}{=} n_F := \#\text{faces of graph},$$

for planar graphs

where we understand a closing circle outside:

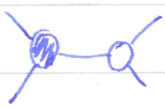
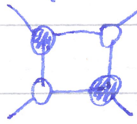
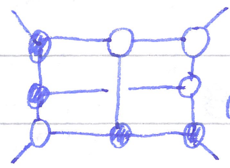
n_F () $= 4$, n_F () $= 5$.

Further let δ denote the number of available δ-functions (not including the momentum conservation)

$$\delta = 2k + 2(n-k) - 4 = 2n - 4$$

to define the number $\hat{d}$ of <u>net</u> degrees of freedom

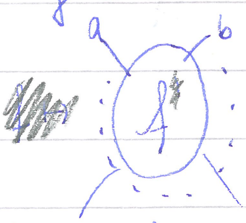
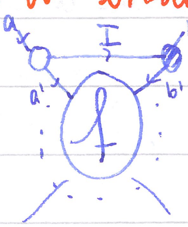
$$\hat{d} := d - \delta . \qquad \begin{cases} \hat{d} < 0: & \text{constraints on external data} \\ \hat{d} = 0: & \text{rational function} \\ \hat{d} > 0: & \text{integrations to be done} \end{cases}$$

▸ <u>Examples:</u>  $\hat{d} = -1$  $\hat{d} = 0$  $\hat{d} = 2$

Remark: On-shell diagrams in general have no relation to any sum of Feynman diagrams!

⇒ Building with BCFW-Bridges

By attaching of a BCFW-bridge we define the operation

 $\longmapsto$  $=:$  .

Momentum conservation fixes

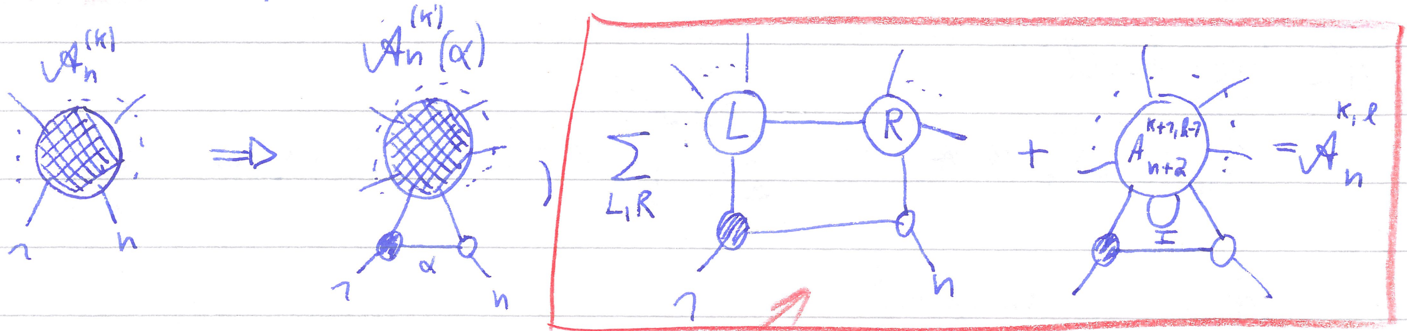
$$\overset{a\rangle}{\overset{\propto}{I\rangle}}\overset{[b}{\overset{\propto}{[I}} = \alpha\, a\rangle[b \qquad a' = a - I = \lambda_a(\tilde{\lambda}_a - \alpha\tilde{\lambda}_b) \qquad b' = b + I = (\lambda_b + \alpha\lambda_a)\tilde{\lambda}_b$$

Hence we deduce that f is a residue of f':

$$f'(\ldots a \ldots b \ldots) = \frac{d\alpha}{\alpha} f(\ldots a' \ldots b' \ldots).$$

On the level of the representation sketched earlier, this shift is $C_b \mapsto C_b + \alpha C_a$.

⇒ All-Loop-recursion relations (planar! N=4)



As measures (to relate to Feynman-loop-integrals),

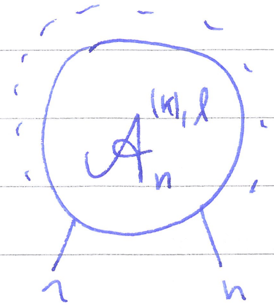
$$\underset{\substack{\uparrow \\ \text{loop momentum}}}{l} = \lambda_I \tilde{\lambda}_I + \alpha \lambda_n \tilde{\lambda}_1 \qquad \frac{d^2\lambda_I \, d^2\tilde{\lambda}_I}{\mathrm{vol}(GL(1))} d\alpha = d^4 l.$$

The problem here is that the $\int d^4 l$ transform in a complicated contour! ($l \in \mathbb{R}^{3,1}$ becomes a quadratic constraint for the λ's).

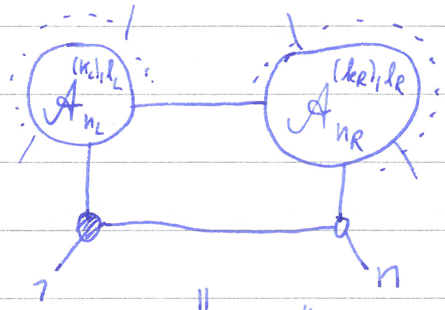
Also we have divergent integrals!

▶ Remark. Here, $A_n \ldots$ denote the integrands, not the (divergent) integrals; i.e. the recursion is for integrands, not the amplitudes!

III Classification, Combinatorics, canonical coordinates & computation



$= \sum_{\substack{n = n_L + n_R - 2 \\ k = k_L + k_R - 1 \\ l = l_L + l_R}}$



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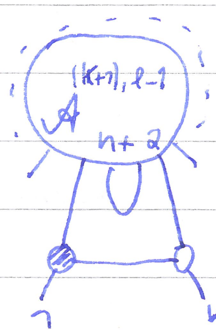


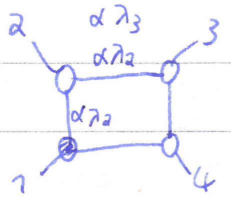
"$\mathcal{A}_L \overset{!!}{\otimes} \mathcal{A}_R$" "$FL(\mathcal{A}(1,\dots,n,I,I))$"

forward limit

$$\boxed{\mathcal{A}_3^{(k),\, l>0} = 0}$$

All tree-level:

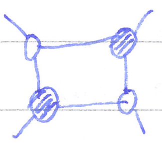
$\mathcal{A}_4^{(1)} = \mathcal{A}_3^{(1)} \otimes \mathcal{A}_3^{(1)} =$



$= 0$ as $\lambda_2 \propto \lambda_3$ is an additional constraint

$\mathcal{A}_n^{(1)} = 0$ in general

$\mathcal{A}_4^{(2)} = \mathcal{A}_3^{(1)} \otimes \mathcal{A}_3^{(2)} + \mathcal{A}_3^{(2)} \otimes \mathcal{A}_3^{(1)} =$



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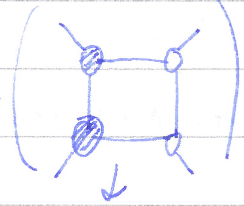
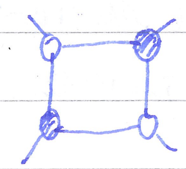


This generalizes to $\mathcal{A}_n^{(2)} = \overbrace{\mathcal{A}_3^{(1)} \otimes \dots \otimes \mathcal{A}_3^{(1)}}^{n-3 \text{ times}} \otimes \mathcal{A}_3^{(2)}$, actually imposes δ-constraint on external vertices

$$\mathcal{A}_n^{(2)} = \frac{\delta^{2\times 4}(\lambda\tilde{\eta})\, \delta^{2\times 2}(\lambda \cdot \tilde{\lambda})}{\langle 12\rangle \langle 23\rangle \dots \langle n1\rangle}$$

is the Park-Taylor-formula for (tree-level) MHV ($k=2$) – amplitudes. If we would have recursed with another bridge, we would have gotten

$\mathcal{A}_4^{(2)} =$



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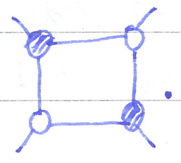

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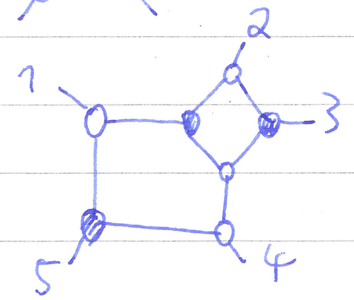
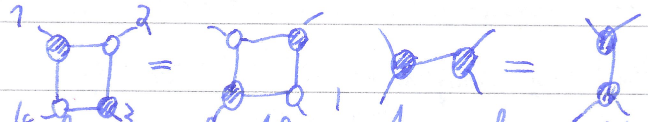
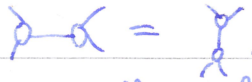
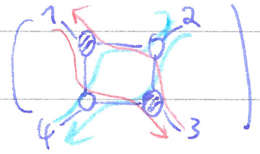
$\mathcal{A}_5^{(2)} = \mathcal{A}_3^{(1)} \otimes \mathcal{A}_4^{(2)} =$



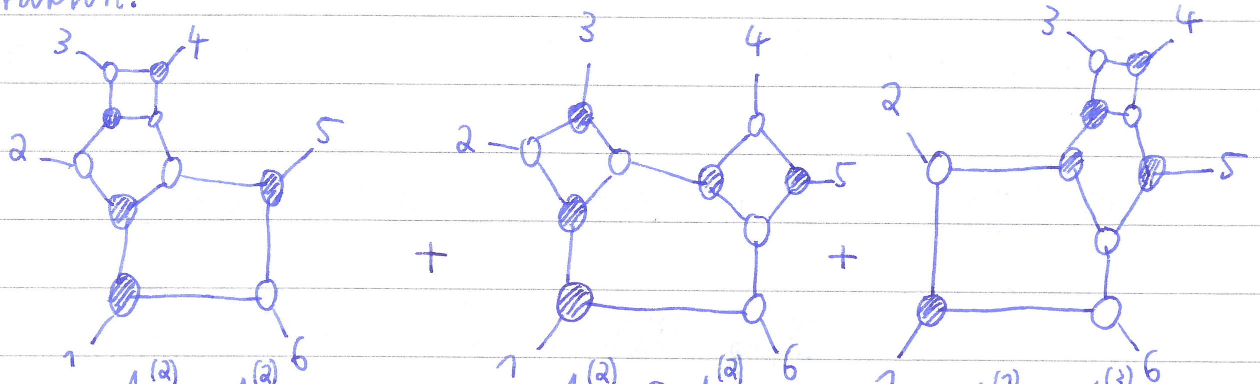
▶ **Remark.** Here we always consider the colour-ordered amplitudes (that's why we have a fixed cyclic order on the external particles).

We have three relations:  and . The equivalence class of these transformations gives a well-defined permutation of the externals by

i) turn right at black   ii) turn left at white

$\sigma\Big($  $\Big) = (13)(24)$

▶ **Theorem.** Two planar! graphs are related by the 3 moves $\Leftrightarrow$ they give the same permutation.

$\mathcal{A}_6^{(3)} =$ 

$\mathcal{A}_5^{(2)} \otimes \mathcal{A}_3^{(2)} \qquad \mathcal{A}_4^{(2)} \otimes \mathcal{A}_4^{(2)} \qquad \mathcal{A}_3^{(1)} \otimes \mathcal{A}_5^{(3)}$

$\downarrow$

$\sigma = (25)(1463)$

Note that the individual contributions in this sum are not cyclically symmetric, this only happens in the sum. In particular, all three graphs give the same permutation after cyclically relabelling the externals suitably!

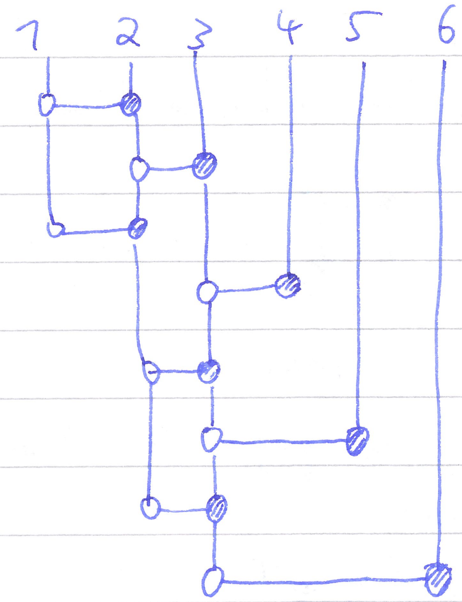
▶ **Observation.** Graph notation is very redundant! All information of (planar) on-shell diagrams is encoded in the permutation.

▶ Remark. Adding a BCFW-Bridge transposes the adjacent edges it attaches to (in the permutation), so for any permutation σ we can find a corresponding on-shell-diagram (and therefore the corresponding function) by repeatedly attaching (adjacent) BCFW-Bridges.

Transpositions

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 7 & 6 & 8 & 9 \end{pmatrix}$$

1. (12)	5	4				
2. (23)		7	4			
3. (12)	7	5	4	6	8	9
4. (34)		5	6	4	8	9
5. (23)		6	5		8	9
6. (35)		6	8		5	9
7. (23)		8	6			9
8. (36)	7	8	9	4	5	6



$$f(\sigma) = \int \frac{d\alpha_1}{\alpha_1} \wedge \ldots \wedge \frac{d\alpha_8}{\alpha_8}\, \delta^{3\times 4}\left(C(\alpha)\cdot\tilde{\eta}\right)\, \delta^{3\times 2}\left(C(\alpha)\tilde{\lambda}\right)\, \delta^{2\times 3}\left(\lambda\cdot C(\alpha)^{\perp}\right)$$

$$C = \begin{pmatrix} 1 & \alpha_1+\alpha_3 & \alpha_2\alpha_3 & 0 & 0 & 0 \\ 0 & 1 & \alpha_2+\alpha_7+\alpha_5 & \alpha_4(\alpha_7+\alpha_5) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 1 & \alpha_4 & \alpha_6 & \alpha_8 \end{pmatrix}$$

$$\underset{\uparrow}{\Rightarrow} f(\sigma) = \mathcal{A}_4^{(2)} \otimes \mathcal{A}_4^{(2)} = f_{\underset{\sigma}{\underset{\parallel}{(457689)}}} = \frac{\delta^{3\times 4}(C\cdot\tilde{\eta})\,\delta^{2\times 2}(\lambda\cdot\tilde{\lambda})}{(123)(234)(345)(561)(612)}$$ with another

Exercise

$$C = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ [23] & [31] & [12] & 0 & 0 & 0 \end{pmatrix}.$$