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Polylogarithms and physical applications

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0.1 The cast of characters: polylogarithms and zeta values

There are many types of polylogarithms. We will not give a general definition, but mention just some of the most common types.

- Goncharov polylogarithms. For $a_1, \ldots, a_n, x \in \mathbb{C}$, we define

$$G(a_1, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t), \tag{1}$$

  with $G(a; x) = \int_0^x \frac{dt}{t-a}$. The integration contours are taken along some paths in the complex plane. They are multi-valued functions. We call n the weight (or transcendentality) of the polylogarithm.

- Multiple polylogarithms defined as power series

$$\mathrm{Li}_{n_1,\ldots,n_k}(x_1, \ldots, x_k) = \sum_{1 \le p_1 < \ldots < p_k} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}. \tag{2}$$

  These power series are convergent in a polydisc $|x_i| < 1$. For $\mathrm{Li}_{n_1,\ldots,n_k}(x_1, \ldots, x_k)$ we call k the depth and $n_1 + \cdots n_k$ the weight.

  These polylogarithms can be written in terms of Goncharov polylogarithms as

$$\mathrm{Li}_{n_1 \cdots n_k}(x_1, \ldots, x_k) = (-1)^k G\Big(\underbrace{0, \ldots, 0}_{n_k - 1} \frac{1}{x_k}, \ldots, \frac{1}{x_2 \cdots x_k}, \underbrace{0, \ldots, 0}_{n_1 - 1}, \frac{1}{x_1 \cdots x_k}; 1\Big). \tag{3}$$

- Another notation is

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t). \tag{4}$$

  Obviously we have

$$G(a_1, \ldots, a_n; x) = I(0; a_n, \ldots, a_1; x), \tag{5}$$

  where the a arguments are reversed in the I function with respect to the G function.

- From these functions we can find particular cases which are simpler. The "classical" polylogarithms are defined by (and studied by Euler, Abel, Kummer)

$$\mathrm{Li}_n(x) = \sum_{p=1}^{\infty} \frac{x^p}{p^n}. \tag{6}$$

We also have

$$\mathrm{Li}_n(x) = -G\Big(\underbrace{0,\ldots,0}_{n-1},\frac{1}{x};1\Big) = -\int_0^x \frac{dt}{t} G\Big(\underbrace{0,\ldots,0}_{n-2},\frac{1}{t};1\Big) = \int_0^x \frac{dt}{t}\,\mathrm{Li}_{n-1}(t). \tag{7}$$

The function $\mathrm{Li}_1(x)$ is just the usual logarithm

$$\mathrm{Li}_1(x) = -G\Big(\frac{1}{x};1\Big) = -\int_0^1 \frac{dt}{t-x^{-1}} = \int_0^x \frac{dt}{1-t} = -\ln(1-x). \tag{8}$$

- The polylogarithm $\mathrm{Li}_n(z)$ can be written as

$$\mathrm{Li}_n(z) = \int_{0\leq 1-t_1\leq t_2\leq \ldots\leq t_n\leq z} \frac{dt_1}{t_1}\wedge\ldots\wedge\frac{dt_n}{t_n}. \tag{9}$$

  Aomoto considered a more general situation. Suppose we have an n-simplex Δ_L defined by a set of hyperplanes $(L_0,\ldots,L_n)$ and an n-form ω_M with logarithmic poles on a set of hyperplanes $(M_0,\ldots,M_n)$. The Aomoto polylogarithm $\Lambda_n(L_0,\ldots,L_n;M_0,\ldots,M_n)$

$$\Lambda_n(L_0,\ldots,L_n;M_0,\ldots,M_n) = \int_{\Delta_L}\omega_M. \tag{10}$$

  If we work in projective space and dualize the hyperplanes to points, then the Aomoto polylogarithm depends on configurations of $2(n+1)$ points in $\mathbb{CP}^{n-1}$.

Why are such functions interesting?

- $\mathrm{Li}_n(1) = \zeta(n)$. More generally, $\mathrm{Li}_{n_1,\ldots,n_k}(1,\ldots,1) = \zeta(n_1,\ldots,n_k)$, where the right hand side is a "multiple zeta value" (or MZV). The multiple zeta values satisfy interesting algebraic relations. One old example, found by Euler, is $\zeta(1,2) = \zeta(3)$. Such identities are interesting for number theorists. The multiple zeta values are also of interest to physicists. They appear in the computation of vacuum diagrams in field theory [REFS] and also in the α' expansion of string theory scattering amplitudes [REFS].

- Iterated integrals appear frequently in practice and one can show that the polylogarithms defined above are universal, in the following sense. An iterated integral $\int^z \omega_1\circ\cdots\circ\omega_n$ of rational 1-forms ω_i on a rational variety X can be written in terms of hyperlogarithms, whose arguments are rational fractions in z (see proposition 18 in ref. [22] for a more precise statement).

- Motives, K-theory, etc.

- In physics, polylogarithms appear in several ways. One way they appear is when computing Feynman integrals.

- Polylogarithms also appear in integrable models. For example, suppose we deform a two-dimensional CFT by a relevant operator. In some cases we obtain massive two-dimensional field theories with integrable S matrix. In order to find this S matrix exactly some assumptions are made. One way to check them is to use the thermodynamic Bethe ansatz (TBA), by formulating the theory on a cylinder of radius r. The ground state energy $E(r)$ can be computed from the knowledge of the mass spectrum and of the S matrix. In the UV limit $r \to 0$ we recover the original CFT whose ground state energy behaves like $\frac{-\pi\tilde{c}}{6r}$, where $\tilde{c}$ is the effective central charge. In many cases of interest the function $E(r)$ can be expressed in terms of dilogarithms in the limit $r \to 0$ and this yields dilogarithm identities. See refs. [17, 15].

0.2 Iterated integrals

A reference for this section is the paper [5] by K-T Chen.

The simplest iterated integral is in one dimension. We introduce a notation

$$\int_a^b f_1(t)dt \circ \cdots \circ f_r(t)dt = \int_a^b \left(\int_a^t f_1(u)du \circ \cdots \circ f_{r-1}(u)du \right) f_r(t)dt. \quad (11)$$

This is a recursive definition.

If α and β are paths (i.e. maps $\alpha, \beta : [0,1] \to X$ where X is some manifold), define the product $\alpha\beta$ to be the path α followed by the path β. Also, define the inverse path $\gamma = \alpha^{-1}$, by $\gamma(t) = \alpha(1-t)$.

If $w_1, \ldots, w_r$ are 1-forms on X, then we define the iterated integral on the path α by

$$\int_\alpha w_1 \circ \cdots \circ w_r = \int_0^1 \alpha^* w_1 \circ \cdots \circ \alpha^* w_r, \quad (12)$$

where $\alpha^* w_i$ is the pullback[1] of the 1-form w_i on the path α.

[1] If you are not familiar with the notion of pullback, here is the definition. If $w = \sum_i f_i dx^i$ is a 1-form on X, with x^i some local coordinates, and α is a map (in our case $\alpha : [0,1] \to X$), then we define the pullback $\alpha^* w = \sum_i f_i \frac{dx^i}{dt} dt$. This is a 1-form on the interval $[0,1]$.

These iterated integrals have the following properties

$$\int_\alpha w_1 \circ \cdots \circ w_r \int_\alpha w_{r+1} \circ \cdots \circ w_{r+s} = \sum_\sigma \int_\alpha w_{\sigma(1)} \circ \cdots \circ w_{\sigma(r+s)}, \tag{13}$$

where the sum is over all (r, s) shuffles. The (r, s) shuffles are permutations σ of $r + s$ letters with $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r + 1) < \cdots < \sigma^{-1}(r + s)$.

The σ^{-1} may look strange but they are correct. The shuffle product of the sets $\{1, \ldots, r\}$ and $\{r + 1, \ldots, r + s\}$ is the set of all permutations of $\{1, \ldots, r + s\}$ such that $1, \ldots, r$ and $r + 1, \ldots, r + s$ always appear ordered. Then, in eq. (13) w_i appears at position $\sigma^{-1}(i)$ in the right-hand side. This implies $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r + 1) < \cdots < \sigma^{-1}(r + s)$.

The sum over all the (r, s) shuffles is the *shuffle product*

$$(w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s}) = \sum_{\sigma \in (r,s) \text{ shuffles}} w_{\sigma(1)} \circ \cdots \circ w_{\sigma(r+s)}. \tag{14}$$

We will justify eq. 13 on a two-dimensional case.

$$\int_0^1 f_1(t_1) dt_1 \int_0^1 f_2(t_2) dt_2 = \int_{0 \le t_1 < t_2 \le 1} dt_1 dt_2 f_1(t_1) f_2(t_2) + \\ \int_{0 \le t_2 < t_1 \le 1} dt_1 dt_2 f_1(t_1) f_2(t_2) = \int f_1(t)dt \circ f_2(t)dt + \int f_2(t)dt \circ f_1(t)dt. \tag{15}$$

This argument can be extended recursively without too much difficulty. The recursive proof uses the following recursive definition of the shuffle product

$$(w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s}) = w_1 \circ ((w_2 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s})) + \\ w_{r+1} \circ ((w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+2} \circ \cdots \circ w_{r+s})). \tag{16}$$

With respect to composition of the paths α and β we have the property

$$\int_{\alpha\beta} w_1 \cdots w_r = \int_\alpha w_1 \cdots w_r + \cdots \int_\alpha w_1 \cdots w_i \int_\beta w_{i+1} \cdots w_r + \cdots \int_\beta w_1 \cdots w_r. \tag{17}$$

When evaluating the iterated integral on the inverse path we get

$$\int_{\alpha^{-1}} w_1 \cdots w_r = (-1)^r \int_\alpha w_r \cdots w_1. \tag{18}$$

Suppose we take a path in α in X and we define a functional of the path α by

$$F[\alpha] = \int_\alpha w_1 \circ \cdots \circ w_k. \tag{19}$$

Then we say that F is independent on the path α if the path can be locally deformed[2] while keeping its end points fixed, without changing the value of F.[3]

If we consider the path independent functional $F[\alpha]$ as a function of the endpoint $\alpha(1) \in X$, then we have

$$dF[\alpha] = w_k \int_\alpha w_1 \circ \cdots \circ w_{k-1}, \tag{20}$$

where w_k is evaluated at the endpoint of the path. This is basically the fundamental theorem of calculus for the case of iterated integrals.

0.3 Transcendental functions. Symbols

Let us now define some transcendental functions as iterated integrals. We start by giving some examples before giving the general definitions.

The simplest iterated integral we will consider is an integral of a differential form $w = d \ln f(x)$, where $x \in X$ and $f(x)$ is a rational fraction. For reasons which will become clear later we impose the restriction that the rational fraction $f(x)$ be a rational fraction with rational coefficients. This differential form can be trivially integrated and the functions defined in this way are logarithms with rational fractions as arguments.

The next example is a twice iterated integral[4]

$$\mathrm{Li}_2(z) = \int_0^z \left(\int_0^t \frac{du}{1-u} \right) \frac{dt}{t} = -\int_0^z dt \frac{\ln(1-t)}{t}. \tag{21}$$

[2]While deforming the path α we avoid crossing any singularities (like poles) that the differential forms w_i may have.

[3]In more mathematical terms, F is independent on the homotopy class (relative to the endpoints) of the path α. A relative homotopy class is defined as the equivalence class of paths related by continuous transformations which keep some part of the manifold fixed, in this case the endpoints.

[4]For now you can think about this integral as being taken along the real axis, for $0 < z < 1$. Of course, this function can be analytically continued for complex values of z.

This function is called the dilogarithm. It has a regular series expansion around $z = 0$, which can be found by using

$$-\ln(1-t) = \sum_{k=1}^{\infty} \frac{t^n}{n} \tag{22}$$

and exchanging the order of summation and integration. We get

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^n}{n^2}. \tag{23}$$

Then, we define recursively

$$\mathrm{Li}_n(z) = \int_0^z dt \frac{\mathrm{Li}_{n-1}(t)}{t}, \tag{24}$$

with $\mathrm{Li}_1(z) = -\ln(1-z)$. The differential relation is

$$z \frac{\partial}{\partial z} \mathrm{Li}_n(z) = \mathrm{Li}_{n-1}(z). \tag{25}$$

One can show recursively that

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \tag{26}$$

In the notation of sec. 0.2 we have

$$\mathrm{Li}_n(z) = -\int_0^z d\ln(1-t) \circ \underbrace{d\ln t \circ \cdots \circ d\ln t}_{n-1}, \tag{27}$$

where the integral is taken along some path from 0 to z.

The knowledge of the differential forms and their ordering is enough to define a multivalued function, if the integral is independent on small variations of the path. We will condense this knowledge in an object called *symbol*, which is denoted by

$$\mathcal{S}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{n-1}. \tag{28}$$

Note that the differential forms are ordered as in the iterated integral and are evaluated at the endpoint of the integration path.

If we have a function written as an iterated integral with differential forms $w_i = d\ln f_i$, then the symbol can be written immediately. But if we don't know such a form, then a way to compute the symbol is to use eq. (20) recursively. Let us see how this works on an example.

Starting from the series expansion

$$\mathrm{Li}_{a,b}(x,y) = \sum_{n>m\geq 1} \frac{x^n}{n^a}\frac{y^m}{m^b}, \tag{29}$$

we easily obtain

$$d\,\mathrm{Li}_{a,b}(x,y) = d\ln x\,\mathrm{Li}_{a-1,b}(x,y) + d\ln y\,\mathrm{Li}_{a,b-1}(x,y). \tag{30}$$

We also get

$$\mathrm{Li}_{a,0}(x,y) = \frac{1}{1-y}\left(y\,\mathrm{Li}_a(x) - \mathrm{Li}_a(xy)\right), \qquad \mathrm{Li}_{0,a}(x,y) = \frac{x}{1-x}\,\mathrm{Li}_b(xy). \tag{31}$$

Using this we find

$$d\,\mathrm{Li}_{a,1}(x,y) = d\ln x\,\mathrm{Li}_{a-1,1}(x,y) - d\ln(1-y)\,\mathrm{Li}_a(x) - d\ln\frac{y}{1-y}\,\mathrm{Li}_a(xy), \tag{32}$$

$$d\,\mathrm{Li}_{1,b}(x,y) = -d\ln(1-x)\,\mathrm{Li}_b(xy) + d\ln y\,\mathrm{Li}_{1,b-1}(x,y), \tag{33}$$

$$d\,\mathrm{Li}_{1,1}(x,y) = -d\ln(1-x)\,\mathrm{Li}_1(xy) - d\ln(1-y)\,\mathrm{Li}_1(x) - d\ln\left(\frac{y}{1-y}\right)\mathrm{Li}_1(xy). \tag{34}$$

So the symbol is

$$\mathcal{S}(\mathrm{Li}_{1,1}(x,y)) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y) + (1-xy)\otimes\left(\frac{y}{1-y}\right). \tag{35}$$

From the definition, it is clear that the symbols satisfy some properties. For example, from $d\ln(xy) = d\ln x + d\ln y$ we find that

$$\cdots\otimes(xy)\otimes\cdots = \cdots\otimes x\otimes\cdots + \cdots\otimes y\otimes\cdots. \tag{36}$$

Also, a purely numeric entry $c\in\mathbb{Q}$ gives zero because $d\ln c = 0$. So we have

$$\cdots\otimes c\otimes\cdots = 0. \tag{37}$$

Using these properties, we find

$$\begin{aligned}\mathcal{S}(\mathrm{Li}_{1,1}(x,y)) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)+ \\ (1-xy)\otimes y - (1-xy)\otimes(1-y).\end{aligned} \tag{38}$$

0.4 Homotopy invariance

Let us start with an integral of a closed one-form w_1. Because w_1 is closed, it is locally exact. This implies via Stokes theorem that the integral

$$\int_\alpha w_1 \tag{39}$$

is independent on small variations of the path α. For example if $w_1 = \frac{dt}{1-t}$, then the integral

$$\int_0^z \frac{dt}{1-t} = -\ln(1-z) \tag{40}$$

along a path from 0 to z defines a multivalued function. If the integration path can be deformed without encountering the point 1, then the value of the integral does not change.

Consider now the case where we have a double iterated integral,

$$I = \int_\alpha w_1 \circ w_2, \tag{41}$$

and define $F(z) = \int^z w_1$. Then, the definition of the iterated integral implies that

$$I = \int_\alpha w_1 \circ w_2 = \int_\alpha F w_2. \tag{42}$$

We obtain that I is independent on small variations of the path α is $d(Fw_2) = 0$. Using $dw_2 = 0$ and $dF = w_1$, we obtain $w_1 \wedge w_2 = 0$. This is an *integrability condition.*

We can also consider a more general case

$$I = \int_\alpha \sum_{i,j} w_i \circ w_j. \tag{43}$$

In this case, a similar argument implies the integrability condition $\sum_{i,j} w_i \wedge w_j$. As an example, consider the symbol of $\mathrm{Li}_{1,1}(x,y)$ in eq. (38). In this case, we obtain the following expression for the integrability condition

$$\left(\frac{-xdy - ydx}{1-xy}\right) \wedge \left(\frac{-dx}{1-x}\right) + \left(\frac{-dx}{1-x}\right) \wedge \left(\frac{-dy}{1-y}\right) + \left(\frac{-xdy - ydx}{1-xy}\right) \wedge \left(\frac{dy}{y}\right) - \left(\frac{-xdy - ydx}{1-xy}\right) \wedge \left(\frac{-dy}{1-y}\right) =$$

$$dx \wedge dy \left(-\frac{x}{(1-xy)(1-x)} + \frac{1}{(1-x)(1-y)} - \frac{1}{1-xy} - \frac{y}{(1-xy)(1-y)}\right) = 0. \tag{44}$$

We have shown that in this case homotopy invariance holds, but in a non-trivial way (detailed cancellations between different terms are required).

In general, for an iterated integral

$$I = \int_\alpha w_1 \circ \cdots \circ w_r, \tag{45}$$

the integrability conditions are $w_i \wedge w_{i+1} = 0$, for $i = 1, \ldots, r-1$.

The most general case is[5]

$$I = \int_\alpha \sum_{i_1,\ldots,i_r} w_{i_1} \circ \cdots \circ w_{i_r}. \tag{46}$$

How should we think about integrability in this general case? Obviously the conditions that $w_{i_l} \wedge w_{i_{l+1}} = 0$ for $l = 1, \ldots, r-1$ are sufficient but not necessary. It may happen that $w_{i_l} \wedge w_{i_{l+1}} \neq 0$ but the remaining w_{i_k} add up to zero.

The right way to express the integrability conditions in this case is to use the notion of symbol again. We have the I is independent on the path if

$$\sum_{l=1}^{r-1} \sum_{i_1,\ldots,i_r} w_{i_1} \otimes \cdots \otimes \left(dw_{i_l} \wedge dw_{i_{l+1}}\right) \otimes \cdots \otimes w_{i_r} = 0. \tag{47}$$

In order to make this more clear, let us work out an example. We have

$$d\,\mathrm{Li}_{2,1}(x,y) = d\ln x\, \mathrm{Li}_{1,1}(x,y) - d\ln(1-y)\, \mathrm{Li}_2(x) - d\ln\left(\frac{y}{1-y}\right) \mathrm{Li}_2(xy). \tag{48}$$

Using this and the symbol of $\mathrm{Li}_{1,1}(x,y)$ already computed we find that

$$\begin{gathered}\mathcal{S}(\mathrm{Li}_{2,1}(x,y)) = (1-xy)\otimes(1-x)\otimes x + (1-x)\otimes(1-y)\otimes x + \\ (1-xy)\otimes y\otimes x - (1-xy)\otimes(1-y)\otimes x + (1-x)\otimes x\otimes(1-y) + \\ (1-xy)\otimes x\otimes y - (1-xy)\otimes x\otimes(1-y) + (1-xy)\otimes y\otimes y - (1-xy)\otimes y\otimes(1-y).\end{gathered} \tag{49}$$

The check of integrability in the first two entries is very similar to the one we did for $\mathrm{Li}_{1,1}(x,y)$. Therefore we will focus on the last two entries. We

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[5]In fact there are some other slightly more general cases (see [5]), like $\int w_1 \circ w_2 + w_{12}$, where w_1, w_2 and w_{12} are one-forms. In this case the integrability condition is $w_1 \wedge w_2 + dw_{12} = 0$. However, we will not consider these cases where we combine iterated integrals of different lengths.

get

$$(1-xy)\otimes\left(\frac{dy}{y}\wedge\frac{dx}{x}-\frac{-dy}{1-y}\wedge\frac{dx}{x}+\frac{dx}{x}\wedge\frac{dy}{y}-\frac{dx}{x}\wedge\frac{-dy}{1-y}\right)+ \\ (1-x)\otimes\left(\frac{-dy}{1-y}\wedge\frac{dx}{x}+\frac{dx}{x}\wedge\frac{-dy}{1-y}\right)=0. \quad (50)$$

So we have shown that the symbol of $\mathrm{Li}_{2,1}(x,y)$ is integrable.

If a multivalued function is defined as an iterated integral of length k with the forms $w_i = d\ln f_i$, $i = 1,\dots,k$, then we say that the function has *transcendentality* k. There are no nontrivial[6] vanishing linear combinations with rational coefficients, of functions of different transcendentality. In particular, this means that the notion of transcendentality weight is well defined (if one could write the same function as iterated integrals of different lengths, it would not be possible to unambiguously define a notion of transcendentality weight).

Now we can finally explain the reason for the restriction that the arguments of the transcendental functions be rational functions with rational coefficients. If we allow e as an argument then we have $\ln e = 1$, where the right-hand side has transcendentality zero while the left-hand side is a transcendentality one function, evaluated at a transcendental value.

0.5 Integrating the symbol

We have shown how to compute the symbol of a transcendental function, but we have not said why this is useful. The symbol is very useful to check identities between transcendental functions since it reduces the check to algebraic manipulation.

But suppose we don't want to check an identity, but we have a complicated combination of transcendental functions (like in refs. [6, 7]) which we suspect can be simplified. Then we can compute the symbol of this combination of functions and, if we are lucky, we notice that the symbol is "simpler" than the original function so it is likely that the result can be simplified (in other words, it is possible to find a simpler combination of transcendental functions which has the same symbol).

This is the question we will address in this section: given a symbol, how do we find the simplest function with that symbol? Before we start

[6] As an example of trivial relations, we have linear combinations of transcendental functions of different transcendentality, where the parts with the same transcendentality cancel among themselves.

the discussion we should mention that a general answer to this question is not known. In the cases of transcendentality two and three some powerful theorems guide us, but for higher transcendentality we will typically only be able to give partial answers.

Moreover, by using this method we discard some information because computing the symbol by the method described in sec. 0.3 consists in taking derivatives. In general, this method will not reproduce terms containing powers of π or terms proportional to zeta values. Such terms can be found by other methods (see examples below), like imposing appropriate analyticity constraints or by using a modified version of the symbol which discards less information (see refs. [4, 8]).

We will organize the discussion according to transcendentality. The transcendentality one functions are logarithms. They have symbols of length one which can be straightforwardly integrated.

The symbols of transcendentality two functions can be decomposed in symmetric and antisymmetric parts. The symmetric part can be integrated to a product of logarithms while the antisymmetric part, if it satisfies the integrability constraints, can be integrated to dilogarithms.

More explicitly, for the symmetric part we can use,

$$x \otimes y + y \otimes x = \mathcal{S}(\ln x \ln y), \qquad x \otimes x = \mathcal{S}\frac{1}{2}\ln^2(x). \tag{51}$$

The antisymmetric part is more complicated. Let us illustrate the procedure on the example of $\mathrm{Li}_{1,1}(x,y)$. Recall that

$$\mathcal{S}(L_{1,1}(x,y)) = (1-xy) \otimes \frac{y(1-x)}{1-y} + (1-x) \otimes (1-y). \tag{52}$$

The RHS can be rewritten by elementary algebraic manipulations

$$\begin{aligned}&\Big(1+\frac{y(1-x)}{1-y}\Big) \otimes \frac{y(1-x)}{1-y} + (1-y) \otimes \frac{y}{1-y} + (1-x) \otimes_S (1-y) = \\ &\mathcal{S}\Big(-\mathrm{Li}_2\Big(-\frac{y(1-x)}{1-y}\Big) - \mathrm{Li}_2(y) - \frac{1}{2}\ln^2(1-y) + \ln(1-x)\ln(1-y)\Big),\end{aligned} \tag{53}$$

where $\otimes_S$ is the symmetric tensor product denoted sometimes by $\odot$.

So we have shown that the difference $*$ defined below

$$-\mathrm{Li}_2\Big(-\frac{y(1-x)}{1-y}\Big) - \mathrm{Li}_2(y) - \frac{1}{2}\ln^2(1-y) + \ln(1-x)\ln(1-y) - \mathrm{Li}_{1,1}(x,y) = * \tag{54}$$

has symbol zero. Since there are no relations between functions of different transcendentality, this means that $*$ is a linear combination of $\pi \times \ln$ and

π^2 terms, with rational coefficients. However, it is easy to show that $*$ has a regular series expansion around $(x, y) = (0, 0)$ with rational coefficients. Since $*$ contains π the only way for the equality to hold at the level of coefficients of the power series is if $* = 0$. Therefore,

$$-\operatorname{Li}_2\Big(-\frac{y(1-x)}{1-y}\Big) - \operatorname{Li}_2(y) - \frac{1}{2}\ln^2(1-y) + \ln(1-x)\ln(1-y) = \operatorname{Li}_{1,1}(x, y). \tag{55}$$

This is the first nontrivial identity we found using symbols.

Note that the function $\operatorname{Li}_2(x)$ has a well-defined power series expansion around $x = 0$, but its symbol is not antisymmetric. We can define a function (Rogers L function)

$$L(x) = \operatorname{Li}_2(x) + \frac{1}{2}\ln(x)\ln(1-x), \tag{56}$$

whose symbol is antisymmetric

$$\mathcal{S}L(x) = -\frac{1}{2}(1-x)\otimes x + \frac{1}{2}x\otimes(1-x) \equiv -\frac{1}{2}(1-x)\wedge x, \tag{57}$$

but at the cost of not having a well-defined power series around $x = 0$.

It is a theorem that all the transcendentality two functions can be expressed as linear combinations over the rational numbers of dilogarithms, $\ln\times\ln$, $\pi\times\ln$ and π^2 terms. Neglecting for now the terms of type $\pi\times\ln$ and π^2, we have shown above that the terms with antisymmetric symbol can be written in terms of Rogers L functions, while the terms with symmetric symbol can be written as $\ln\times\ln$.

If we allow ourselves to use the logarithm identities $\ln(xy) = \ln x + \ln y$ (which are true up to πi), then a symmetric symbol can be integrated uniquely in terms of logarithms.

For the dilogarithms the situation is more complicated since they satisfy some nontrivial identities. The most important among them is the five-term identity. It has been discovered and rediscovered by many people among which Spence, Abel, Hill, Kummer, Schaeffer, etc.

We present here this five-term identity in a way which makes its $\mathbb{Z}_5$ symmetry manifest. Define a sequence a_n recursively by $1 - a_n = a_{n-1}a_{n+1}$. It is easy to show that this recursion relation has a periodicity five (try it!). A slightly modified version of this recursion is a simple example of cluster coordinate mutation, which will be discussed in more detail in sec. 0.11.

Now compute the following combination

$$\mathcal{S}\left(\sum_{n=1}^{5} L(a_n)\right) = \sum_{n=1}^{5} -\frac{1}{2}(1-a_n)\wedge a_n = -\frac{1}{2}\sum_{n=1}^{5}(a_{n-1}a_{n+1})\wedge a_n = 0, \tag{58}$$

where L is Rogers function. The left-over "subleading functional transcendental part" can only be a constant, which turns out to be, $\zeta(2) = \frac{\pi^2}{6}$, so we get

$$\sum_{n=1}^{5} L(a_n) = \sum_{n=1}^{5} (\mathrm{Li}_2(a_n) + \ln a_{n-1} \ln a_n) = \frac{\pi^2}{6}. \tag{59}$$

In order to get the more familiar form of the five-term identity we set

$$a_1 = x, \quad a_2 = \frac{1-x}{1-xy}, \quad a_3 = \frac{1-y}{1-xy}, \quad a_4 = y, \quad a_5 = 1-xy. \tag{60}$$

In fact, all the dilogarithm identities are consequence of the five-term identity (see ref. [3]).

Before moving on to higher transcendentality, let us recapitulate the main lessons we learned so far in this section. First, we saw that we can separate a part of the symbol (the antisymmetric part for transcendentality two) which can not be written as products of lower transcendentality functions. Second, we showed that, due to dilogarithm identities, there is no unique canonical way to write a transcendental function. In order to extend the methods above to higher transcendentality we need to answer two questions: "What is the higher transcendentality counterpart of antisymmetrization?" and "What are the higher transcendentality analogs of dilogarithms and what are the relations between them?"

Let us start with the first question. We have presented in sec. 0.2 a formula for how to take the product of two functions defined by iterated integrals. This can be particularized to the case of transcendental functions, as follows. If two functions F and G have symbols $a_1 \otimes \cdots \otimes a_n$ and $b_1 \otimes \cdots \otimes b_m$, then the symbol of the product FG is given by the shuffle product of the symbols

$$(a_1 \otimes \cdots \otimes a_n) \sqcup\!\sqcup (b_1 \otimes \cdots \otimes b_m) = a_1 \otimes ((a_2 \otimes \cdots \otimes a_n) \sqcup\!\sqcup (b_1 \otimes \cdots \otimes b_m)) + b_1 \otimes ((a_1 \otimes \cdots \otimes a_n) \sqcup\!\sqcup (b_2 \otimes \cdots \otimes b_m)). \tag{61}$$

We define an operation ρ

$$\rho(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \rho(a_2 \otimes \cdots \otimes a_n) - a_n \otimes \rho(a_1 \otimes \cdots \otimes a_{n-1}). \tag{62}$$

Prove this!

This operator annihilates all the shuffle products

$$\rho((a_1 \otimes \cdots \otimes a_n) \sqcup\!\sqcup (b_1 \otimes \cdots \otimes b_m)) = 0. \tag{63}$$

There are other projectors which can be used to extract the different kinds of products as well (see refs. [23, 9] for more details).

Still, even after applying the ρ projection the symbol can be too complicated to handle. One idea is to integrate only parts of it. If we pick appropriate pieces the integrability condition still holds and therefore those shorter symbols correspond to actual functions. In this way we produce several functions (more accurately equivalence classes of functions under the symbol map) out of a single function. Algebraically this has the structure of a coproduct.

In the case of $\mathrm{Li}_n(x)$, this procedure is easy to carry out: we split the symbol of $\mathrm{Li}_n(x)$ in two parts, one of length k and the other of length $n-k$ for $k = 0, \ldots, n$ and them we integrate them to functions:

$$\Delta \,\mathrm{Li}_n(x) = \sum_{k=0}^{n} -[\underbrace{(1-x)\otimes \cdots \otimes x}_{k}] \otimes [\underbrace{x \otimes \cdots \otimes x}_{n-k}]$$
$$\to \sum_{k=0}^{n} \mathrm{Li}_k(x) \otimes \frac{\ln^{n-k}(x)}{(n-k)!} = \mathbf{1} \otimes \mathrm{Li}_n(x) + \sum_{k=1}^{n-1} \mathrm{Li}_k(x) \otimes \frac{\ln^{n-k}(x)}{(n-k)!} + \mathrm{Li}_n(x) \otimes \mathbf{1}. \tag{64}$$

This procedure can be turned around. We start by defining a "coproduct" on functions (the reason for the quotation marks is that in order to show that this is a coproduct we need to understand all the algebraic relations between the transcendental function to which it is applied and this understanding is not available yet). This coproduct is coassociative, and by applying it n times we obtain the symbol.

In the following we will mostly be interested in yet another coproduct, called δ which differs from Δ by neglecting the products and also terms of type $\mathbf{1} \otimes *$ and $* \otimes \mathbf{1}$. The procedure is best described on an example. We start with the symbol of the $\mathrm{Li}_3(x)$ function, then we apply the projection ρ and we split the resulting symbol in pieces of length two at the beginning and pieces of length one at the end.[7] Finally, we apply ρ to each of the two groups.

$$\mathrm{Li}_3(x) \xrightarrow{\mathcal{S}} -(1-x)\otimes x \otimes x \xrightarrow{\rho} x \otimes (1-x) \wedge x \xrightarrow{\pi_{2,1}} [x\otimes(1-x)]\otimes x - [x \otimes x] \otimes (1-x)$$
$$\xrightarrow{\rho\otimes\rho} [-(1-x)\wedge x] \otimes x = -\{x\}_2 \otimes x, \tag{65}$$

where we have used the shorthand notation $\{x\}_2 = (1-x) \wedge x$. Also, we have denoted by $\pi_{2,1}$ the operation which groups together the first two

[7] We could also do a $(1,2)$ split instead of a $(2,1)$ split but it turns out that when applying the operations $(\rho \otimes \rho) \circ \pi_{1,2} \circ \rho$ on a Li_3 symbol we obtain zero. Therefore, such an operation can not detect trilogarithms.

arguments; this notation is useful for describing the projection $\rho\otimes\rho$, where the first ρ acts on the first two arguments and the last second ρ acts on the last argument.

Let us do the same for the $\mathrm{Li}_{2,1}(x,y)$ function

$$\begin{aligned}&\mathrm{Li}_{2,1}(x,y)\xrightarrow{\mathcal{S}}(1-xy)\otimes(1-x)\otimes x+(1-x)\otimes(1-y)\otimes x+\\&(1-xy)\otimes y\otimes x-(1-xy)\otimes(1-y)\otimes x+(1-x)\otimes x\otimes(1-y)+\\&(1-xy)\otimes x\otimes y-(1-xy)\otimes x\otimes(1-y)+(1-xy)\otimes y\otimes y-(1-xy)\otimes y\otimes(1-y)\xrightarrow{(\rho\otimes\rho)\circ\pi_{2,1}\circ\rho}\\&\left(\left\{-\frac{y(1-x)}{1-y}\right\}_2+\{y\}_2\right)\otimes x-(\{x\}_2+\{y\}_2)\otimes(1-xy)+\{xy\}_2\otimes y+(-\{xy\}_2+\{x\}_2)\otimes(1-y).\end{aligned} \tag{66}$$

Using the five-term identity

$$\{x\}_2+\left\{\frac{1-x}{1-xy}\right\}_2+\left\{\frac{1-y}{1-xy}\right\}_2+\{y\}_2+\{1-xy\}_2=0, \tag{67}$$

and also $\{1-x\}_2=-\{x\}_2$ and $\{x^{-1}\}_2=-\{x\}_2$ we can rewrite the expression above as

$$\begin{aligned}&-\{x\}_2\otimes x+\{1-xy\}_2\otimes(1-xy)+\{xy\}_2\otimes xy+\\&\left\{\frac{x(1-y)}{1-xy}\right\}_2\otimes\frac{x(1-y)}{1-xy}-\left\{\frac{1-y}{1-xy}\right\}_2\otimes\frac{1-y}{1-xy}+\{1-y\}_2\otimes(1-y).\end{aligned} \tag{68}$$

Notice that now we only have terms of type $\{x\}_2\otimes x$ which we will denote by $\{x\}_3$. We have shown that the same kind of terms arise when applying this sequence of operations to trilogarithms so we are led to conclude that, when expressed in terms classical polylogarithms, $\mathrm{Li}_{2,1}(x,y)$ contains the following combination

$$-\mathrm{Li}_3(xy)+\mathrm{Li}_3\left(\frac{1-y}{1-xy}\right)-\mathrm{Li}_3\left(\frac{x(1-y)}{1-xy}\right)-\mathrm{Li}_3(1-xy)+\mathrm{Li}_3(x)-\mathrm{Li}_3(1-y). \tag{69}$$

In fact, this representation is not unique, since under the combined sequence of operations described above, the following quantity

$$\mathrm{Li}_3(z)+\mathrm{Li}_3(1-z)+\mathrm{Li}_3(1-z^{-1}) \tag{70}$$

projects to zero. Indeed, in order to eliminate the possibility of π's or ζ values of appearing, it is a good idea to use this identity to replace the Li_3's in eq. (69) by Li_3 which have a regular power expansion around $(x, y) = (0, 0)$. After doing this we subtract the symbol of the group of Li_3 from the symbol of $\mathrm{Li}_{2,1}$ and study the remaining part. The remaining functions can only be of type $\mathrm{Li}_2 \times \ln$ and $\ln \times \ln \times \ln$. They can be easily found and the full rewriting of $\mathrm{Li}_{2,1}$ is

$$\mathrm{Li}_{2,1}(x,y) = -\mathrm{Li}_3\left(-\frac{(1-x)y}{1-y}\right) - \mathrm{Li}_3\left(\frac{x(1-y)}{1-xy}\right) - \mathrm{Li}_3\left(\frac{(1-x)y}{1-xy}\right) + \mathrm{Li}_3\left(\frac{xy}{xy-1}\right) -$$
$$\mathrm{Li}_2(x)\log(1-xy) - \mathrm{Li}_2(y)\log(1-x) + \mathrm{Li}_3(x) + \mathrm{Li}_3(y) + \mathrm{Li}_3\left(\frac{y}{y-1}\right) + \frac{1}{6}\log^3(1-xy) -$$
$$\frac{1}{2}\log(1-x)\log^2(1-xy) - \frac{1}{2}\log(1-y)\log^2(1-xy) -$$
$$\frac{1}{2}\log(1-x)\log^2(1-y) + \log(1-x)\log(1-y)\log(1-xy). \quad (71)$$

This can be cross-checked by computing the power expansion around $(x, y) = (0, 0)$ and comparing to the one of $\mathrm{Li}_{2,1}(x, y)$. It is an example of how paying attention to the analytic structure of the function under study helps us fix the terms which can not be obtained from the symbol.

0.6 Coproduct

In sec. 0.5 we hinted that the operation of partial integration has the structure of a coproduct. However in order for the definition to be consistent we need to show that it is compatible with the algebraic relations between the transcendental functions. Since these relations are known only conjecturally, the existence of a coproduct is also a conjecture. However, the functions we discussed can be lifted to motivic versions, $I \to I^{\mathcal{M}}$ for which the coproduct can be defined rigorously (see ref. [20]). The coproduct acts as follows on the functions $I^{\mathcal{M}}$:

$$\Delta I^{\mathcal{M}}(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{k=0}^{n} \sum_{0=i_0<\cdots<i_{k+1}=n+1} I^{\mathcal{M}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes$$
$$\prod_{p=0}^{k} I^{\mathcal{M}}(a_{i_p}; a_{i_p+1}, \cdots, a_{i_{p+1}-1}; a_{i_{p+1}}). \quad (72)$$

This has a nice graphical interpretation which can be explained on an example (see fig. 1). We draw a semicircle and arrange the points $a_0, \ldots, a_{n+1}$ on it such that a_0 and a_{n+1} are at the beginning and at the end. Then we pick k points $a_{i_1}, \ldots, a_{i_k}$ on this semicircle. The first term in the coproduct

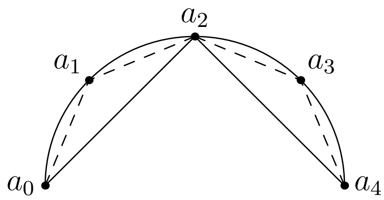


Figure 1: The term $I^{\mathcal{M}}(a_0; a_2; a_4) \otimes I^{\mathcal{M}}(a_0; a_1; a_2) I^{\mathcal{M}}(a_2; a_3; a_4)$ in the coproduct of $I^{\mathcal{M}}(a_0; a_1, a_2, a_3; a_4)$.

is $I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1})$. In between two of these k points there may be other points on the semicircle and they form $k+1$ strings. The second term in the coproduct is given by the product of $I^{\mathcal{M}}$'s with these strings as arguments.

Let us present an example of coproduct computation. Recall that

$$\mathrm{Li}_{1,1}(x, y) = G(y^{-1}, (xy)^{-1}; 1) = I(0; (xy)^{-1}, y^{-1}; 1). \tag{73}$$

The reduced coproduct $\Delta'(x) = \Delta(x) - \mathbf{1} \otimes x - x \otimes \mathbf{1}$ can be found using the general formula

$$\Delta' I(a_0; a_1, a_2; a_3) = I(a_0; a_1; a_3) \otimes I(a_1; a_2; a_3) + I(a_0; a_2; a_3) \otimes I(a_0; a_1; a_2). \tag{74}$$

We get

$$\Delta' \mathrm{Li}_{1,1}(x, y) = \Delta' I(0; (xy)^{-1}, y^{-1}; 1) = I(0; (xy)^{-1}; 1) \otimes I((xy)^{-1}; y^{-1}; 1) + I(0; y^{-1}; 1) \otimes I(0; (x \tag{75}$$

Let us compute all the weight one functions $I(a; b; c)$

$$I(a; b; c) = \int_a^c \frac{dt}{t-b} = \ln\left(\frac{c-b}{a-b}\right). \tag{76}$$

This is not well-defined if $a = b$ or $b = c$. If $a = c$ the answer is zero. In order to get a well-defined result we regularize $a \to a + \epsilon$, $c \to c + \epsilon$, expand in powers of $\ln \epsilon$ and drop the singular terms in $\ln \epsilon$. Then we have

$$I(a; b; c) \to \int_{a+\epsilon}^{c+\epsilon} \frac{dt}{t-b} = \ln\left(\frac{c-b+\epsilon}{a-b+\epsilon}\right), \tag{77}$$

so

$$I(a; b; c) = \begin{cases} \ln\left(\frac{c-b}{a-b}\right), & a \neq b, c \neq b, \\ \ln(c-b), & a = b, c \neq b, \\ \ln\left(\frac{1}{a-b}\right), & a \neq b, c = b, \\ \ln(1) = 0, & a = b = c \end{cases}. \tag{78}$$

Then, we find

$$\Delta' \operatorname{Li}_{1,1}(x,y) = \ln(1-xy) \otimes \ln\left(\frac{1-y}{1-x^{-1}}\right) + \ln(1-y) \otimes \ln(1-x). \quad (79)$$

0.7 Mathematical Preliminaries on Polylogarithm Functions

In this section we provide some more mathematical details on transcendental functions and explain how to partially integrate them. We denote by $\mathcal{L}_n$ the abelian group (under addition) of transcendental functions of weight n. An important character in this story is the Bloch group B_n, also called the classical polylogarithm group: it is the subgroup of $\mathcal{L}_n$ generated by the classical polylogarithm functions Li_n and their products.

We first consider the simplest kind of transcendental function, the logarithm. If we are working modulo $2\pi i$, then we have that $\ln z + \ln w = \ln(zw)$, for any $z, w \in \mathbb{C}$. In order to express this simple functional relation formally, define $\mathbb{Z}[\mathbb{C}^*]$ to be the free abelian group generated by $\{z\}$, with rational coefficients and z non-zero complex numbers. Concretely, elements of this group are $\{z\} + \{w\}$ and the group operation is defined in the obvious way. Then, we can quotient this group by the relations satisfied by the logarithm to obtain the logarithm group B_1,

$$B_1 = \mathbb{Z}[\mathbb{C}^*]/(\{z\} + \{w\} - \{zw\}). \quad (80)$$

This group is isomorphic to the multiplicative group of complex numbers, $\mathbb{C}^\times$.

The next simplest transcendental functions are the dilogarithms, Li_2. It can be shown that any transcendentality two function can be written as a linear combination of Li_2 functions and products of logarithms.

The dilogarithms satisfy a simple five-term functional relation. One way to express this functional relation is to consider five points on $\mathbb{CP}^1$ with coordinates $z_1, \ldots, z_5$. From any four such points we can form a cross-ratio $r(z_1, \ldots, \hat{z}_i, \ldots z_5)$, where the hatted argument is missing. We use the definition $r(i,j,k,l) = \frac{z_{ij} z_{kl}}{z_{jk} z_{li}}$. Then the five-term identity can be written as

$$\sum_{i=1}^{5} (-1)^i \operatorname{Li}_2(r(z_1, \ldots, \hat{z}_i, \ldots, z_5)) = \text{logs}, \quad (81)$$

where we have denoted by logs the terms which can be written uniquely in terms of logarithms. There is a theorem (see ref. [3]) that all the relations between dilogarithms are consequences of the five-term relations. We can now define the Bloch group B_2 by analogy to the logarithm case. We first define $\mathbb{Z}[\mathbb{C}]$ to be the free abelian group generated by $\{z\}_2$, where z is a non-zero complex number. Then, we quotient be the five-term relations and the quotient is denoted by B_2

$$B_2 = \mathbb{Z}[\mathbb{C}]/(\text{five-term relations}). \tag{82}$$

In this case we have a group morphism δ, $B_2 \xrightarrow{\delta} \Lambda^2\mathbb{C}^*$ which is defined by $\delta(\{z\}_2) = (1-z) \wedge z$. To check that this is a group morphism we need to show that $\delta(\text{five-term relation}) = 0$ or

$$\sum_{i=1}^{5} (-1)^i (1 - r(z_1, \dots, \hat{z}_i, \dots, z_5)) \wedge r(z_1, \dots, \hat{z}_i, \dots, z_5) = 0, \tag{83}$$

which can be done by a short computation.

Let us now discuss Li_3 functions. There is a theorem stating that all transcendentality three functions can be written as a linear combination of Li_3 and products of lower transcendentality functions (see ref. [21]).

Just like in the previous cases, we first need to find the functional relations satisfied by Li_3 functions. The identity satisfied by Li_3 is very similar to the one satisfied by Li_2 and can be described in terms of configurations of seven points on $\mathbb{CP}^2$. It is convenient to describe each of these points in terms of their homogeneous $v_i \in \mathbb{C}^3$ coordinates, with $i = 1, \dots, 7$. For three such vectors v_i, v_j, v_k we can define a three-bracket $\langle \cdot, \cdot, \cdot \rangle : \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}$ by the volume of the parallelepiped generated by them $\langle i, j, k \rangle = \text{Vol}(v_i, v_j, v_k)$.

Given six points in $\mathbb{CP}^3$, we can form a cross-ratio

$$r_3(1, 2, 3, 4, 5, 6) = \frac{\langle 124 \rangle \langle 235 \rangle \langle 316 \rangle}{\langle 125 \rangle \langle 236 \rangle \langle 314 \rangle}. \tag{84}$$

Such cross-ratios have been introduced and extensively used in ref. [21] and we also discuss their geometric interpretation in sec. .2. The Li_3 functional relations can be expressed in terms of this cross-ratio as

$$\sum_{i=1}^{7} (-1)^i \, \text{Alt}_6 \, \text{Li}_3(r_3(1, \dots, \hat{i}, \dots, 7)) \approx 0, \tag{85}$$

where Alt_6 mean antisymmetrization in the six points on which r_3 depends and $\approx$ means that we have omitted the terms which are products of lower transcendentality functions.

Now we define

$$B_3 = \mathbb{Z}[\mathbb{C}]/(\text{seven-term relations}). \tag{86}$$

There is a morphism $\delta : B_3 \to B_2 \otimes \mathbb{C}^*$, $\delta(\{x\}_3) = \{x\}_2 \otimes x$. In order to show that this morphism is well-defined, we need to show that that δ annihilates the seven-term relations.

It may seem that we can continue in the same way to higher transcendentality. However, this is not the case. At transcendentality four there are new functions which can not be expressed in terms of Li_4 and products of lower transcendentality functions. We can define B_n for $n \geq 4$ in the same way as before, but there is a bigger group $\mathcal{L}_n$ which is the abelian group related to weight n polylogs, some of which are not classical polylogs.

We defined B_n to be the abelian groups generated by classical polylogs and $\mathcal{L}_n$ to be the abelian groups of all polylogs of weight n. Now we want to characterize them. The most mathematically concise way to describe their (conjectural!) connection is by an exact sequence, which for $n = 4$ reads

$$0 \to B_4 \to \mathcal{L}_4 \to \Lambda^2 B_2 \to 0. \tag{87}$$

An exact sequence is a sequence of maps between spaces such that the image of a map falls in the kernel of the next one. In the example above, the first arrow says that B_4 maps to $\mathcal{L}_4$ injectively, which is obvious since B_4 is contained in $\mathcal{L}_4$. The last arrow says that the map $\mathcal{L}_4 \to \Lambda^2 B_2$ is surjective. This is less obvious, but it means that for any element of $\Lambda^2 B_2$ one can find a weight four polylog with that $\Lambda^2 B_2$ projection.

Finally, the rest of the sequence means that $\ker(\mathcal{L}_4 \to \Lambda^2 B_2) = B_4$. This means that if a weight four polylog has zero $\Lambda^2 B_2$ projection, which is to say it belongs to $\ker(\mathcal{L}_4 \to \Lambda^2 B_2)$, then it is a classical polylog, and vice-versa.

Let us explain in more detail how to compute the $\Lambda^2 B_2$ component, following the steps described in sec. 0.5. We start with a length four symbol and we apply the following sequence of operations

$$a\otimes b\otimes c\otimes d \xrightarrow{\rho} a\otimes b\otimes c\otimes d - a\otimes b\otimes d\otimes c - a\otimes d\otimes b\otimes c + a\otimes d\otimes c\otimes b - d\otimes a\otimes b\otimes c + d\otimes a\otimes c\otimes b + d\otimes c\otimes a\otimes b - d\otimes c\otimes b\otimes a \xrightarrow{\pi_{2,2}} [a\otimes b]\otimes[c\otimes d] - [a\otimes b]\otimes[d\otimes c] - [a\otimes d]\otimes[b\otimes c] + [a\otimes d]\otimes[c\otimes b] - [d\otimes a]\otimes[b\otimes c] + [d\otimes a]\otimes[c\otimes b] + [d\otimes c]\otimes[a\otimes b] - [d\otimes c]\otimes[b\otimes a] \xrightarrow{\rho\otimes\rho} 2[a\wedge b]\otimes[c\wedge d] - 2[c\wedge d]\otimes[a\wedge b] = 2[a\wedge b]\wedge[c\wedge d]. \tag{88}$$

If the symbol $a \otimes b \otimes c \otimes d$ is integrable, then the terms $[a \wedge b]$ and $[c \wedge d]$ are also integrable to elements of B_2 and the full result belongs to the space $\Lambda^2 B_2$.

It is easy to check that classical four-logarithms, or elements of B_4 map to zero in $\Lambda^2 B_2$. Goncharov has conjectured that the converse is also true, which is a much more nontrivial statement. This conjecture has received a very nontrivial confirmation in the computations of ref. [23].

So, when we first encounter a transcendentality four function we should compute its $\Lambda^2 B_2$ projection. Then we should compute its $B_3 \otimes \mathbb{C}^*$ projection by using the same procedure as in eq. (88) except replacing $\pi_{2,2}$ by $\pi_{3,1}$. If the $\Lambda^2 B_2$ projection is zero, we should be able to reorganize the $B_3 \otimes \mathbb{C}^*$ projection in the same way as we did for the $\mathrm{Li}_{2,1}$ in eq. (68) after which we will be able to recognize the Li_4 appearing in the result. However, if the $\Lambda^2 B_2$ projection is non-zero, we can still compute the $B_3 \otimes \mathbb{C}^*$ but we will typically not be able to say more. This is the situation we are in for seven- and higher-point remainder functions (see ref. [18]).

For higher transcendentality there are similar exact sequences. For example, at transcendentality five we have (again conjecturally!)

$$0 \to B_5 \to \mathcal{L}_5 \to B_2 \otimes B_3 \to 0. \tag{89}$$

For transcendentality six we have a similar exact sequence

$$0 \to B_6 \to \mathcal{L}_6 \to B_3 \wedge B_3 \oplus B_2 \otimes B_4 \to \Lambda^3 B_2 \to 0. \tag{90}$$

This is not a short exact sequence anymore so it is a bit harder to extract information from it.

What is the origin of the particular combinations $B_3 \wedge B_3$ and $B_2 \otimes B_4$ like they appear above? The key is that the sum of all the $\mathcal{L}_n$ yields a graded algebra which we denote by $\mathcal{L}_\bullet$ and which has a structure of a "motivic Lie coalgebra". This means that there is a co-commutator

$$\delta : \mathcal{L}_\bullet \to \Lambda^2 \mathcal{L}_\bullet, \tag{91}$$

and this can generate the whole structure.

0.8 More details on transcendentality four

In this section we will present more details about the transcendentality four computations, illustrating the general features on the multiple polylogarithms $\mathrm{Li}_{2,2}$, $\mathrm{Li}_{1,3}$ and $\mathrm{Li}_{3,1}$. At transcendentality four we will compute two pieces of "motivic content", $\Lambda^2 B_2$ and $B_3 \otimes \mathbb{C}^*$. We will also show that these two parts are not independent but are related by a constraint which follows from integrability.

It is not too hard to compute the $B_3 \otimes \mathbb{C}^*$ and $\Lambda^2 B_2$ components of these functions. We find

$$\begin{gathered}\mathrm{Li}_{2,2}(x,y) \xrightarrow{B_3\otimes\mathbb{C}^*} \Big(2\{y\}_3 - 2\{x\}_3\Big) \otimes (xy-1)+ \\ \left(\{1-xy\}_3 - \left\{\frac{x-1}{xy-1}\right\}_3 + \left\{\frac{(x-1)y}{xy-1}\right\}_3 + \{1-x\}_3 - \{y\}_3\right) \otimes x+ \\ \left(\left\{\frac{y-1}{xy-1}\right\}_3 - \left\{\frac{x(y-1)}{xy-1}\right\}_3 + \left\{\frac{xy}{xy-1}\right\}_3 + \{x\}_3 - \{1-y\}_3\right) \otimes y,\end{gathered} \tag{92}$$

$$\mathrm{Li}_{2,2}(x,y) \xrightarrow{\Lambda^2 B_2} 2\{x\}_2 \wedge \{y\}_2 - 2\{x\}_2 \wedge \{xy\}_2 + 2\{y\}_2 \wedge \{xy\}_2. \tag{93}$$

We construct a map $B_3 \otimes \mathbb{C}^* \to B_2 \otimes (\Lambda^2\mathbb{C}^*)$,

$$\{x\}_3 \otimes y \mapsto \{x\}_2 \otimes (x \wedge y) \tag{94}$$

and another map $\Lambda^2 B_2 \to B_2 \otimes (\Lambda^2\mathbb{C}^*)$,

$$\{x\}_2 \wedge \{y\}_2 \mapsto \{x\}_2 \otimes ((1-y) \wedge y) - \{y\}_2 \otimes ((1-x) \wedge x). \tag{95}$$

It is a good exercise to check that

$$\begin{gathered}\mathrm{Li}_{2,2}(x,y) \xrightarrow{(B_2\otimes(\Lambda^2\mathbb{C}^*))\&(B_3\otimes\mathbb{C}^*)} \{x\}_2 \otimes ((1-xy) \wedge xy - (1-y) \wedge y)+ \\ \{y\}_2 \otimes (-(1-xy)\wedge xy + (1-x)\wedge x) + \{xy\}_2 \otimes (-(1-xy)\wedge xy + (1-y)\wedge y).\end{gathered} \tag{96}$$

This computation uses in an essential way the five-term dilogarithm identity

$$\{x\}_2 + \left\{\frac{1-x}{1-xy}\right\}_2 + \left\{\frac{1-y}{1-xy}\right\}_2 + \{y\}_2 - \{xy\}_2 = 0 \tag{97}$$

and the identities $\{x\}_2 = -\{1-x\}_2 = -\{x^{-1}\}_2$.

We can also compute the projection $\Lambda^2 B_2 \to B_2 \otimes (\Lambda^2\mathbb{C}^*)$ of $\mathrm{Li}_{2,2}(x,y)$. Doing so, we obtain the same result as for the projection of $B_3 \otimes \mathbb{C}^*$, up to a multiplicative factor of (-2).

The same can be done for the other transcendentality four functions $\mathrm{Li}_{1,3}$ and $\mathrm{Li}_{3,1}$. We find

$$\begin{gathered}\mathrm{Li}_{1,3}(x,y) \xrightarrow{B_3\otimes\mathbb{C}^*} \Big(\{x\}_3 - \{y\}_3\Big) \otimes (xy-1) + \Big(\{y\}_3 - \{xy\}_3\Big) \otimes (x-1)+ \\ \left(\{1-xy\}_3 - \left\{\frac{x-1}{xy-1}\right\}_3 + \left\{\frac{(x-1)y}{xy-1}\right\}_3 + \{1-x\}_3 - \{y\}_3\right) \otimes y\end{gathered} \tag{98}$$

$$\mathrm{Li}_{1,3}(x,y) \xrightarrow{\Lambda^2 B_2} 2\{x\}_2 \wedge \{xy\}_2 \tag{99}$$

$$\mathrm{Li}_{3,1}(x,y) \xrightarrow{B_3\otimes\mathbb{C}^*} \Big(\{x\}_3-\{y\}_3\Big)\otimes(xy-1)+\Big(\{y\}_3-\{xy\}_3\Big)\otimes(x-1)+ \\ \left(\{1-xy\}_3-\left\{\frac{x-1}{xy-1}\right\}_3+\left\{\frac{(x-1)y}{xy-1}\right\}_3+\{1-x\}_3-\{y\}_3\right)\otimes y \tag{100}$$

$$\mathrm{Li}_{3,1}(x,y) \xrightarrow{\Lambda^2 B_2} -2\{y\}_2\wedge\{xy\}_2 \tag{101}$$

The $\Lambda^2 B_2$ and $B_3\otimes\mathbb{C}^*$ parts are sufficient to determine a transcendentality four function, up to products. If we are given only an element of $\{x\}_2\wedge\{y\}_2\in\Lambda^2 B_2$ then we can construct an element of $B_3\otimes\mathbb{C}^*$ compatible with it (see ref. [19, eq. 27])

$$\frac{1}{12}\sum_{\sigma,\tau}(-1)^{|\sigma|+|\tau|}\left\{\frac{\sigma(x)}{\tau(y)}\right\}_3\otimes\frac{1-\sigma(x)}{1-\tau(y)}, \tag{102}$$

where we sum over functions σ,τ which act as permutations of arguments $x, x^{-1}, 1-x, (1-x)^{-1}, 1-x^{-1}, (1-x^{-1})^{-1}$ and $|\sigma|$ is the signature of the permutation.

0.9 Zeta values

Let us discuss the zeta values, which are easier to understand than polylogarithms, but are still pretty mysterious.

The multiple zeta values form a commutative algebra

$$\zeta(m)\zeta(n)=\zeta(m,n)+\zeta(n,m)+\zeta(n+m). \tag{103}$$

We denote by $\mathcal{Z}$ the space of $\mathbb{Q}$-linear combinations of zeta values.

We know that

$$\zeta(2n)=(-1)^{n-1}\frac{B_{2n}}{2(2n)!}(2\pi)^{2n}, \tag{104}$$

where B_n are the Bernoulli numbers,

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k t^k}{k!}. \tag{105}$$

Therefore, the even zeta values are expressible as powers of π^2 with rational coefficients $\zeta(2n)\in\mathbb{Q}[\pi^2]$ (this is a notation for the ring of polynomials with coefficients in $\mathbb{Q}$ and variable π^2).

Very little is known about the odd zeta values. Apéry showed that $\zeta(3) \notin \mathbb{Q}$ with an elementary but magical proof. The rest is a mystery. We can't even prove that $\zeta(5)$ is irrational or that $\pi^2\zeta(3)$ and $\zeta(5)$ are linearly independent.

In order to describe the relations between the zeta values, we introduce a free graded Lie algebra $\mathcal{F}(3, 5, \ldots)$, generated by elements e_{2n+1} with degree $-(2n + 1)$. Denote by $\mathcal{UF}(3, 5, \ldots)$ its universal enveloping algebra and by $\mathcal{UF}(3, 5, \ldots)^\vee$ the dual of the universal enveloping algebra. Then, we have the following

Conjecture. *The space $\mathcal{Z}$ is graded by weight, and $\mathcal{Z}_\bullet = \mathbb{Q}[\pi^2] \oplus_{\mathbb{Q}} \mathcal{UF}(3, 5, \ldots)^\vee_\bullet$.*

The conjecture can be formulated in terms of primitive elements $\mathcal{PZ}$ of $\mathcal{Z}$, that is elements which can not be written as products of elements of lower weight. To eliminate them we take the quotient $\mathcal{Z}/(\mathcal{Z}_{>0} \cdot \mathcal{Z}_{>0})$

$$\mathcal{PZ} = \mathcal{Z}/(\mathcal{Z}_{>0} \cdot \mathcal{Z}_{>0}). \tag{106}$$

Then the conjecture is equivalent to

$$\mathcal{PZ} = \langle \pi^2 \rangle \oplus \mathcal{F}(3, 5, \ldots)^\vee, \tag{107}$$

where $\langle \pi^2 \rangle$ is the one-dimensional $\mathbb{Q}$ vector space generated by π^2.

This conjecture implies that all the odd zeta values $\zeta(2n+1)$ are linearly independent (since they have different grading).

0.10 Kinematics

We study scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory with $SU(N)$ gauge group in the planar limit. We denote the coupling of the theory by g and we will be considering the scattering amplitudes in a perturbative expansion in g. Consider an n-particle scattering process. The particle labeled by i is described by its on-shell momentum p_i (with $p_i^2 = 0$), its helicity, a gauge algebra generator T^{a_i}.

In the planar limit $N \to \infty$, $g^2 N = \lambda$ fixed, only single-trace terms survive in the scattering amplitudes. If we look at one of these single-trace terms, we see that the scattered particles are cyclically ordered. We can therefore introduce a dual space with coordinates x such that the momenta p_i are expressed as $p_i = x_{i-1} - x_i$.

The $\mathcal{N} = 4$ super-Yang-Mills theory is superconformal invariant. However, scattering amplitudes are not well-defined in the absence of a regularization and after regularization some of the superconformal symmetry is broken. Besides this superconformal symmetry, the $\mathcal{N} = 4$ super-Yang-Mills theory also has a surprising *dual* superconformal symmetry, whose bosonic subgroup acts the dual coordinates x. In the following we will mostly be interested in this conformal subgroup of the dual superconformal group.

The complexified and compactified dual space can be represented as the $G(2,4)$ Grassmannian of two-planes in $\mathbb{C}^4$ containing the origin. Therefore, to each point in dual space we can associate a two-plane in $\mathbb{C}^4$. Two points in dual space are light-like separated if their corresponding planes intersect in a line. If we projectivize this construction, to a line in $\mathbb{C}^4$ corresponds a point in $\mathbb{CP}^3$. We can do this for all pairs of points (x_{i-1}, x_i) and associate to each of them a point $Z_i \in \mathbb{CP}^3$. So instead of describing the kinematics by giving the momenta p_i subject to on-shell conditions $p_i^2 = 0$ and momentum conservation $\sum_{i=1}^n p_i = 0$, we can describe it by giving n points $Z_i \in \mathbb{CP}^3$. The variables Z_i are known as momentum twistors and were introduced in ref. [24]. Unlike for the variables p_i or x_i, the momentum twistors are unconstrained.

The $\mathbb{C}^4$ space in which the two-planes are embedded is endowed with a non-degenerate bilinear form which we will denote by $\cdot$ (Warning! this $\mathbb{C}^4$ space is not the complexified Minkowski space). Given a two-plane X_i corresponding to a point x_i we can construct an orthogonal two-plane $X_i^\perp$ and by the same construction as for the momentum twistors Z_i we can construct conjugate momentum twistors W_i. Then, $Z_i \in X_i$ since $Z_i \in X_{i-1} \cap X_i$ and $W_i \in X_i^\perp$ since $W_i \in X_{i-1}^\perp \cap X_i^\perp$ so we can conclude that $Z_i \cdot W_i = 0$. Similarly, $Z_{i-1} \cdot W_i = Z_{i+1} \cdot W_i = 0$.

Parity acts as the discrete transformation $Z_i \leftrightarrow W_i$. If a quantity f depends on momentum twistors and is parity invariant, then we must have that

$$f(Z_1, \ldots, Z_n) = f(W_1, \ldots, W_n). \tag{108}$$

The complexified dual conformal group acts as $SL(4,\mathbb{C})$ on the momentum twistors $Z \to ZM$, $W \to M^{-1}W$, where $M \in SL(4,\mathbb{C})$. In order to make $SL(4,\mathbb{C})$ invariants, we can form four-brackets $\langle ijkl \rangle = \omega(v_i, v_j, v_k, v_l)$, where v_i is a vector in $\mathbb{C}^4$ corresponding to Z_i and ω is a volume form which is preserved by the action of $SL(4,\mathbb{C})$.

0.11 Introduction to cluster algebras

In this section we present some useful facts about cluster algebras. Cluster algebras have been introduced in a series of papers [11, 12, 2, 14] by Fomin and Zelevinsky.

We can informally define the cluster algebras as follows: they are commutative algebras constructed from distinguished generators (called *cluster variables*) which are grouped into non-disjoint sets of constant cardinality (called *clusters*), which are constructed recursively by an operation called *mutation* from an initial cluster. The number of variables in a cluster is called the rank of the cluster algebra.

As an example, take the A_2 cluster algebra defined by the following data:

- cluster variables: $x_m, \quad m \in \mathbb{Z}$
- clusters: $\{x_m, x_{m+1}\}$
- initial cluster: $\{x_1, x_2\}$
- rank: 2
- exchange relations: $x_{m-1}x_{m+1} = 1 + x_m$
- mutation: $\{x_{m-1}, x_m\} \to \{x_m, x_{m+1}\}$.

Using the exchange relations we find that

$$x_3 = \frac{1+x_2}{x_1}, \quad x_4 = \frac{1+x_1+x_2}{x_1x_2}, \quad x_5 = \frac{1+x_1}{x_2}, \quad x_6 = x_1, x_7 = x_2. \tag{109}$$

Therefore, the sequence x_m is periodic with period five and the number of cluster variables is finite.

When expressing the cluster variables x_m in terms of (x_1, x_2), we encounter two unexpected features (which also hold true for general cluster algebras). First, the denominators of the cluster variables are always monomials. In general, we expect the cluster variables to be rational fractions of the initial cluster variables, but in fact the denominator is always a monomial. This is known under the name of "Laurent phenomenon" (see. [11]). The second observation is that the numerator is a polynomial with positive coefficients.

Another example of rank two cluster algebra is the $A_{(b,c)}$ algebra, which has a different exchange relation

$$x_{m-1}x_{m+1} = \begin{cases} 1 + x_m^b, & m \text{ is odd,} \\ 1 + x_m^c, & m \text{ is even} \end{cases} . \tag{110}$$

Here b, c are positive integers.

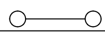
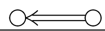
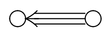
(b,c)	Cartan matrix	Lie algebra	Dynkin diagram	period
$(1,1)$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	A_2		5
$(1,2)$	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	C_2		7
$(1,3)$	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$	G_2		8

Table 1: Correspondence between finite cluster algebras and simple Lie algebras at rank two.

This cluster algebra has generically an infinite number of cluster variables. It can be shown that it has a finite number of cluster variables if and only if $bc \leq 3$. If we now form the matrices

$$\begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix}, \tag{111}$$

we notice that the finite cluster algebras correspond to Cartan matrices of simple Lie algebras (see tab. 1).

Let us now describe the link between quivers and cluster algebras. A quiver is an oriented graph. In the following we will restrict to connected, finite quivers without loops (arrows with the same origin and target) and two-cycles (pairs of arrows going in opposite directions between two vertices).

For a quiver with a given vertex k we define a new quiver obtained by mutating at vertex k. The new quiver is obtained by applying the following operations on the initial quiver:

- for each path $i \to k \to j$ we add an arrow $i \to j$
- reverse all the arrows on the edges incident with k
- remove all the two-cycles that may have formed.

The mutation at k is an involution; when applied twice in succession we obtain the initial cluster.

Each quiver of the restricted type defined above is in one-to-one correspondence with skew-symmetric matrices, once we fix an ordering of the vertices. The skew-symmetric matrix is defined as

$$b_{ij} = (\#\text{arrows } i \to j) - (\#\text{arrows } j \to i). \tag{112}$$

Since only one of the terms above is nonvanishing, $b_{ij} = -b_{ji}$. Under a mutation at vertex k the matrix b transforms to b' given by

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i,j\}, \\ b_{ij}, & \text{if } b_{ik}b_{kj} \leq 0, \\ b_{ij} + b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} < 0 \end{cases}. \tag{113}$$

If we start with a quiver with n vertices and associate to each vertex i a variable x_i, we can use the skew-symmetric matrix b to define a mutation relation at the vertex k by

$$x_k x'_k = \prod_{i|b_{ik}>0} x_i^{b_{ik}} + \prod_{i|b_{ik}<0} x_i^{-b_{ik}}, \tag{114}$$

with the understanding that an empty product is set to one. The mutation at k changes x_k to x'_k defined by eq. (114) and leaves the other cluster variables unchanged.

The A_2 cluster algebra can be expressed by a quiver $x_1 \to x_2$. Then, a mutation at x_1 replaces it by $x'_1 = \frac{1+x_2}{x_1} \equiv x_3$ and reverses the arrow. A mutation at x_2 replaces it by $x'_2 = \frac{1+x_1}{x_2} \equiv x_5$.

The $A_{(b,c)}$ cluster algebras can not be obtained from a quiver as described above, but they can be obtained from a *valued quiver*. If we denote by $\mathcal{V}$ the set of vertices and by $\mathcal{E}$ the set of edges of a quiver, then a valued quiver is a quiver together with two functions $v : \mathcal{V} \to \mathbb{N}^2$ and $d : \mathcal{E} \to \mathbb{N}$ such that for each arrow e_{ij}, $i \to j$ we have $d(i)v(e_{ij})_1 = d_j v(e_{ij})_2$, where $v(e_{ij})_{1,2}$ are the components of $v(e_{ij})$. The the matrix b is defined as

$$b_{ij} = \begin{cases} 0, & \text{no arrow between } i \text{ and } j, \\ v(e)_1 & e \text{ is an arrow between } i \text{ and } j, \\ -v(e)_2 & e \text{ is an arrow between } j \text{ and } i, \end{cases}. \tag{115}$$

Note that in this case the matrix b is not skew-symmetric anymore, but it is *skew-symmetrizable*. This means that there is a diagonal matrix $D = \operatorname{diag}(d(1), \dots)$ such that Db is skew-symmetric. Now the algebra $A_{(b,c)}$ can be represented by a seed quiver $x_1 \xrightarrow{(b,c)} x_2$. After a mutation at x_1 we obtain the quiver $\frac{1+x_2^c}{x_1} \xleftarrow{(c,b)} x_2$, while after a mutation at x_2 we obtain the quiver $x_1 \xleftarrow{(c,b)} \frac{1+x_1^b}{x_2}$. This reproduces the exchange rule of the $A_{(b,c)}$ cluster algebra.

We will be mostly interested in a special class of cluster algebras, named *cluster algebras of geometric type*. They are also described by quivers, but part of the vertices are special and called *frozen vertices*. The quiver is special in that we do not allow arrows between the frozen vertices. Also, we do not allow mutations in the frozen vertices. The associated variables to the frozen vertices are called *coefficients* instead of *cluster variables*. We define the *principal part* of such a quiver to be the quiver obtained by erasing the frozen vertices and the edges incident with them.

In the case of cluster algebras of geometric type we can define an analog of the b matrix as well. If the algebra has rank n (n unfrozen vertices) and m frozen vertices, we can naively define $(n+m)\times(n+m)$ matrix b. However, since there are no links between the frozen vertices this matrix will have an $m\times m$ block filled with zeros. Instead of working with this full matrix it is more economical to work with a $n\times(n+m)$ submatrix of the full $(n+m)\times(n+m)$ matrix.

In the following we will show how cluster variables arise from Grassmannians $G(k,n)$ (the Grassmannian $G(k,n)$ is the space of k-planes thought the origin of $\mathbb{C}^n$). We can associate to each point in $G(k,n)$ an equivalence class of $k\times n$ matrices of maximal rank k (all matrices which differ by a left action of $GL(k)$ are in the same equivalence class). This parametrization by $k\times n$ matrices arises as follows: a point in $G(k,n)$ is a k-plane in $\mathbb{C}^n$. In order to describe this k-plane we can arbitrarily pick k independent vectors in $\mathbb{C}^n$ which span the k-plane. Using these k n-vectors we can build a $k\times n$ matrix. However, if we chose another set of vectors which are obtained from the initial ones by a $GL(k)$ transformation which preserves the k-plane, we are describing the same point in the Grassmannian. This is why we need to identify the $k\times n$ matrices which differ by a $GL(k)$ action.

Dually, a $k\times n$ matrix can be thought of as n ordered points in $\mathbb{C}^k$, the points having coordinates given by the columns of the matrix. Using the $SL(k)$ subgroup of $GL(k)$ we can transform these k-vectors by the same linear transformation. Moreover, by the action of $GL(1)=GL(k)/SL(k)$ we can rescale all the k-vectors by the same amount so we should more properly consider them as points in $\mathbb{CP}^{k-1}$ rather than vectors in $\mathbb{C}^k$. We have therefore related points in the Grassmannian $G(k,n)$ to configurations of n ordered points in $\mathbb{CP}^{k-1}$.

Given a $k\times n$ matrix with $k\leq n$ we can form $\binom{n}{k}$ minors of type $k\times k$. They can be labeled by k integers $i_1,\ldots,i_k\in\{1,\ldots,n\}$, corresponding to the columns of the initial $k\times n$ matrix. We will denote the determinants of these minors by $(i_1,\ldots,i_k)$. These determinants are also known as Plücker coordinates. They satisfy Plücker relations

$$(i,j,I)(k,l,I)=(i,k,I)(j,l,I)+(i,l,I)(j,k,I), \tag{116}$$

where I is a multi-index with $k-2$ entries. The Plücker relations define an embedding, called Plücker embedding, of the Grassmannian into a projective space of dimension $\binom{n}{k}$.

Notice that the Plücker relations in eq. (116) look very similar to the exchange relations in a cluster algebra (see eq. (114, for example). Indeed, we will start with cluster whose variables are Plücker coordinates and by mutation we will generate more complicated cluster coordinates.

0.12 The cluster algebra for $G(k,n)$

The Grassmannian $G(k,n)$ has a cluster algebra structure which was described in ref. [16] (this construction is also reviewed in ref. [25]).

The construction of ref. [16] uses the definition of the Grassmannian $G(k,n)$ as a coset of $SL(n,\mathbb{C})$ by a parabolic subgroup P. In the basis of $\mathbb{C}^n$ where the first k basis vectors lie in the k-plane which determines a point of $G(k,n)$, the subgroup P contains all the matrices of $SL(n,\mathbb{C})$ with a $k\times(n-k)$ zero block in the upper-right block.

For our purposes it is sufficient to consider the description of the Grassmannian as equivalence classes of $k\times n$ matrices, where two matrices are equivalent if they differ by the left action of a $GL(k)$ matrix. If the leftmost $k\times k$ minor is non-singular, i.e. $\langle 1,\ldots,k\rangle \neq 0$ then, by left multiplication with an appropriate $GL(k)$ matrix, we can transform it to the identity matrix. After this operation the representative $k\times n$ matrix has the form $(\mathbf{1}_k, Y)$, where $\mathbf{1}_k$ is the $k\times k$ identity matrix and Y is a $k\times l$ matrix with $l=n-k$. The entries y_{ij}, $1\leq i\leq k$, $1\leq j\leq l$ of the matrix Y are coordinates on the cell of the Grassmannian where $\langle 1,\ldots,k\rangle\neq 0$.

Now we define a matrix F_{ij} for $1\leq i\leq k$, $1\leq j\leq l$, which is the biggest square matrix which fits inside Y and whose lower-left corner is at position (i,j) inside Y. Then we define $l(i,j)=\min(i-1,n-j-k)$ and

$$f_{ij}=(-1)^{(k-i)(l(i,j)-1)}\det F_{ij}. \tag{117}$$

Let us express f_{ij} in terms of k-brackets. There are two cases to consider: $i\leq l-j+1$ and $i>l-j+1$. By adding rows and columns to the matrix F_{ij} to make it a $k\times k$ matrix, we find

$$f_{ij}=\begin{cases}\frac{\langle i+1,\ldots,k,k+j,\ldots,i+j+k-1\rangle}{\langle 1,\ldots,k\rangle}, & i\leq l-j+1,\\ \frac{\langle 1,\ldots,i+j-l-1,i+1,\ldots,k,k+j,\ldots,n\rangle}{\langle 1,\ldots,k\rangle}, & i>l-j+1\end{cases}. \tag{118}$$

In the definition of f_{ij} we have divided by $\langle 1,\ldots,k\rangle$ so that the expression derived above holds even when $\langle 1,\ldots,k\rangle\neq 1$.

According to ref. [16], the initial quiver for the $G(k,n)$ cluster algebra is given by[8]

$$
\begin{tikzcd}
f_{1l} \arrow[r] \arrow[d] & \cdots \arrow[r] & f_{13} \arrow[r] \arrow[d] & f_{12} \arrow[r] \arrow[d] & \boxed{f_{11}} \\
f_{2l} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[ul] & f_{23} \arrow[r] \arrow[d] & f_{22} \arrow[r] \arrow[d] \arrow[ul] & \boxed{f_{21}} \arrow[ul] \\
\vdots \arrow[d] & \vdots & \vdots \arrow[d] & \vdots \arrow[d] \arrow[ul] & \vdots \arrow[ul] \\
\boxed{f_{kl}} & \cdots & \boxed{f_{k3}} & \boxed{f_{k2}} \arrow[ul] & \boxed{f_{k1}} \arrow[ul]
\end{tikzcd}
\tag{119}
$$

In order to obtain the quivers in sec. 0.11, we need to make one last change to the quiver above. We rescale all the coordinates, frozen and unfrozen, by $\langle 1,\ldots,k\rangle$. This produces a frozen variable $\langle 1,\ldots,k\rangle$ which connects to the node labeled by f_{1l} by an ingoing arrow. After this modification all the unfrozen vertices of the initial quiver have an equal number of ingoing and outgoing arrows.

Let us start with some simple examples. For four points in $\mathbb{CP}^1$ the quiver diagram has one central node connected to four external nodes (these external nodes are the frozen variables).[9] The central node has two arrows

[8]Here we are presented a flipped version of the quiver and with the arrows reversed with respect to the quivers of refs. [16, 25].

[9]See ref. [16] for the construction of Grassmannian cluster algebras. The quivers diagrams we present below are derived in this reference.

coming in and two arrows going out.

$$
\begin{tikzcd}
 & \boxed{34} \arrow[d] & \\
\boxed{23} & 13 \arrow[l] \arrow[r] & \boxed{14} \\
 & \boxed{12} \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \boxed{34} & \\
\boxed{23} \arrow[r] & 24 \arrow[u] \arrow[d] & \boxed{14} \arrow[l] \\
 & \boxed{12} &
\end{tikzcd}
\tag{120}
$$

The cluster coordinates we have used so far are called A coordinates. They have the drawback that they are not invariant under rescaling of coordinates. We can define scaling invariant quantities (called an X-coordinate) associated to any unfrozen node by taking the ratio of the product of A coordinates which can be reached by going against the arrows coming in by the product of A coordinates which can be reached by following the arrows going out. A mutation reverses the arrows and therefore transforms the X coordinate to its inverse. As an example, the X coordinates for the central vertex in eq. (120) are $\frac{(12)(34)}{(14)(23)}$ and $\frac{(14)(23)}{(12)(34)}$, respectively.

Before moving on, let us make a useful observation. A point in the Grassmannian $G(k, n)$ is a k-plane through the origin of $\mathbb{C}^n$ which can equally be described by an orthogonal $n-k$-plane through the origin of $\mathbb{C}^n$. This correspondence is one-to-one so we are led to conclude that $G(k, n) = G(n-k, n)$. In terms of configurations of points this means that the configurations of n ordered points in $\mathbb{CP}^{k-1}$ are the same as configurations of n ordered points in $\mathbb{CP}^{n-k-1}$. Therefore we can restrict to $k \leq \frac{n}{2}$ without loss of generality.

According to the discussion in the previous paragraph, we don't need to go further than $\mathbb{CP}^1$ for $n = 4$. Now let us move on to five points. Just like for four points, we only need to study configurations in $\mathbb{CP}^1$. Thinking about the Plücker identities (or using the results of ref. [16]), we see that we can start with a cluster whose quiver diagram looks like below.

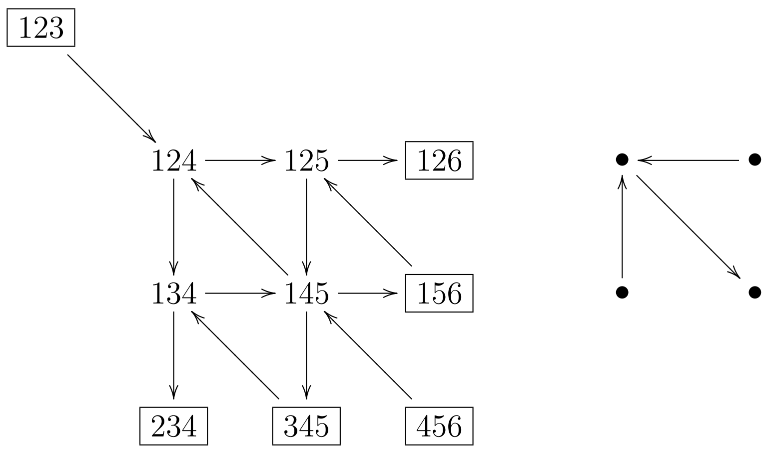
$$
\begin{tikzcd}
\boxed{12} \arrow[dr] & & & \\
 & 13 \arrow[r] \arrow[d] & 14 \arrow[r] \arrow[d] & \boxed{15} \\
 & \boxed{23} & \boxed{34} \arrow[ul] & \boxed{45} \arrow[ul]
\end{tikzcd}
\tag{121}
$$

Then we can do a mutation on the node (14), for example. After this we obtain a similar quiver diagram where the frozen vertex (15) is special instead of (34). Just like in the four-point case the arrows containing the mutated node get reversed and the link between (13) and (34) gets deleted and replaced with a link (13) $\to$ (15). It is easy to see that by mutating one gets the five similar quivers and nothing more.

By the duality explained above, the same kind of quiver is associated to configurations of five points in $\mathbb{CP}^2$, we just need to replace the labels by their complement as $12 \implies 345$, $23 \implies 145$, etc.

The principal part of the quiver in the case of configurations of four points in $\mathbb{CP}^1$ is just one vertex, the central one. This is the Dynkin diagram of the A_1 Lie algebra.[10] The principal part of the quiver for configurations of five points in $\mathbb{CP}^1$ is the same as the Dynkin diagram of A_2 Lie algebra. This cluster algebra is in fact the same as the A_2 algebra we defined at the beginning of this section and the appearance of the A_2 Dynkin diagram provides the motivation for the name. The A_n cluster algebras appear in describing the configurations of $n+3$ points in $\mathbb{CP}^1$.

More exotic cases appear for six points in $\mathbb{CP}^2$, where we obtain a D_4 Dynkin diagram. We can start with an initial quiver at the left below and mutate at vertex 124 to obtain the principal part of the quiver shown at right, which is the same as the Dynkin diagram of D_4.



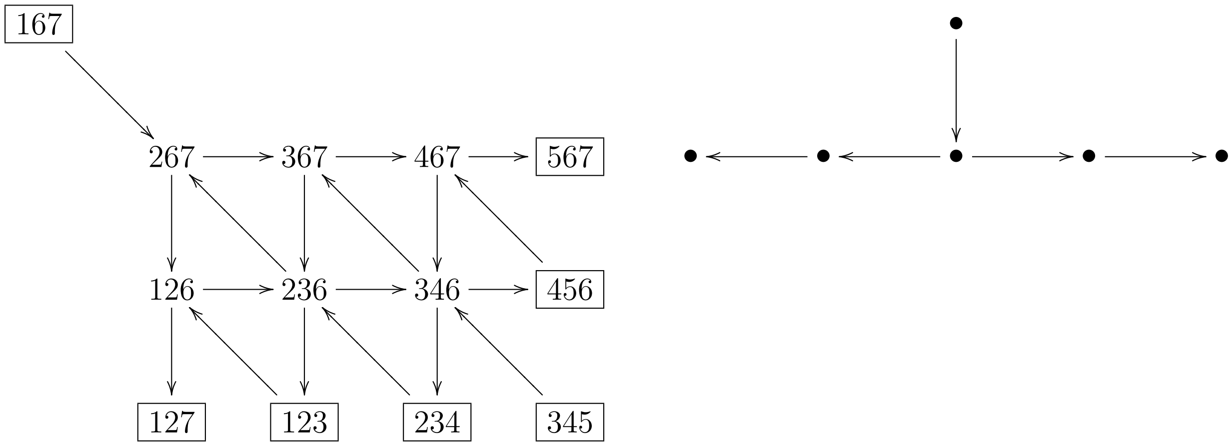
(122)

For seven points in $\mathbb{CP}^2$ we present below the initial quiver. After a sequence of mutations at vertices 126, 467, 367, 236, 267, 126, 467, 126, 346 we obtain the E_6 quiver in the right part of the figure below.

[10] This is the simplest possible cluster algebra, but it is a bit too simple which is why we have not used it for illustration purposes. It has two clusters of one element each, $\{x\}$ and $\{x^{-1}\}$. Under mutations we have the transformation $x \to x^{-1}$.

A_n	B_n, C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	105	8

Table 2: The number of clusters for cluster algebras of finite type.



(123)

Finally, for eight points in $\mathbb{CP}^2$ the principal part of the quiver can be brought into the form of an E_8 Dynkin diagram by a sequence of mutations (see ref. [26] for more details).

In ref. [12], Fomin and Zelevinsky showed that a cluster is of finite type (i.e. it has a finite number of cluster variables), if the principal part of its quiver can be made to be a Dynkin diagram after a sequence of mutations. Further, if the principal part of the quiver contains a subgraph which is an affine Dynkin diagram, then the cluster algebra is of infinite type. We have seen above that cluster algebras arising from $G(2, n)$ and $G(3, 6)$, $G(3, 7)$ and $G(3, 8)$ are of finite type. In ref. [26], Scott has shown that all the other $G(k, n)$ with $2 \leq k \leq \frac{n}{2}$ are of infinite type.

This has striking implications for scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. There, the relevant Grassmannian is $G(4, n)$, for $n \geq 6$. If $n = 6$ we obtain $G(4, 6) = G(2, 6)$ which is of finite type. If $n = 7$ we obtain $G(4, 7) = G(3, 7)$ which is again of finite type. However, starting at eight-point the cluster algebras are not of finite type anymore. In table. 2 we list the number of clusters for different cluster algebras of finite type (see ref. [13])

A natural question is what kind of A coordinates appear for the simplest cluster algebra of infinite type which is physically relevant, i.e. $G(4, 8)$. Besides the usual Plücker determinants, we also find more complicated quantities like

$$\langle 12(345) \cap (678) \rangle \equiv \langle 1345 \rangle \langle 2678 \rangle - \langle 2345 \rangle \langle 1678 \rangle. \tag{124}$$

The notation with $\cap$ emphasizes the following geometrical fact: the composite bracket $\langle 12(345) \cap (678) \rangle$ vanishes whenever the projective line $(345) \cap (678)$ obtained by intersecting two projective planes (345) and (678) and the points 1 and 2 lie in the same projective plane. Another type of composite bracket which appears is

$$\langle 12(345) \cap (567) \rangle, \tag{125}$$

which already appears for seven-point when expressed in $\mathbb{CP}^3$ language. This $\cap$ notation has been introduced in ref. [1].

However, since the number of possible A coordinates is infinite, we are bound to find more and more complicated expressions. One miraculous feature of the mutations is that the denominator can always be canceled by the numerator, after using Plücker identities. Therefore, the A coordinates always seem to be *polynomials* in the Plücker coordinates. This is an analog of the Laurent phenomenon, but this time we obtain polynomials. As an example which appears for $G(4, 8)$, we have the following identity

Is this a theorem? A conjecture?

$$\frac{\langle 1237 \rangle \langle 1245 \rangle \langle 1678 \rangle + \langle 1278 \rangle \langle 45(671) \cap (123) \rangle}{\langle 1267 \rangle} = \langle 45(781) \cap (123) \rangle. \tag{126}$$

Here the left-hand side is the expression obtained following a mutation, while the right-hand side is the expression where the denominator has been canceled.

Even more complicated A coordinates can be generated. As an example, we also find

$$-\langle (123) \cap (345), (567) \cap (781) \rangle. \tag{127}$$

This vanishes when the lines $(123) \cap (345)$ and $(567) \cap (781)$ intersect. This is equivalent to saying that the lines $(345) \cap (567)$ and $(781) \cap (123)$ intersect. Finally, an even more complicated A coordinate reads

$$\begin{aligned}\langle 1246 \rangle \langle 1256 \rangle \langle 1378 \rangle \langle 3457 \rangle - \langle 1246 \rangle \langle 1257 \rangle \langle 1378 \rangle \langle 3456 \rangle - \\ \langle 1246 \rangle \langle 1278 \rangle \langle 1356 \rangle \langle 3457 \rangle + \langle 1278 \rangle \langle 1257 \rangle \langle 1346 \rangle \langle 3456 \rangle + \\ \langle 1236 \rangle \langle 1278 \rangle \langle 1457 \rangle \langle 3456 \rangle.\end{aligned} \tag{128}$$

Its geometrical interpretation is obscure.

In principle all these A coordinates can appear in the symbol of the eight-point remainder or ratio functions. Nevertheless, at low loop orders only a small number of A coordinates appears. It would be interesting to investigate when and if such more complicated entries appear in the symbol.

Notice that the seeds we have been using break the cyclic symmetry of the configuration of points. In order to see that the cyclic symmetry is preserved we need to show that by mutations one can reach another quiver whose labels are permuted by one unit. For the case of $G(3,7)$ described above, this can not be done in fewer than six mutations, since all the unfrozen A coordinates need to change. Indeed it is not hard to show that after mutating in the vertices which are initially labeled by (126), (267), (236), (367), (346) and (467), we obtain the cluster with the vertex labels shifted by one (123) $\to$ (234), etc. This proves the cyclic symmetry.

0.13 Poisson brackets

One can define a Poisson bracket on the cluster X coordinates. It is enough to define the Poisson bracket between the X coordinates in a given cluster. It is defined as

$$\{X_i, X_j\} = b_{ij} X_i X_j, \tag{129}$$

where $b_{ij} = -b_{ji}$ is the b matrix of the cluster. It is not hard to check that under mutations we obtain

$$\{X'_i, X'_j\} = b'_{ij} X'_i X'_j, \tag{130}$$

where X'_i and b'_{ij} are obtained from X_i and b_{ij}, respectively. Therefore the Poisson structure is preserved by mutations.

The Poisson structure is easiest to understand for $G(2,n)$ cluster algebras (see ref. [10] for a discussion). To a configuration of n points in $\mathbb{CP}^1$ with a cyclic ordering we associate a convex polygon. Each of the vertices of this polygon corresponds to one of the n points.

Then consider a complete triangulation of the polygon. Each of the $n-3$ diagonals in this triangulation determines a quadrilateral and therefore four points in $\mathbb{CP}^1$. Suppose a diagonal E determines a quadrilateral with vertices i, j, k, l where the ordering is the same as the ordering of the initial polygon. Using these four points we can form a cross-ratio $r(i,j,k,l) = \frac{z_{ij} z_{kl}}{z_{jk} z_{il}}$. We have $r(i,j,k,l) = r(k,l,i,j)$ which implies that the cross-ratio is uniquely determined by the diagonal E and we don't have to chose an orientation.

If we flip the diagonal E then the initial cross-ratio goes to its inverse, but the cross-ratios corresponding to neighboring quadrilaterals change in a more complicated way. In fact, they transform in the same way as cluster

X coordinates, if the matrix b_{ij} is defined as follows. Two diagonals E and F in a given triangulation are called adjacent if they are the sides of one of the triangles of the triangulation. If the diagonals are adjacent we set $b_{EF} = 1$ if the diagonal E comes before F when listing the diagonals at the common vertex in clockwise order. Otherwise we set $b_{EF} = -1$. If two diagonals E and F are not adjacent we set $\epsilon_{EF} = 0$.

As an example we can consider a pentagon triangulated by diagonals a which joins vertices 1 and 4 and diagonal b joining vertices 1 and 3. Initially we have the cross-ratios $X_a = r(1,3,4,5)$, $X_b = r(1,2,3,4)$. Now perform the mutation corresponding to flipping the diagonal b. After the flip the diagonal b joins vertices 2 and 4 and the new cross-ratios are $X'_a = r(1,2,4,5)$ and $X'_b = r(2,3,4,1)$. It is easy to show that

$$X'_a = X_a(1 + X_b^{-1})^{-1}, \qquad X'_b = X_b^{-1}. \tag{131}$$

This corresponds to $\epsilon_{ab} = 1$ (before the mutation).

Consider the $\Lambda^2 B_2$ projection of the seven-point remainder function. In $\mathbb{CP}^2$ language it is given by

$$\begin{aligned}
&-\left\{-\frac{\langle 2\times 3, 4\times 6, 7\times 1\rangle}{\langle 167\rangle\langle 234\rangle}\right\}_2 \wedge \left\{-\frac{\langle 7\times 1, 2\times 3, 4\times 5\rangle}{\langle 127\rangle\langle 345\rangle}\right\}_2 \\
&\qquad -\left\{-\frac{\langle 2\times 3, 4\times 6, 7\times 1\rangle}{\langle 167\rangle\langle 234\rangle}\right\}_2 \wedge \left\{-\frac{\langle 234\rangle\langle 456\rangle}{\langle 246\rangle\langle 345\rangle}\right\}_2 \\
&\qquad -\left\{-\frac{\langle 2\times 3, 4\times 6, 7\times 1\rangle}{\langle 167\rangle\langle 234\rangle}\right\}_2 \wedge \left\{-\frac{\langle 146\rangle\langle 567\rangle}{\langle 167\rangle\langle 456\rangle}\right\}_2 \\
&\quad -\left\{-\frac{\langle 2\times 3, 4\times 6, 7\times 1\rangle}{\langle 167\rangle\langle 234\rangle}\right\}_2 \wedge \left\{-\frac{\langle 5\times 6, 7\times 1, 2\times 3\rangle}{\langle 123\rangle\langle 567\rangle}\right\}_2 \\
&+\left\{-\frac{\langle 137\rangle\langle 467\rangle}{\langle 167\rangle\langle 347\rangle}\right\}_2 \wedge \left\{-\frac{\langle 123\rangle\langle 347\rangle}{\langle 137\rangle\langle 234\rangle}\right\}_2 - \left\{-\frac{\langle 137\rangle\langle 467\rangle}{\langle 167\rangle\langle 347\rangle}\right\}_2 \wedge \left\{-\frac{\langle 347\rangle\langle 456\rangle}{\langle 345\rangle\langle 467\rangle}\right\}_2 \\
&\qquad\qquad + \text{cyclic permutations of } 1, 2, \ldots, 7.
\end{aligned} \tag{132}$$

We can show that for every term $\{-x\}_2 \wedge \{-y\}_2 \in \Lambda^2 B_2$ listed above we can find at least one cluster such that both x and y belong to it. This means that their Poisson bracket is simple.[11] In fact, we find that these Poisson brackets are zero. In order to prove this, for every pair (x, y) we need to exhibit a quiver graph which contains them and which is such that there are no arrows between x and y.

[11]The Poisson bracket between cluster coordinates can become complicated. As an example, consider the A_2 cluster algebra. There, we have $\{x_1, x_2\} = x_1 x_2$. The Poisson bracket between coordinates in different clusters can be computed more easily than for $G(k, n)$ and even then we obtain more complicated quantities.

.1 Glossary of mathematical notions

The source for the Lie algebra constructions discussed in this section is ref. [27]. In the following we will work over some base field $\mathbb{K}$.

Definition .1.1 (Lie algebra homomorphism)**.** A map $\phi : L \to L'$, between two Lie algebras L, L' is a Lie algebra homomorphism if it is linear and $\phi([x,y]) = [\phi(x), \phi(y)]'$ for all $x, y \in L$. Note that the bracket $[\cdot,\cdot]$ is computed in L while the bracket $[\cdot,\cdot]'$ is computed in L'.

Definition .1.2 (Universal enveloping algebra)**.** Given a Lie algebra L, the universal enveloping algebra $U(L)$ is an associative algebra with unit such that:

1. there exists a map $\epsilon : L \to U(L)$ which is a Lie algebra homomorphism. The condition that ϵ is a Lie algebra homomorphism reads $\epsilon([x,y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$.
2. if A is any associative algebra with unit and $\alpha : L \to A$ is a Lie algebra homomorphism, then there is a unique homomorphism $\beta : U(L) \to A$ of associative algebras such that $\alpha = \beta \circ \epsilon$.

Definition .1.3 (Tensor product of algebras)**.** If A, B are algebras (over the same field) then they are also vector spaces and we can form their tensor product $A \otimes B$ as vector fields. This tensor product can be made into an algebra by defining a product by

$$(a \otimes b)(c \otimes d) = (ac) \otimes (bd), \tag{133}$$

for $a, c \in A$ and $b, d \in B$ and extending it to the whole $A \otimes B$ by linearity. If A, B are associative then $A \otimes B$ is associative. If A, B have units 1_A and 1_B, then $A \otimes B$ has unit $1_A \otimes 1_B$.

Definition .1.4 (Tensor algebra)**.** If V is a vector space over a field $\mathbb{K}$, we define $T^nV = \underbrace{V \otimes \cdots \otimes V}_{n}$, with $T^0V = \mathbb{K}$. The tensor algebra of V, denoted by TV, is

$$TV = \oplus_{n=0}^{\infty} T^nV. \tag{134}$$

Definition .1.5 (Universal enveloping algebra, alternative definition)**.** The universal enveloping algebra $U(L)$ of the Lie algebra L can be defined as a coset, as follows. Let I be the two-sided ideal of the tensor algebra TL generated by the elements $x \otimes y - y \otimes x - [x,y]$, for $x, y \in L$. Then the universal enveloping algebra $U(L)$ of L is defined as

$$U(L) = TL/I. \tag{135}$$

Lemma .1.1. *If L, L' are two Lie algebras and $\phi : L \to L'$ is a Lie algebra homomorphism, then ϕ induces an associative algebra homomorphism $U(L) \to U(L')$. Similarly, if L is a Lie algebra and A is an algebra and $\phi : L \to A$ is a Lie algebra homomorphism, then ϕ induces an associative algebra homomorphism $U(L) \to A$.*

Theorem .1.1. *If L, L' are Lie algebras, then we have the following isomorphism of universal enveloping algebras*

$$U(L \oplus L') \cong U(L) \otimes U(L'). \tag{136}$$

Proof. We will establish an isomorphism between $U(L \oplus L')$ and $U(L) \otimes U(L')$.

Recall that we have $\epsilon : L \to U(L)$ and $\epsilon' : L \to U(L')$, which embed the Lie algebras L, L' in their universal enveloping algebra. We also denote by ϵ the map $\epsilon : L \to U(L \oplus L')$ and by ϵ' the map $\epsilon' : L' \to U(L \oplus L')$. From lemma .1.1, we see that they induce homomorphisms $\chi : U(L) \to U(L \oplus L')$, $\chi' : U(L') \to U(L \oplus L')$.

Since $[x, x'] = 0$ for $x \in L$ and $x' \in L'$, we get that $\chi(u)\chi'(u') = \chi'(u')\chi(u)$ for any $u \in U(L)$, $u' \in U(L')$. Therefore, the product $\chi(u)\chi'(u')$ is commutative and non-ambiguous. We define a map

$$\psi : U(L) \otimes U(L') \to U(L \oplus L'), \qquad \psi(u \otimes u') = \chi(u)\chi'(u'), \tag{137}$$

for all $u \in L$, $u' \in L'$ and then extend it to the whole $L \otimes L'$ by linearity.

We have a homomorphism $L \oplus L' \to U(L) \otimes U(L')$ defined by $x \to \epsilon(\pi(x)) \otimes 1 + 1 \otimes \epsilon'(\pi'(x))$, where π is the projection on L and π' is the projection on L'. This is a Lie algebra homomorphism and, by the lemma .1.1 induces a homomorphism $\phi : U(L \oplus L') \to U(L) \otimes U(L')$.

It remains to show that ϕ and χ are inverse of one another. □

Let L be a Lie algebra over a field $\mathbb{K}$. The map $L \to U(L) \otimes U(L)$, $x \mapsto x \otimes 1 + 1 \otimes x$ defines a Lie algebra homomorphism. By lemma .1.1 it induces an associative algebra homomorphism $\Delta : U(L) \to U(L) \otimes U(L)$. Define $\varepsilon : U(L) \to \mathbb{K}$ by $\varepsilon(1) = 1$ and $\varepsilon(x) = 0$ for $x \in L$ and extend as an algebra homomorphism.

We have

$$(\varepsilon \otimes \mathrm{Id}) \circ (x \otimes 1 + 1 \otimes x) = 1 \otimes x, \qquad x \in L. \tag{138}$$

Since $\mathbb{K} \otimes L \cong L$ we can identify $1 \otimes x$ with x, so $(\varepsilon \otimes \Delta) \circ \Delta : U(L) \to U(L)$ is the identity. The same holds for $(\mathrm{Id} \otimes \varepsilon) \circ \Delta$.

Definition .1.6 (Coalgebra)**.** Let C be a vector space over a field $\mathbb{K}$. A map $\Delta : C \to C \otimes C$ called comultiplication and a map $\epsilon : C \to \mathbb{K}$ called counit satisfying $(\varepsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id} = (\mathrm{Id} \otimes \varepsilon) \circ \Delta$, is called a coalgebra.

Definition .1.7 (Bialgebra)**.** If C is an algebra and a coalgebra and Δ, ε are algebra homomorphisms, then C is a bialgebra.

Definition .1.8 (Coassociativity)**.** A comultiplication Δ is coassociative if $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$.

Example of coassociative comultiplication. Take x such that $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then,

$$\begin{aligned}(\Delta \otimes 1) \circ \Delta(x) = (\Delta \otimes 1)(x \otimes 1 + 1 \otimes x) = \Delta(x) \otimes 1 + \Delta(1) \otimes x = \\ x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x = (1 \otimes \Delta) \circ \Delta(x), \quad (139)\end{aligned}$$

where we have used $\Delta(1) = 1 \otimes 1$, which follows form the condition that Δ is an algebra homomorphism.

Definition .1.9 (Graded Lie algebra)**.** Let L be a Lie algebra over the field $\mathbb{K}$ and $U(L)$ its universal enveloping algebra, $\epsilon : L \to U(L)$. Define $U_n(L)$ to be the subspace of $U(L)$ generated by products of at most n elements of L. Then we have

$$U_0(L) = \mathbb{K}, \qquad U_1(L) = \mathbb{K} \oplus \epsilon(L), \tag{140}$$

$$U_0(L) \subset U_1(L) \cdots U_n(L) \subset \cdots , \tag{141}$$

$$U_m(L) \cdot U_n(L) \subset U_{n+m}(L). \tag{142}$$

We define

$$\mathrm{gr}_n\, U(L) = U_n(L)/U_{n-1}(L), \qquad \mathrm{gr}\, U(L) = \oplus_n \mathrm{gr}_n\, U(L). \tag{143}$$

The multiplication on $\mathrm{gr}\, U(L)$ is induced by the multiplication on $U(L)$.

If $x, y \in L$, then $\varepsilon(x), \varepsilon(y) \in U(L)$ and we have $\varepsilon(x)\varepsilon(y) - \varepsilon(x)\varepsilon(y) = \varepsilon([x, y]) \in U_1 L$. In $\mathrm{gr}_2\, U(L)$ the RHS of the previous equation vanishes which implies that the gr_2 is commutative. The same holds for higher grading so $\mathrm{gr}\, U(L)$ is a commutative algebra.

Definition .1.10 (Primitive element)**.** Given a bialgebra L, an element $x \in L$ is called primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Lemma .1.2. *Let L be a Lie algebra with comultiplication $\Delta : U(L) \to U(L) \otimes U(L)$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x$ if $x \in L$. If $x \in U(L)$ is primitive, then $x \in L$.*

Let X be a set. We set $X_1 := X$. Then X_2 consists of all expressions ab with $a, b \in X_1$, so $X_2 = X_1 \times X_2$. We define

$$X_n = \coprod_{p+q=n} X_p \times X_q. \tag{144}$$

Set $M_X = \coprod_{n=1}^{\infty} X_n$. An element $w \in M_x$ is called a non-associative word. Multiplication on M_X is concatenation of non-associative words.

We define A_X to be the vector space of finite formal linear combinations of elements of M_X. The multiplication on M_X extends by bilinearity to make A_X into an algebra.

Definition .1.11. Let $I \subset A_X$ be the two-sided ideal generated by aa and by $(ab)c + (bc)a + (ca)b$, for $a, b, c \in A_X$. We set

$$L_X = A_X / I \tag{145}$$

and call L_X the free Lie algebra of X.

Lemma .1.3. *The ideal I defined above is graded, that is for $a \in I$ and a decomposition $a = \sum a_n$ into homogeneous components a_n, then $a_n \in I$ for all a_n in the decomposition of a. Since the ideal I is graded, the quotient $L_X = A_X/I$ is a graded algebra as well.*

Definition .1.12 (Nilpotent Lie algebra)**.** A lie algebra L is nilpotent if the series L, $[L, L]$, $[[L, L], L]$, etc. becomes zero eventually. The reason for the name is that, given $a \in L$, the linear operator ad_a is nilpotent.

Definition .1.13 (Direct Limit)**.** Let A_i be a family of algebraic objects (groups, rings, modules, algebras) indexed by an ordered set I. Let $\phi_{ij} : A_i \to A_j$ be homomorphisms defined for all $i \leq j$, with the properties that $\phi_{ii} = \mathrm{Id}|_{A_i}$, and $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$ for all $i \leq j \leq k$. We define the direct limit $\varinjlim A_i$ of the A_i as the disjoint union of A_i modulo an equivalence relation $\sim$,

$$\varinjlim_{i \in I} A_i = \bigsqcup_i A_i / \sim . \tag{146}$$

The equivalence relation $\sim$ is defined by taking, for any $x_i \in A_i$, $x_i \sim \phi_{ij}(x_i)$ (x_i is equivalent to all its images under the maps ϕ_{ij}).

Definition .1.14 (Directed partially ordered set)**.** A directed partially ordered set I us a set with a partial order $\leq$ such that for any $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

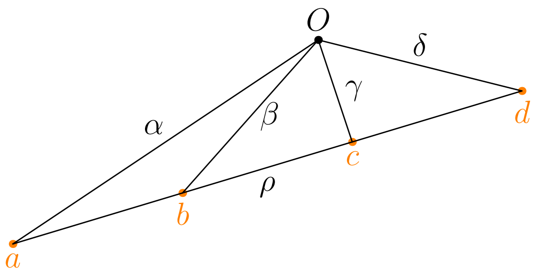


Figure 2: The cross-ratio of four lines in $\mathbb{CP}^2$.

Definition .1.15 (Inverse Limit)**.** Let A_i be a family of algebraic objects (groups, rings, modules, algebras) indexed by a directed partially ordered set I. Let $\phi_{ij} : A_j \to A_i$ be homomorphisms defined for all $i \leq j$, with the properties that $\phi_{ii} = \mathrm{Id}\,|_{A_i}$, and $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$ for all $i \leq j \leq k$. Then the inverse limit is

$$\varprojlim_{i\in I} A_i = \left\{ \vec{a} \in \prod_{i\in I} A_i \;\middle|\; a_i = \phi_{ij}(a_j),\ \forall i \leq j, i, j \in I \right\} \tag{147}$$

Definition .1.16 (Pro-nilpotent)**.** An algebra A is called pronilpotent if it can be written as an inverse limit of nilpotent algebras.

.2 Elements of projective geometry

The simplest type of cross-ratio is the cross-ratio of four points (a, b, c, d) in $\mathbb{CP}^1$. If the points have have coordinates (z_a, z_b, z_c, z_d), then their cross-ratio is

$$r(a, b, c, d) = \frac{z_{ab} z_{cd}}{z_{bc} z_{da}}. \tag{148}$$

In the following we will try to reduce more complicated situations to configurations of four points on a projective line.

By duality, a point in $\mathbb{CP}^2$ is in correspondence with a line in $\mathbb{CP}^2$. Therefore, we can talk about the cross-ratio of four lines in $\mathbb{CP}^2$ (see fig. 2).

The cross-ratios of four lines $(\alpha, \beta, \gamma, \delta)$ can be related to the cross-ratio of four points by taking an arbitrary line ρ and computing the intersection points $a = \rho \cap \alpha$, $b = \rho \cap \beta$, $c = \rho \cap \gamma$, $d = \rho \cap \delta$. Then, the cross-ratio of the points (a, b, c, d) on ρ is independent on ρ and is equal to the cross-ratio of the lines $(\alpha, \beta, \gamma, \delta)$

$$r(\alpha, \beta, \gamma, \delta) = r(a, b, c, d). \tag{149}$$

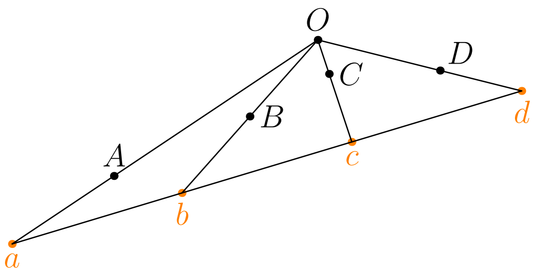


Figure 3: The cross-ratio of four lines determined by their common intersection point O and another point on each on of them.

If the lines are defined by pairs of points $\alpha = (OA)$, $\beta = (OB)$, $\gamma = (OC)$, $\delta = (OD)$, as in fig. 3, then the cross-ratio of the four lines is

$$r(\alpha, \beta, \gamma, \delta) = r(a, b, c, d) = (O|A, B, C, D) \equiv \frac{\langle OAB\rangle\langle OCD\rangle}{\langle OBC\rangle\langle ODA\rangle}, \tag{150}$$

where $\langle XYZ\rangle$ is proportional to the oriented area of the triangle $\Delta(X, Y, Z)$.

If the four points A, B, C, D do not belong to a line we can't generically define their cross-ratio. However, given a conic $\mathcal{C}$ such that A, B, C, D belong[12] to $\mathcal{C}$, then we can define their cross-ratio as follows: pick a point X on the conic $\mathcal{C}$. Then, by Chasles' theorem the cross-ratio of the lines (XA), (XB), (XC) and (XD) is independent on the point X and is defined to be the cross-ratio of the points A, B, C, D (with respect to the conic $\mathcal{C}$). See fig. 4.

Let us now discuss the triple ratio of six points in $\mathbb{CP}^2$ which was introduced by Goncharov. We take the six points to be A, B, C, X, Y, Z. Numerically, this triple ratio is given by

$$r_3(A, B, C; X, Y, Z) = \frac{\langle ABX\rangle\langle BCY\rangle\langle CAZ\rangle}{\langle ABY\rangle\langle BCZ\rangle\langle CAX\rangle}. \tag{151}$$

It turns out that this ratio has several geometrical interpretations. Consider first the situation in fig. 5. There, we have four lines which are dashed and blue: $\alpha = (CB)$, $\beta = (Cb)$, $\gamma = (Cc)$, $\delta = (Cd)$, where $b = (AX) \cap (BY)$, $c = A$ and $d = (CZ) \cap (AX)$. Their cross-ratio, obtained by intersecting with the line (AX), is given by

$$r(\alpha, \beta, \gamma, \delta) = r(a, b, c, d) = (C|B, (AX) \cap (BY), A, Z). \tag{152}$$

[12]Any conic is determined by five points. Given four points there is an infinity of conics which contain them.

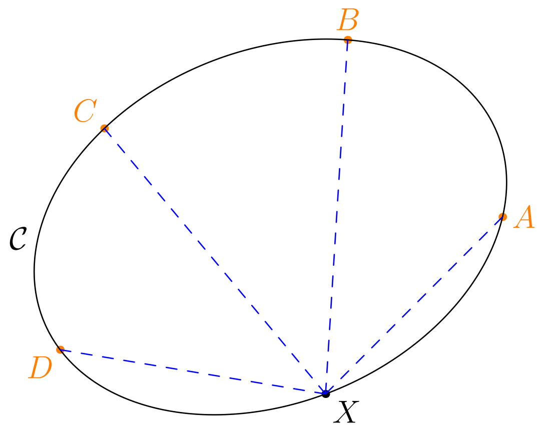


Figure 4: The cross-ratio of points A, B, C, D with respect to the conic $\mathcal{C}$.

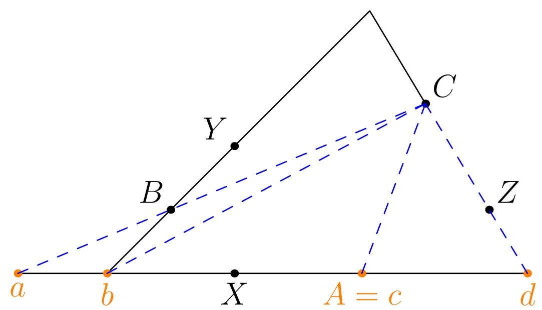


Figure 5: Triple ratio, expressed as a cross-ratio of points on the line (AX).

But, instead of considering the intersections of the lines $(\alpha, \beta, \gamma, \delta)$ with the line (AX) as above, we can consider the intersection with the line (BY). The intersection points are

$$a' = \alpha \cap (BY) = B, \tag{153}$$
$$b' = \beta \cap (BY) = b = (AX) \cap (BY), \tag{154}$$
$$c' = \gamma \cap (BY) = (CA) \cap (BY), \tag{155}$$
$$d' = \delta \cap (BY) = (CZ) \cap (BY). \tag{156}$$

The corresponding figure is fig. 6. If we denote by $\alpha' = (AB)$, $\beta' = (AX)$, $\gamma' = (AC)$, $\delta' = (Ad')$, we have

$$r(a, b, c, d) = r(\alpha, \beta, \gamma, \delta) = r(a', b', c', d') = \\ = r(\alpha', \beta', \gamma', \delta') = (A|B, X, C, (BY) \cap (CZ)). \tag{157}$$

Now we can repeat the previous procedure. We compute the cross-ratio $r(\alpha', \beta', \gamma', \delta')$ by considering the intersection with (CZ). The intersection

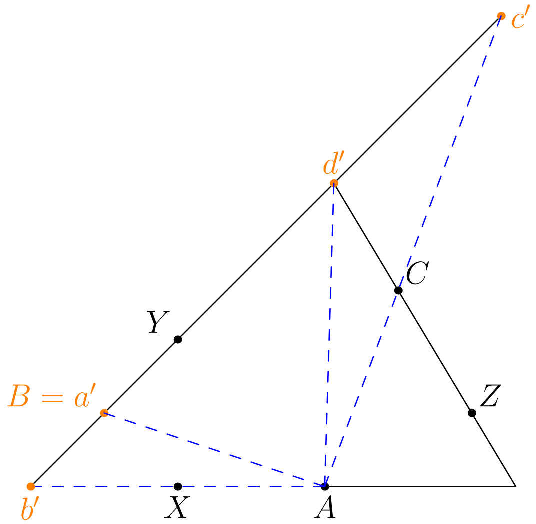


Figure 6: Triple ratio, expressed as a cross-ratio of points on the line (BY).

points are

$$a'' = \alpha' \cap (CZ) = (AB) \cap (CZ), \tag{158}$$
$$b'' = \beta' \cap (CZ) = (AX) \cap (CZ), \tag{159}$$
$$c'' = \gamma' \cap (CZ) = C, \tag{160}$$
$$d'' = \delta' \cap (CZ) = (BY) \cap (CZ). \tag{161}$$

See fig. 7 for a geometrical representation. If we define the lines $\alpha'' = (BA)$, $\beta'' = (Bb'')$, $\gamma'' = (BC)$, $\delta'' = (Bd'')$, we have

$$(B|A, (CZ)\cap(AX), C, Y) = r(\alpha'', \beta'', \gamma'', \delta'') = r(a'', b'', c'', d'') = r(\alpha', \beta', \gamma', \delta'). \tag{162}$$

We have therefore shown that

$$(A|B, X, C, (BY)\cap(CZ)) = (B|A, (CZ)\cap(AX), C, Y) = (C|B, (AX)\cap(BY), A, Z). \tag{163}$$

Notice that this is also implied by the symmetry $r_3(A, B, C; X, Y, Z) = r_3(B, C, A; Y, Z, X)$.

Let us now show that the invariant $(A|B, X, C, (BY) \cap (CZ))$ has the same zeros and poles as $r_3(A, B, C; X, Y, Z)$. Form the definition, we know that $(A|B, X, C, (BY) \cap (CZ))$ vanishes when $\langle ABX\rangle = 0$ or $\langle AC(BY) \cap (CZ)\rangle = 0$. The second three-bracket vanishes if $\langle BCY\rangle = 0$ or $\langle CAZ\rangle = 0$. In the first case B, C, Y are collinear and therefore $(BY) \cap (CZ) = C$ so

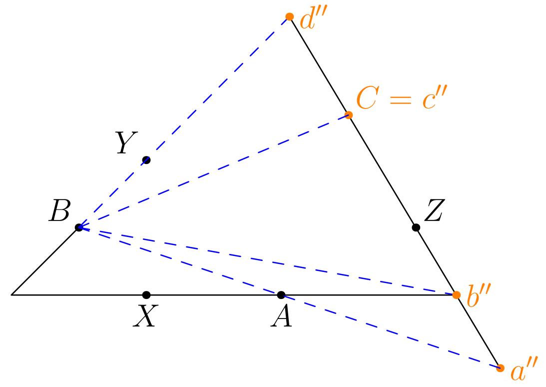


Figure 7: Triple ratio, expressed as a cross-ratio of points on the line (CZ).

we have $\langle AC(BY) \cap (CZ)\rangle = \langle ACC\rangle = 0$. In the second case, when $\langle CAZ\rangle = 0$ we have that $A \in (CZ)$, $C \in (CZ)$ and $P \equiv (BY) \cap (CZ) \in (CZ)$. Since all the entries of the three-bracket are collinear, we find that $\langle AC(BY) \cap (CZ)\rangle = 0$. We have shown that $(A|B, X, C, (BY) \cap (CZ))$ vanishes if $\langle ABX\rangle = 0$ or $\langle BCY\rangle = 0$ or $\langle CAZ\rangle = 0$ which is the same as the numerator of $r_3(A, B, C; X, Y, Z)$. In order to find the poles we reason in the same way.

Notice that in fig. 5, we have five points (a, b, X, c, d) on the line (AX). From five points $(z_1, \ldots, z_5)$ in $\mathbb{CP}^1$ we can produce a dilogarithm identity

$$\sum_{i=1}^{5} (-1)^i \{r(z_1, \ldots, \widehat{z_i}, \ldots, z_5)\}_2 = 0. \tag{164}$$

This motivates us to find the expressions in terms of three-brackets for the other cross-ratios that can be constructed from these five points on (AX) (see fig. 5):

$$r(b, X, A, d) = \frac{\langle BXY\rangle\langle ACZ\rangle}{\langle A \times X, B \times Y, C \times Z\rangle}, \tag{165}$$

$$r(a, X, A, d) = (C|B, X, A, Z), \tag{166}$$

$$r(a, b, A, d) = r_3(A, B, C; X, Y, Z), \tag{167}$$

$$r(a, b, X, d) = r_3(X, B, C; A, Y, Z), \tag{168}$$

$$r(a, b, X, A) = (B|C, Y, X, A). \tag{169}$$

This provides a geometric proof for the following dilogarithm identity

$$-\left\{\frac{\langle BXY\rangle\langle ACZ\rangle}{\langle A\times X, B\times Y, C\times Z\rangle}\right\}_2+\left\{\frac{\langle CBX\rangle\langle CAZ\rangle}{\langle CXA\rangle\langle CZB\rangle}\right\}_2-\left\{\frac{\langle ABX\rangle\langle BCY\rangle\langle CAZ\rangle}{\langle ABY\rangle\langle BCZ\rangle\langle CAX\rangle}\right\}_2 + \left\{\frac{\langle XBA\rangle\langle BCY\rangle\langle CXZ\rangle}{\langle XBY\rangle\langle BCZ\rangle\langle CXA\rangle}\right\}_2 - \left\{\frac{\langle BCY\rangle\langle BXA\rangle}{\langle BYX\rangle\langle BAC\rangle}\right\}_2 = 0. \quad (170)$$

This kind of identities are useful to check that the $B_2 \wedge \mathbb{C}^*$ projection of the 40-term trilogarithm identity is zero. For example, one of the dilogarithm identities which is useful is

$$-\left\{-\frac{\langle 123\rangle\langle 456\rangle}{\langle 1\times 2, 3\times 4, 5\times 6\rangle}\right\}_2-\left\{-\frac{\langle 125\rangle\langle 134\rangle}{\langle 123\rangle\langle 145\rangle}\right\}_2-\left\{-\frac{\langle 123\rangle\langle 156\rangle\langle 345\rangle}{\langle 125\rangle\langle 134\rangle\langle 356\rangle}\right\}_2+ \left\{-\frac{\langle 124\rangle\langle 156\rangle\langle 345\rangle}{\langle 125\rangle\langle 134\rangle\langle 456\rangle}\right\}_2-\left\{-\frac{\langle 156\rangle\langle 345\rangle}{\langle 135\rangle\langle 456\rangle}\right\}_2 = 0, \quad (171)$$

can be interpreted geometrically as five points $(3, 4, (15)\cap(34), (12)\cap(34), (34)\cap(56))$ on the line (34).

$$\left\{-\frac{\langle 156\rangle\langle 234\rangle}{\langle 1\times 2, 3\times 4, 5\times 6\rangle}\right\}_2-\left\{-\frac{\langle 136\rangle\langle 234\rangle}{\langle 123\rangle\langle 346\rangle}\right\}_2-\left\{-\frac{\langle 156\rangle\langle 236\rangle}{\langle 126\rangle\langle 356\rangle}\right\}_2 +\left\{-\frac{\langle 123\rangle\langle 156\rangle\langle 346\rangle}{\langle 126\rangle\langle 134\rangle\langle 356\rangle}\right\}_2-\left\{-\frac{\langle 123\rangle\langle 256\rangle\langle 346\rangle}{\langle 126\rangle\langle 234\rangle\langle 356\rangle}\right\}_2 = 0 \quad (172)$$

can be interpreted geometrically as five points $(1, 2, (12)\cap(34), (12)\cap(36), (12)\cap(56))$ on the line (12).

$$-\left\{-\frac{\langle 156\rangle\langle 234\rangle}{\langle 1\times 2, 3\times 4, 5\times 6\rangle}\right\}_2+\left\{-\frac{\langle 145\rangle\langle 234\rangle}{\langle 124\rangle\langle 345\rangle}\right\}_2+\left\{-\frac{\langle 156\rangle\langle 245\rangle}{\langle 125\rangle\langle 456\rangle}\right\}_2- \left\{-\frac{\langle 124\rangle\langle 156\rangle\langle 345\rangle}{\langle 125\rangle\langle 134\rangle\langle 456\rangle}\right\}_2+\left\{-\frac{\langle 124\rangle\langle 256\rangle\langle 345\rangle}{\langle 125\rangle\langle 234\rangle\langle 456\rangle}\right\}_2 = 0 \quad (173)$$

can be interpreted geometrically as five points $(1, 2, (12)\cap(34), (12)\cap(45), (12)\cap(56))$ on the line (12).

$$\left\{-\frac{\langle 123\rangle\langle 456\rangle}{\langle 1\times 2, 3\times 4, 5\times 6\rangle}\right\}_2+\left\{-\frac{\langle 125\rangle\langle 234\rangle}{\langle 123\rangle\langle 245\rangle}\right\}_2+\left\{-\frac{\langle 123\rangle\langle 256\rangle\langle 345\rangle}{\langle 125\rangle\langle 234\rangle\langle 356\rangle}\right\}_2- \left\{-\frac{\langle 124\rangle\langle 256\rangle\langle 345\rangle}{\langle 125\rangle\langle 234\rangle\langle 456\rangle}\right\}_2+\left\{-\frac{\langle 256\rangle\langle 345\rangle}{\langle 235\rangle\langle 456\rangle}\right\}_2 = 0, \quad (174)$$

can be interpreted geometrically as five points $(3, 4, (12)\cap(34), (25)\cap(34), (34)\cap(56))$ on the line (34).

The identities above are the identities needed to show the vanishing of terms of type $* \otimes \langle 123\rangle$ in the projection to $B_2 \otimes \mathbb{C}^*$ of the 40-term trilogarithm identity. For the terms of type $* \otimes \langle 124\rangle$ the same identities are sufficient, but there is another, simpler identity too, written below

$$-\left\{-\frac{\langle 126\rangle\langle 145\rangle}{\langle 124\rangle\langle 156\rangle}\right\}_2 + \left\{-\frac{\langle 126\rangle\langle 245\rangle}{\langle 124\rangle\langle 256\rangle}\right\}_2 - \left\{-\frac{\langle 146\rangle\langle 245\rangle}{\langle 124\rangle\langle 456\rangle}\right\}_2 + \left\{-\frac{\langle 156\rangle\langle 245\rangle}{\langle 125\rangle\langle 456\rangle}\right\}_2 - \left\{-\frac{\langle 156\rangle\langle 246\rangle}{\langle 126\rangle\langle 456\rangle}\right\}_2 = 0. \quad (175)$$

This identity is special because it does not depend on point 3 at all. It can be more geometrically written as

$$\{(1|2654)\}_2 + \{(2|1456)\}_2 + \{(4|1652)\}_2 + \{(5|1246)\}_2 + \{(6|1542)\}_2 = 0. \quad (176)$$

Curiously, this simple-looking identity has a slightly more obscure geometrical interpretation. Through the five points 1, 2, 4, 5, 6 passes a unique conic $\mathcal{C}$. The cross-ratio $(1|2654)$ is the cross-ratio of the points $(2, 6, 5, 4)$ with respect to the conic $\mathcal{C}$. But we can pick another point $X \in \mathcal{C}$ and we have, by Chasles' theorem, that $(X|2654) = (1|2654)$. Then the previous identity becomes

$$-\{(X|2456)\}_2+\{(X|1456)\}_2-\{(X|1256)\}_2+\{(X|1246)\}_2-\{(X|1245)\}_2 = 0, \quad (177)$$

which is the usual form of the dilogarithm identity, where the cross-ratios are cross-ratios of the lines $(X1)$, $(X2)$, $(X4)$, $(X5)$, $(X6)$.

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