Upwinding in finite element systems

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1D convection diffusion

- Problems with small viscosity. Model problem on $[-1, 1]$:

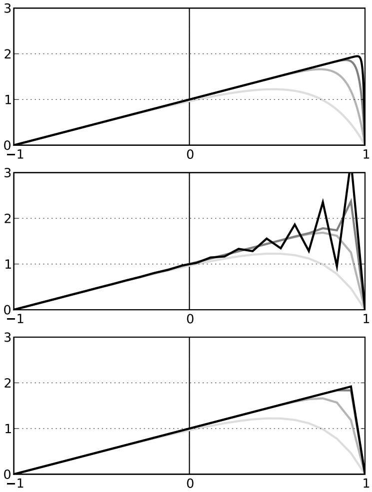
$$-\alpha u'' + \beta u' = f, \tag{1}$$
$$u(-1) = 0, \ u(1) = 0. \tag{2}$$

- Galerkin with continuous piecewise affine $u_h \in X_h$ s.t.:

$$\forall v_h \in X_h \quad \alpha \int u_h' v_h' + \int \beta u_h' v_h = \int f v_h. \tag{3}$$

- Unstable when $\alpha < \beta h$.

Numerical results



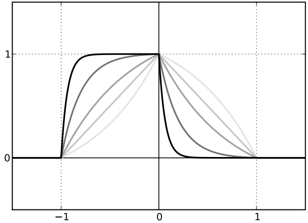
Exponential fitting. Allen-Southwell 1955.

- Petrov Galerkin method: $u_h \in X_h$ s.t $\forall v_h \in Y_h$:

$$\alpha \int u_h' v_h' + \int \beta u_h' v_h = \int f v_h. \qquad (4)$$

- X_h continuous piecewise affine.
  Y_h continuous piecewise exponential (adjoint problem):

$$-\alpha v_h'' - \beta v_h' = 0 \quad \text{on each subinterval.} \qquad (5)$$



- Stable, accurate away from boundary layers.

3D discrete de Rham sequences

- Hilbert spaces of scalar and vector fields:

$$\mathrm{H}_{\mathsf{op}} = \{u \in \mathrm{L}^2 \;:\; \mathsf{op}\, u \in \mathrm{L}^2\}.$$

- 3D de Rham sequence, Hilbert style:

$$\mathrm{H}_{\mathsf{grad}} \xrightarrow{\mathsf{grad}} \mathrm{H}_{\mathsf{curl}} \xrightarrow{\mathsf{curl}} \mathrm{H}_{\mathsf{div}} \xrightarrow{\mathsf{div}} \mathrm{H}$$

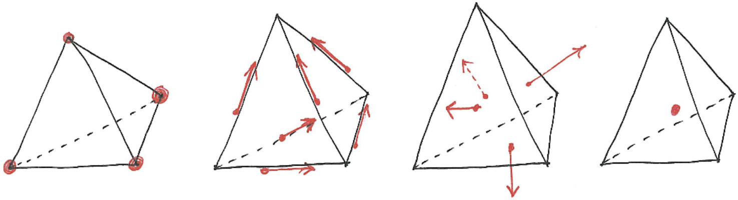
- Goal: define good finite dimensional subcomplexes:

$$\begin{array}{ccccccc}
\mathrm{H}_{\mathsf{grad}} & \xrightarrow{\mathsf{grad}} & \mathrm{H}_{\mathsf{curl}} & \xrightarrow{\mathsf{curl}} & \mathrm{H}_{\mathsf{div}} & \xrightarrow{\mathsf{div}} & \mathrm{H} \\
\cup\uparrow & & \cup\uparrow & & \cup\uparrow & & \cup\uparrow \\
X_h^0 & \xrightarrow{\mathsf{grad}} & X_h^1 & \xrightarrow{\mathsf{curl}} & X_h^2 & \xrightarrow{\mathsf{div}} & X_h^3
\end{array}$$

Nédélec-Raviart-Thomas elements

Lowest order elements:

Space	Local form	Parameters	DoF	Assigned
$\mathrm{H}_{\mathsf{grad}}$	$a \cdot x + b$	$a \in \mathbb{R}^3, b \in \mathbb{R}$	$u(x)$	vertexes
$\mathrm{H}_{\mathsf{curl}}$	$a \times x + b$	$a \in \mathbb{R}^3, b \in \mathbb{R}^3$	$\int u \cdot \tau$	edges
$\mathrm{H}_{\mathsf{div}}$	$a\, x + b$	$a \in \mathbb{R}\ , b \in \mathbb{R}^3$	$\int u \cdot \nu$	faces
H	b	$b \in \mathbb{R}$	$\int u$	tetrahedra



Commuting interpolation

- Matching degrees of freedom yields commuting interpolators:

$$
\begin{array}{ccccccc}
\mathcal{F} & \xrightarrow{\text{grad}} & \mathcal{V} & \xrightarrow{\text{curl}} & \mathcal{V} & \xrightarrow{\text{div}} & \mathcal{F} \\
\downarrow{\scriptstyle I_h^0} & & \downarrow{\scriptstyle I_h^1} & & \downarrow{\scriptstyle I_h^2} & & \downarrow{\scriptstyle I_h^3} \\
X_h^0 & \xrightarrow{\text{grad}} & X_h^1 & \xrightarrow{\text{curl}} & X_h^2 & \xrightarrow{\text{div}} & X_h^3
\end{array}
$$

- FEEC reviewed in Arnold-Falk-Winther 10:
  – Lowest order equivalent to Whitney forms (Bossavit 88).
  – Eigenvalue convergence for Maxwell / Hodge-Laplacian, (Dodziuk-Patodi 76, Boffi 12, C.- Winther 12).

Recursive harmonic extension

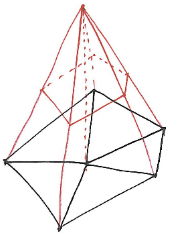
- Low order mixed FE are locally harmonic:
  Basis can be defined by recursive harmonic extension.
- Solve $\mathrm{d}^\star \mathrm{d} u = 0$ and $\mathrm{d}^\star u = 0$ at each level.
- Nédélec: $\operatorname{div} u = 0$.
  Raviart-Thomas: $\operatorname{curl} u = 0$.
  Constants: $\operatorname{grad} u = 0$.
- Works also for Whitney forms,
  and lowest order tensor-product.

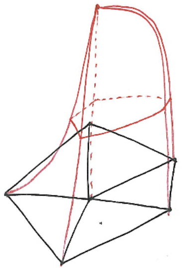
Upwinded harmonic extension

- Multidimensional convection diffusion on S:
  given α, β ($\operatorname{div} \beta = 0$) and f, find u s.t.:

$$-\alpha \Delta u + \beta \cdot \operatorname{grad} u = f, \;\; u|_{\partial S} = 0. \tag{6}$$

  Introduce upwinding in harmonic extension:





Generalization to differential forms

- Define one-form $\beta \simeq g(\beta, \cdot)$ and covariant exterior derivative:

$$u \mapsto \mathrm{d}_\beta u = \mathrm{d}u + \beta \wedge u. \tag{7}$$

$$\mathrm{grad}\, u + \beta u, \quad \mathrm{curl}\, u + \beta \times u, \quad \mathrm{div}\, u + \beta \cdot u. \tag{8}$$

- Convective harmonic functions: $\mathrm{d}_\beta^\star \mathrm{d}u = 0$ and $\mathrm{d}_\beta^\star u = 0$.
  If $\beta = \mathrm{d}\gamma$ equivalent conditions:

$$\mathrm{d}^\star \exp(-\gamma)\mathrm{d}u = 0, \quad \mathrm{d}^\star \exp(-\gamma) u = 0. \tag{9}$$

  Harmonic for weighted L^2 product!
- Gauge theory: group $\mathbb{R}_+$ acting on $\mathbb{R}$.
  $\mathbb{U}$ acting on $\mathbb{C}$ relates to wave equations.

Finite element systems

- Fix a cellular complex $\mathcal{T}$.
- A finite element system A is
  $A^k(T) \subseteq \Omega^k(T)$ for $k \in \mathbb{N}$ and $T \in \mathcal{T}$ of all dimensions.
  – exterior derivative, $\mathrm{d} : A^k(T) \to A^{k+1}(T)$.
  – pullback, $i : T' \subseteq T$ gives $i^\star : A^k(T) \to A^k(T')$.
- Inverse limit, define:

$$A^k(\mathcal{T}) = \{u \in \bigoplus_{T \in \mathcal{T}} A^k(T) \ : \ T' \subseteq T \Rightarrow u_T|_{T'} = u_{T'}\}. \quad (10)$$

(encodes a form of continuity).

Compatible finite element systems

Equivalent in a given FES:

- Discrete harmonic extension is well defined. For $u \in A^k(\partial T)$:

$$u \in A^k(T) \ : \ \mathrm{d}u \perp \mathrm{d}A^k_0(T) \text{ and } u \perp \mathrm{d}A^{k-1}_0(T). \qquad (11)$$

- Trace is onto: $A^k(T) \to A^k(\partial T)$.
  Exact sequence (with or without BC).

$$0 \to \mathbb{R} \to A^0(T) \to A^1(T) \to \ldots \to A^d(T) \to 0. \qquad (12)$$

- Projection based interpolation (Demkowicz-Buffa 05) is well defined.
- The FES has a commuting interpolation operator.

Applications

Gives spaces of locally harmonic forms (C. 08).
Unifies some constructions by agglomeration:

- Generalizes Kuznetzov-Repin 05 (Hdiv).
- Gives dual spaces (Hdiv-Hrot) in 2D (Buffa-C. 07).
- Can be applied recursively (Pasciak-Vassilevski 08).

Here: locally harmonic forms for exponentially weighted L^2 product
gives a conforming discrete de Rham complex,
with a commuting interpolator,
adapted to convection diffusion.

Setting for analysis

On $S = [0, T] \times U$, $\beta = (1, 0)$.

$$-\alpha \Delta u + \beta \cdot \operatorname{grad} u + u = f. \tag{13}$$

Product grid: intervals on $[0, T]$ simplices on U. Petrov Galerkin.
Standard trial.
Test: upwinding gives tensor products of:

- upwinded exponentials on $[0, T]$.
- piecewise linears on U.

Continuity and Inf-Sup

- Norm, depends on α:

$$\|u\|_\alpha^2 = \|u\|^2_{\mathrm{H}^{1/2}(0,T)\otimes \mathrm{L}^2(U)} + \int |u|^2 + \alpha |\operatorname{grad} u|^2. \qquad (14)$$

  Sangalli 08, 1D a posteriori.

- Continuity up to $|\log h|^2$. Price for not using $\mathrm{H}^{1/2}_{00}(0, T)$.
- For quasi-uniform meshes, for $\alpha \leq h/C|\log h|$ we have:

$$\inf_{u \in X_h} \sup_{v \in Y_h^\alpha} \frac{|a_\alpha(u, v)|}{\|u\|_\alpha \|v\|_\alpha} \geq 1/|\log \alpha|(|\log \alpha| + |\log h|^6). \qquad (15)$$

A limiting case

- Neglect streamline diffusion, gives parabolic:

$$\beta \cdot \operatorname{grad} u - \alpha \Delta_U u + u = f. \tag{16}$$

- Extreme upwinding:
  Testfunctions are piecewise constant in time.
  Gives Crank-Nicolson scheme.
- Baiocchi-Brezzi 1983: $\mathrm{H}^{1/2}$ in time.
- Continuous optimal testfunction:

$$v = \mathcal{H}u + \lambda u. \tag{17}$$

  using Hilbert transform and large enough λ.
  Can be projected onto testspace.

Hilbert transform

- Definition:
$$u \mapsto \frac{1}{x} * u. \tag{18}$$
- Fourier transform:
$$x \mapsto 1/x, \text{ gives } \xi \mapsto i\,\mathsf{sign}\,\xi. \tag{19}$$
- Usage:
$$\int \dot{u}\mathcal{H}u = \int \xi\,\mathsf{sign}(\xi)|\hat{u}(\xi)|^2 \mathrm{d}\xi = |u|^2_{\mathrm{H}^{1/2}} \tag{20}$$
- Continuous on L^2.

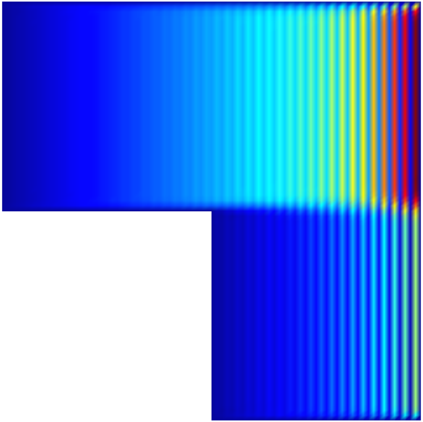
Optimal test function

- $v_0 = \mathcal{H}u + \lambda u$.
  v_1: projection onto piecewise constants in time.
  v_2: solve $\alpha \dot{v}_2 + \beta v_2 = \beta v_1$, $v_2(0) = 0$.
  Is in upwinded test space! except for outflow boundary.
  v_3 corrected on last strip of elements.
- Map $v_0 \mapsto v_2$, from $\mathrm{H}^{1/2}$ to $\mathrm{H}^{1/2}$ norm $|\log \alpha|^{1/2}$.
  Projection onto piecewise constants continuous
  $\mathrm{H}^{1/2} \to \mathrm{B}^{1/2,2}_{\infty}$, then smoothing effect of ODE.
- Hilbert transform on piecewise linears:

$$\|\mathcal{H}u\|_{\mathrm{L}^\infty} \leq C|\log h|^{1/2}\|u\|_{\mathrm{H}^{1/2}} \tag{21}$$

- arXiv...

Numerics



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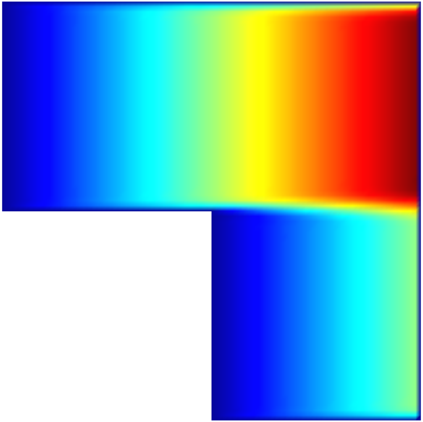
Numerics



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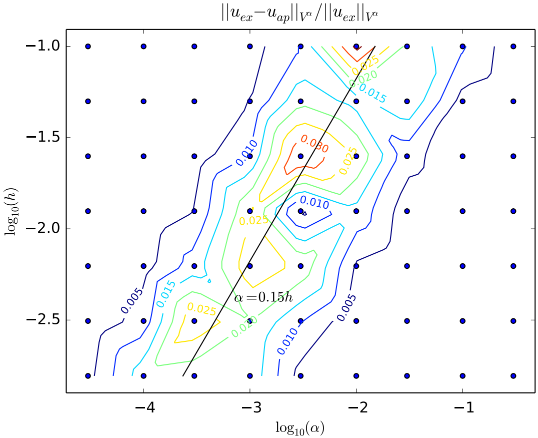
More numerics



Compares exact and discrete unwinding.
Adapted sub-mesh with one extra point per cell.

Examples of FES

- Fix $p \in \mathbb{N}^*$. For all cells T:

$$A^k(T) = \{u \in \mathbb{PA}_p^k(T)\}. \tag{22}$$

- Raviart-Thomas 77, Nédélec 80, Hiptmair 99, Arnold-Falk-Winther 06.
  Koszul operator (or Poincaré operator):

$$(\kappa u)_x(\xi_1, \ldots, \xi_k) = u_x(x, \xi_1, \ldots, \xi_k). \tag{23}$$

  Fix $p \in \mathbb{N}^*$. For all simplexes T:

$$A^k(T) = \{u \in \mathbb{PA}_p^k(T) \ : \ \kappa u \in \mathbb{PA}_p^{k-1}(T)\} \tag{24}$$

Smoothing

- Schöberl 08, C. 07, Arnold-Falk-Winther 06, C.-Winther 08.
- Bramble-Hilbert:

$$\|u - I_h u\|_{\mathrm{L}^q(T)} \preceq \sum_{T' \subseteq T} h_T^{\ell + (\dim T - \dim T')/q} \|\nabla^\ell u\|_{\mathrm{L}^q(T')}, \quad (25)$$

- Smoothing by convolution on $\mathbb{R}^d$. Bell function ϕ, scaled ϕ^δ. For a cell T' of any dimension:

$$\|\phi^\delta * u\|_{\mathrm{L}^q(T')} \preceq \delta^{(\dim T' - d)/q} \|u\|_{\mathrm{L}^q(\mathcal{V}_\delta(T'))}, \quad (26)$$

- Fix ϵ. $u \mapsto u_h = I_h(\phi^{\epsilon h} * u)$ is:
  – $\mathrm{L}^q(M) \to \mathrm{L}^q(M)$ stable,
  – order optimal $\|u - u_h\| \preceq h^\ell \|\nabla^\ell u\|$,
  – choose ϵ so that $\|u - u_h\| \leq 1/2 \|u\|$ when u discrete.

Applications

- Hilbert L^2 setting. Gives stable commuting projections therefore eigenvalue convergence for Hodge Laplace.
- Locality gives translation estimate (Karlsen-Karper 10):

$$\|v_h - \tau_\xi v_h\|_{\mathrm{L}^2} \preceq (|\xi| + |\xi|^{1/2} h^{1/2}) \|\mathrm{d} v_h\|_{\mathrm{L}^2}. \qquad (27)$$

  for $v_h \in X_h^k$, L^2 orthogonal to $\mathrm{d}X_h^{k-1}$.
- L^q estimates give Sobolev injection (Buffa-Ortner 08), C.-Scheid 11, for Maxwell-Klein-Gordon, e.g.

$$\|v_h\|_{\mathrm{L}^6} \preceq \|\operatorname{curl} v_h\|_{\mathrm{L}^2}, \qquad (28)$$

  for $v_h \in X_h^1$, Nédélec edge element, discrete divergence free.