Nonconforming Virtual Elements for second order elliptic problems

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+ fruitful discussions: F. Brezzi & L.D. Marini

Building Bridges: Connections and Challenges in
Modern Approaches to Numerical Partial Differential Equations
Durham, July 14th, 2014

VEM: A brand new method

- *Born* as *evolution* of Mimetic Finite Difference (MFD)
  - ▷ difficult to construct high order approximations
    - ▷ [Manzini & Lipnikov(14)]
  - ▷ analysis cumbersome and not always feasible
- Often, MFD can be *recast* as *VEM*

- [Beirao,Brezzi, Cangiani, Marini, Manzini,Russo (13)]
- Plate Bending [Brezzi,Marini (13)]
- Elasticity [Beirao, Brezzi,Marini (13)]
- Elliptic 3D and more: [Ahmad, A. Alsaedi,Brezzi,Marini,Russo (14)]
- Mixed $H(\text{div})$-2D: [Brezzi,Falk, Marini(14),....]
- ......

Toy Model problem: the Poisson Problem

Let $\Omega \in \mathbb{R}^d, d = 2, 3$ convex. Given $f \in L^2(\Omega)$, find $u \in H^2(\Omega)$ s.t.

$$-\Delta u = f \qquad \text{in } \Omega \qquad u = 0 \qquad \text{on } \partial\Omega.$$

- Variational Formulation: $V = H_0^1(\Omega)$

$$\text{Find } u \in V \text{ s.t} \quad \int_\Omega \nabla u \nabla w d\Omega = \int_\Omega f w d\Omega \qquad \forall w \in V.$$

- Conforming FEM: $V_h^{conf} \subset V$

$$\text{find } u_h^c \in V_h^{conf} \quad : \quad \int_\Omega \nabla u_h^c \nabla w_h^c d\Omega = \int_\Omega f w_h^c d\Omega \;\; \forall w_h^c \in V_h^{conf}$$

- Nonconforming FEM $V_h^{nc} \not\subset V$

$$\text{find } u_h \in V_h^{nc} \not\subset V \quad s.t. \quad \sum_K \int_K \nabla u_h \nabla v_h = \sum_K \int_K f w_h \;\; \forall w_h \in V_h^{nc}$$

Toy Model problem: the Poisson Problem

Let $\Omega \in \mathbb{R}^d, d = 2, 3$ convex. Given $f \in L^2(\Omega)$, find $u \in H^2(\Omega)$ s.t.

$$-\Delta u = f \qquad \text{in } \Omega \qquad u = 0 \qquad \text{on } \partial\Omega.$$

- Variational Formulation: $V = H_0^1(\Omega)$

Find $u \in V$ s.t $\quad a(u, v) := \int_\Omega \nabla u \nabla w d\Omega = \int_\Omega f w d\Omega \qquad \forall\, w \in V.$

- Conforming FEM: $V_h^{conf} \subset V$

$$\text{find } u_h^c \in V_h^{conf} \quad : \quad \int_\Omega \nabla u_h^c \nabla w_h^c d\Omega = \int_\Omega f w_h^c d\Omega \;\; \forall\, w_h^c \in V_h^{conf}$$

- Nonconforming FEM $V_h^{nc} \not\subset V$

$$\text{find } u_h \in V_h^{nc} \not\subset V \quad s.t. \quad \sum_K \int_K \nabla u_h \nabla v_h = \sum_K \int_K f w_h \;\; \forall\, w_h \in V_h^{nc}$$

Nonconforming FEM: a brief overview...

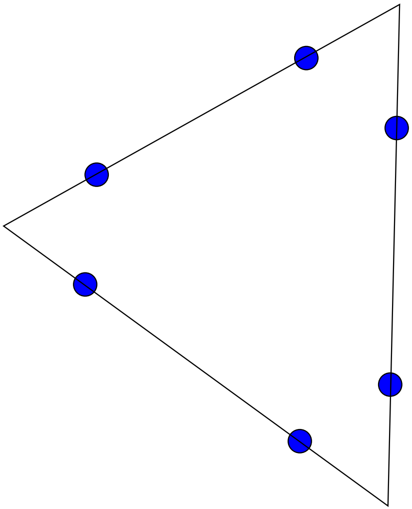
- Variational Crime: $V_h^{nc} \not\subset V$ [Strang (73,74)]

Benefits in continuum mechanics:

- ▷ Stokes $k = 1$ [Coruziex-Raviart (73)]
- ▷ Fourth order [Lascaux -Lasaint(75)]
- ▷ Stokes $k = 2$ 2D, 3D[Fortin Soulie (85),Fortin (85)]
- ▷ Hybridization Hellan-Herrmann-Johnson [Comodi [89] any k
- ▷ Stokes $k = 3$ [Crouziex-Falk (89)]
- ▷ Stokes $k = 1, \ K = \square$ [Rannacher-Turek (92)]
- ▷ Stokes k [Matthies-Tobiska (05),Baran-Stoyan (06)]
- ▷ Elliptic a-posteriori [Ainsworth-Rakin (08)]

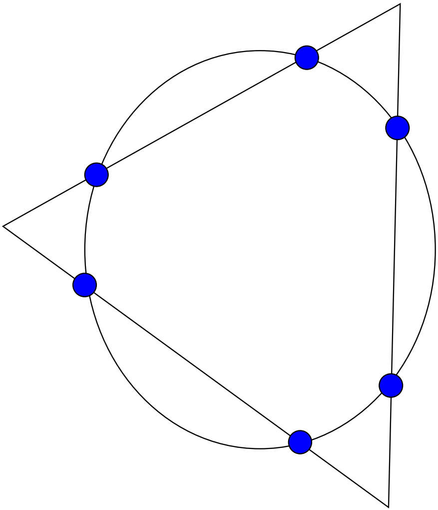
- Construction of spaces $V_h^{nc} \not\subset V$ highly depends on:
  - ▷ degree k and shape of element K
  - ▷ extensions to 3D not simple for k even

Non-conforming finite elements $k = 2$



[Fortin & Soulie (83)]

Non-conforming finite elements $k = 2$



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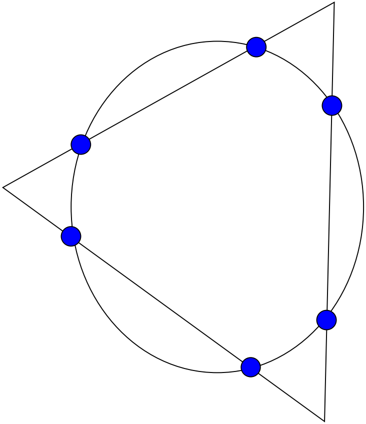
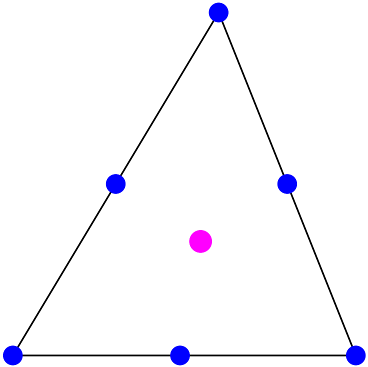




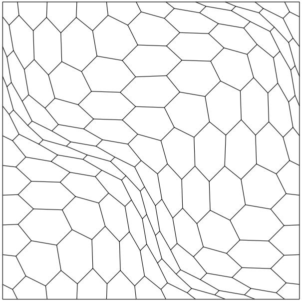
[Fortin & Soulie (83)]

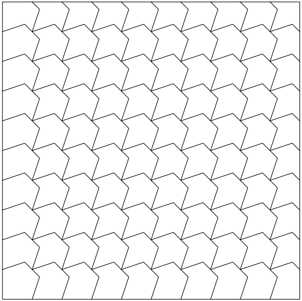
VEM & FEM in a few words..

- Similarities:
  - same starting point, i.e., variational formulation of the given problem;
  - for fixed $k \geq 1$ $\mathbb{P}^k \subset V_h$ (spaces of polynomials of a given degree are included.
- Differences
  - grids made of polygons of arbitrary shape can be used;
  - easy to construct high-order (& high regularity approximations.
- Note: VEM offers more flexibility (specially in mesh handling) but in principle the covergence would not be better than the equivalent FEM
- But: VEM might provide a working element where FEM fails to do so...?

Nonconforming VEM

- $\{\mathscr{T}_h\}_h$ partition into elements K (now *polygons*!)
- $\mathcal{E}_h$ skeleton of partition: edges $(d = 2)$; faces $(d = 3)$ and $\mathcal{E}_h^{\partial} = \mathcal{E}_h \cap \partial\Omega$





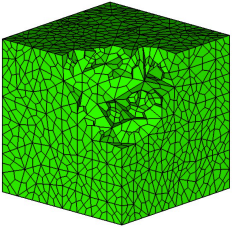
Nonconforming VEM

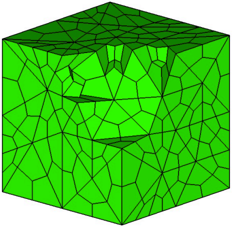
- $\{\mathscr{T}_h\}_h$ partition into elements K (now *polygons*!)

  We assume *shape regularity* for $\mathscr{T}_h$: $\exists\, \varrho > 0$ s.t:

  - K star-shaped w.r.t all the points of a sphere of radius $\geq \varrho h_K$;
  - $e \in \mathcal{E}_h$ star-shaped w.r.t. all points of a disk of radius $\geq \varrho h_e$.
  - for every K and for every $e \subset \partial K \quad : \quad h_e \geq \varrho h_K$

- $\mathcal{E}_h$ skeleton of partition: edges $(d = 2)$; faces $(d = 3)$ and $\mathcal{E}_h^\partial = \mathcal{E}_h \cap \partial\Omega$





$$[\![v]\!] := v^+ \mathbf{n}_e^+ + v^- \mathbf{n}_e^- \quad \text{on } e \in \mathcal{E}_h \setminus \partial\Omega \quad \text{and} \quad [\![v]\!] := v\mathbf{n}_e \text{ on } e \in \mathcal{E}_h \cap \partial\Omega\,,$$

Nonconforming VEM: general plan

Let $k \geq 1$ be fixed

$$\begin{cases} \text{Find } u_h \in V_h^k \qquad \text{such that:} \\ a_h(u_h, v_h) = < f_h, v_h > \qquad \forall\, v_h \in V_h^k \end{cases} \qquad (P)$$

Ingredients:

- Definition-Construction of $V_h^k \not\subset V$
- Definition-Construction of $a_h : V_h^k \times V_h^k \longrightarrow \mathbb{R}$
- Definition-Construction of $f_h \in V_h'$

How: To Guarantee (P) has unique solution u_h and optimal convergence....

- ▷ "we look for sufficient conditions on a_h and V_h that ensure all the good properties that you would have with *standard FE*"
- ▷ Here: we also aim at avoiding *patologies* compared to *nonconforming* FE

Construction of local element space $V_h^k(K)$: fixed $k \geq 1$

- $V_h^k(K)$ associated to polygon/polyhedra K; $n := \#$ edges/faces of K

Recall the definition of conforming VEM:

$$V_h^{conf}(K) = \{ v \in H^1(K) \cap C^0(\partial K) : \Delta v \in \mathbb{P}^{k-2}(K), \quad \boxed{v|_e \in \mathbb{P}^k(e)} \; \forall e \subset \partial K \}$$

- Can we still ask $v|_e$ to be a polynomial and enforcing *non-conformity* ??

  $\longrightarrow$ leads to constructions dependent on k and n being odd or even X

Construction of local element space $V_h^k(K)$: fixed $k \geq 1$

- $V_h^k(K)$ associated to polygon/polyhedra K; $n := \#$ edges/faces of K

$$V_h^k(K) = \left\{ v \in H^1(K) \; : \quad \Delta v \in \mathbb{P}^{k-2}(K), \quad \frac{\partial v}{\partial n} \in \mathbb{P}^{k-1}(e) \;\; \forall e \subset \partial K \; \right\},$$

$$\dim(V_h^k(K)) = \begin{cases} nk + k(k-1)/2 & \text{for } d = 2, \\ nk(k+1)/2 + k(k^2-1)/6 & \text{for } d = 3, \end{cases}$$

Dofs:

- $\mathcal{M}_e^{k-1}(v_h) = \frac{1}{|e|} \int_e v_h \, q_{k-1} \, ds, \quad \forall q_{k-1} \in \mathbb{P}^{k-1}(e) \; \forall e \subset \partial K$
- $\mathcal{M}_K^{k-2}(v_h) = \frac{1}{|K|} \int_K v_h \, p_{k-2} \, \mathrm{dx}, \quad \forall p_{k-2} \in \mathbb{P}^{k-2}(K)$

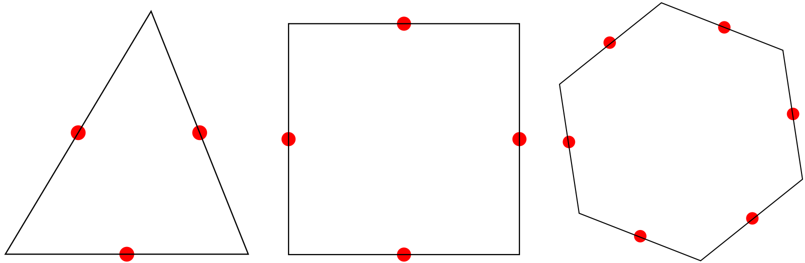
- Note $\dim(V_h^k(K)) = \#$ Dofs
- same dofs as MFD [Manzini & Lipnikov(14)]

Construction of local element space $k = 1$

$$k = 1 \qquad V_h^1(K) = \left\{ v \in H^1(K) \;:\; \Delta v = 0, \quad \frac{\partial v}{\partial n} \in \mathbb{P}^0(e) \;\forall e \subset \partial K \right\}$$

- $\frac{\partial v}{\partial n} =$ constant on each $e \longrightarrow n$ conditions
- $\Delta v = 0$ in $K \longrightarrow 1$ condition

But: $v \in V_h^1(K)$ can be determined if $\int_{\partial K} \frac{\partial v}{\partial \mathbf{n}} = 0 \longrightarrow$ -1 condition.

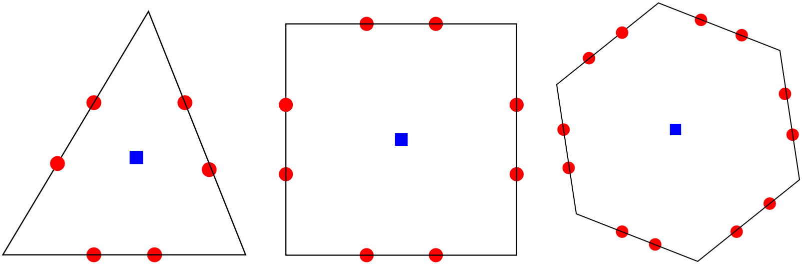


[Crouziex-Raviart (73)] [Rannacher-Turek (92)]

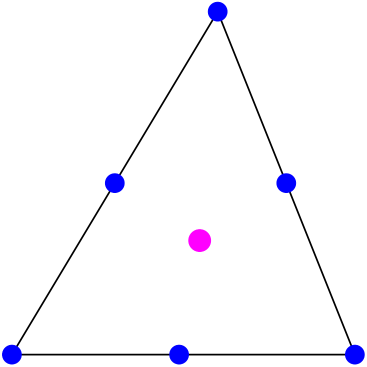
Construction of local element space $k = 2$

$$k = 2 \quad V_h^2(K) = \left\{ v \in H^1(K) : \quad \Delta v \in \mathbb{P}^0(K), \quad \frac{\partial v}{\partial n} \in \mathbb{P}^1(e) \ \forall e \subset \partial K \ \right\},$$

- Δv =constant in K $\longrightarrow$ 1 condition
- $\frac{\partial v}{\partial n} \in \mathbb{P}^1(e)$ on each e $\longrightarrow$ $n \cdot \dim(\mathbb{P}^1(e)) = \mathbf{n} \cdot d$ conditions



Non-conforming VEM vs FEM $k = 2$



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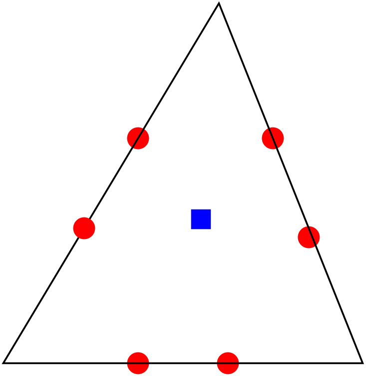

VEM

Construction of local element space: Unisolvence

$$V_h^k(K) = \{ v \in H^1(K) : \quad \Delta v \in \mathbb{P}^{k-2}(K) \quad \frac{\partial v}{\partial n} \in \mathbb{P}^{k-1}(e) \ \forall e \subset \partial K \ \}$$

The degrees of freedom dofs are unisolvent for $V_h^k(K)$.

Idea or Reason:

- $\dim(V_h^k(K)) = \#$ Dofs ✓
- If $v_h \in V_h^k(K)$ s.t. $\mathcal{M}_e^{k-1}(v_h) = 0 \ \forall\, e \subset \partial K \quad \& \quad \mathcal{M}_K^{k-2}(v_h) = 0 \stackrel{?}{\Longrightarrow} \quad v_h \equiv 0$??

Construction of local element space: Unisolvence

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Idea or Reason: $\dim(V_h^k(K)) = \#$ Dofs ✓

- If $v_h \in V_h^k(K)$ s.t. $\mathcal{M}_K^{k-2}(v_h) = 0$ & $\mathcal{M}_e^{k-1}(v_h) = 0\ \forall\, e \subset \partial K \overset{?}{\Longrightarrow} v_h \equiv 0$??

$$\mathcal{M}_K^{k-2}(v_h) = \frac{1}{|K|}\int_K v_h\, p_{k-2}\, \mathrm{d}x = 0 \qquad \mathcal{M}_e^{k-1}(v_h) = \frac{1}{|e|}\int_e v_h\, q_{k-1}\, ds = 0$$

$$\int_K |\nabla v_h|^2 dx = -\int_K v_h \underbrace{\Delta v_h}_{\in \mathbb{P}^{k-2}(K)} d\mathbf{x} + \sum_{e \in \partial K} \int_e v_h \underbrace{\frac{\partial v_h}{\partial \mathbf{n}}}_{\in \mathbb{P}^{k-1}(e)} ds$$

(Divergence Theorem)

Construction of local element space: Unisolvence

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$$\int_K |\nabla v_h|^2 dx = \underbrace{-\int_K v_h \boxed{\Delta\, v_h} d\mathbf{x}}_{= \mathcal{M}_K^{k-2}(v_h)} + \sum_{e\in\partial K} \underbrace{\int_e v_h \boxed{\frac{\partial v_h}{\partial \mathbf{n}}} ds}_{\mathcal{M}_e^{k-1}(v_h)} = 0$$

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Idea or Reason: $\dim(V_h^k(K)) = \#$ Dofs ✓

- If $v_h \in V_h^k(K)$ s.t. $\mathcal{M}_K^{k-2}(v_h) = 0 \quad \& \quad \mathcal{M}_e^{k-1}(v_h) = 0 \ \forall\, e \subset \partial K \implies v_h \equiv 0$ ✓

$$\int_K |\nabla v_h|^2 dx = \underbrace{-\int_K v_h \boxed{\Delta v_h} d\mathbf{x}}_{= \mathcal{M}_K^{k-2}(v_h)} + \sum_{e \in \partial K} \underbrace{\int_e v_h \boxed{\frac{\partial v_h}{\partial \mathbf{n}}} ds}_{\mathcal{M}_e^{k-1}(v_h)} = 0$$

(Divergence Theorem)

$$= \mathcal{M}_K^{k-2}(v_h) + \sum_{e \in \partial K} \mathcal{M}_e^{k-1}(v_h) = 0$$

$$\implies |\nabla v_h| \equiv 0 \implies v_h = \text{constant in } K$$

- But $\mathcal{M}_e^0(v_h) = 0$ on each $e \subset \partial K \implies v_h \equiv 0$ in K

Construction of *(global)* virtual element space: Notation

$$H^s(\mathscr{T}_h) = \prod_{K\in\mathscr{T}_h} H^s(K) = \left\{ v \in L^2(\Omega) \ : \ v|_K \in H^s(K) \right\}, \qquad s > 0,$$

broken H^1-semi-norm: $$|v|^2_{1,h} := \sum_{K\in\mathscr{T}_h} \|\nabla v\|^2_{0,K} \quad \forall\, v \in H^1(\mathscr{T}_h)$$

- $|v|^2_{1,h}$ is a norm for $v \in H^1_0(\Omega)$

Construction of *(global)* virtual element space: Notation

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broken H^1-semi-norm: $\qquad |v|_{1,h}^2 := \sum_{K\in\mathscr{T}_h} \|\nabla v\|_{0,K}^2 \quad \forall\, v \in H^1(\mathscr{T}_h)$

- $|v|_{1,h}^2$ is a norm for $v \in H_0^1(\Omega)$

▷ A space with some *continuity built in...*

$$H^{1,nc}(\mathscr{T}_h; k) = \left\{ v \in H^1(\mathscr{T}_h) \;:\; \int_e [\![v]\!] \cdot \mathbf{n}_e\, q\, ds = 0 \;\; \forall\, q \in \mathbb{P}^{k-1}(e),\; \forall e \in \mathcal{E}_h \right\}.$$

- $|v|_{1,h}^2$ is a norm for $v \in H^{1,nc}(\mathscr{T}_h; 1)$

Construction of *(global)* virtual element space

$$H^{1,nc}(\mathscr{T}_h;k) = \left\{ v \in H^1(\mathscr{T}_h) \ : \ \int_e [\![v]\!] \cdot \mathbf{n}_e \, q \, ds = 0 \ \ \forall\, q \in \mathbb{P}^{k-1}(e), \ \forall e \in \mathcal{E}_h \right\}.$$

$$\boxed{V_h^k = \{ v \in H^{1,\mathrm{nc}}(\mathscr{T}_h;k) \ : \ (v_h)|_K \in V_h^k(K) \quad \forall\, K \in \mathscr{T}_h \}}$$

$$\dim(V_h) = \begin{cases} nk + N_{\mathrm{element}}\, k(k-1)/2 & \text{for } d = 2 \\ nk(k+1)/2 + N_{\mathrm{element}}\, k(k^2-1)/6 & \text{for } d = 3 \end{cases}$$

- edge/face moments: $\mathcal{M}_e^{k-1}(v_h) = \dfrac{1}{|e|} \displaystyle\int_e v_h \, q_{k-1} \, ds \qquad \forall q_{k-1} \in \mathbb{P}^{k-1}(e)$
- volume moments: $\mathcal{M}_K^{k-2}(v_h) = \dfrac{1}{|K|} \displaystyle\int_K v_h \, p_{k-2} \, \mathrm{d}x \qquad \forall p_{k-2} \in \mathbb{P}^{k-2}(K)$

- Unisolvence ✓

Construction of bilinear form $a_h^K : V_h^k(K) \times V_h^k(K) \longrightarrow \mathbb{R}$

$$a_h(u_h, v_h) = \sum_{K \in \mathscr{T}_h} a_h^K(u_h, v_h) \qquad \forall u_h, v_h \in V_h^k, \qquad a(u,v) = \sum_{K \in \mathscr{T}_h} a^K(u,v)$$

Aim:

- computable (we do not have basis functions, only dofs!)
- *continuity and stability*

▷ possible guide: exact on polynomials $\mathbb{P}^k(K)$ (patch test..)

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Can we compute $a^K(v_h, p_k) = a^K(p_k, v_h)$ with $p_k \in \mathbb{P}^k(K)$?

$$k = 1: \qquad a^K(v_h, p_k) = \underbrace{-\int_K v_h \boxed{\Delta p_1} d\mathbf{x}}_{} + \sum_{e \in \partial K} \underbrace{\int_e v_h \boxed{\frac{\partial p_1}{\partial \mathbf{n}}} ds}_{}$$

$$= \quad 0 \quad + \sum_{e \in \partial K} \mathcal{M}_e^0(v_h)$$

Construction of bilinear form $a_h^K : V_h^k(K) \times V_h^k(K) \longrightarrow \mathbb{R}$

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Can we compute $a^K(v_h, p_k) = a^K(p_k, v_h)$ with $p_k \in \mathbb{P}^k(K)$?

$$k = 2 : \qquad a^K(v_h, p_k) = \underbrace{-\int_K v_h \boxed{\Delta p_2} d\mathbf{x}} + \sum_{e \in \partial K} \underbrace{\int_e v_h \boxed{\frac{\partial p_2}{\partial \mathbf{n}}} ds}$$

$$= \mathcal{M}_K^0(v_h) + \sum_{e \in \partial K} \mathcal{M}_e^1(v_h)$$

Construction of bilinear form $a_h^K : V_h^k(K) \times V_h^k(K) \longrightarrow \mathbb{R}$

$$a_h(u_h, v_h) = \sum_{K \in \mathscr{T}_h} a_h^K(u_h, v_h) \qquad \forall u_h, v_h \in V_h^k, \qquad a(u,v) = \sum_{K \in \mathscr{T}_h} a^K(u,v)$$

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Can we compute $a^K(v_h, p_k) = a^K(p_k, v_h)$ *with* $p_k \in \mathbb{P}^k(K)$ *?*

$$a^K(v_h, p_k) = -\underbrace{\int_K v_h \boxed{\Delta p_k} d\mathbf{x}} + \sum_{e \in \partial K} \underbrace{\int_e v_h \boxed{\frac{\partial p_k}{\partial \mathbf{n}}} ds}$$

$$= \quad \mathcal{M}_K^{k-2}(v_h) \quad + \sum_{e \in \partial K} \mathcal{M}_e^{k-1}(v_h)$$

$$\forall \ v_h \in V_h^k \quad p_k \in \mathbb{P}^k(K) \qquad a^K(v_h, p_k) \quad \text{is fully computable}$$

Construction of bilinear form: ingredients

$$\forall \;\; v_h \in V_h^k \quad p_k \in \mathbb{P}^k(K) \qquad a^K(v_h, p_k) \quad \text{is fully computable}$$

- $\Pi^\nabla : H^1(K) \longrightarrow \mathbb{P}^k(K) \qquad a^K(\Pi^a v_h - v_h, q_k) = 0 \quad q_k \in \mathbb{P}^k(K)$

$$\int_K \nabla(\Pi^\nabla(v_h) - v_h)\,\nabla q_k \,\mathrm{d}x = 0 \qquad \forall q_k \in \mathbb{P}^\kappa(K),\; v_h \in V_h^k(K)$$

$$\int_{\partial K} (\Pi^\nabla(v_h) - v_h)\, ds = 0 \qquad \text{if } k = 1, \qquad \int_K (\Pi^\nabla(v_h) - v_h)\,\mathrm{d}x = 0 \qquad \text{if } k \geq 2$$

- ▷ $\Pi^a(v_h) = v_h \quad \forall\, v_h \in \mathbb{P}^k(K)$
- ▷ $I - \Pi^a$ *captures the nonpolynomial part*

$$a_h^K(u_h, v_h) := \underbrace{a^K(\Pi^a(u_h), \Pi^a(v_h))}_{\text{polynomial part}} + \underbrace{S^K(u_h - \Pi^a(u_h), v - \Pi^a(v_h))}_{\textit{Stabilization}}$$

Construction of bilinear form

	Polynomials	Others
	$a^K = a_h^K$	$a^K = a_h^K$
	$a^K = a_h^K$	S^K

$$a_h^K(u_h, v_h) := \underbrace{a^K(\Pi^a(u_h), \Pi^a(v_h))}_{\text{polynomial part}} + \underbrace{S^K(u_h - \Pi^a(u_h), v - \Pi^a(v_h))}_{\textit{Stabilization}}$$

Construction of bilinear form

Polynomials | Others

$$
\left[\begin{array}{c|c}
a^K = a_h^K & a^K = a_h^K \\
\hline
a^K = a_h^K & S^K
\end{array}\right]
$$

$$a_h^K(u_h, v_h) := \underbrace{a^K(\Pi^a(u_h), \Pi^a(v_h))}_{\text{polynomial part}} + \underbrace{S^K(u_h - \Pi^a(u_h), v - \Pi^a(v_h))}_{\textit{Stabilization}}$$

$$c^* a^K(v_h, v_h) \leq \mathcal{S}^K(v_h, v_h) \leq c_* a^K(v_h, v_h) \qquad \forall v_h \in \ker(\Pi^a)$$

$$\text{Take :} \qquad \mathcal{S}^K(v_h, v_h) \simeq h^{d-2}\mathbf{v}^t\mathbf{v} \simeq h^{d-2}\|\mathbf{v}\|_{\ell_2}$$

Construction of bilinear form: Definition and Properties

$$a_h^K(u_h, v_h) := \underbrace{a^K(\Pi^a(u_h), \Pi^a(v_h))}_{\text{polynomial part}} + \underbrace{S^K(u_h - \Pi^a(u_h), v - \Pi^a(v_h))}_{\textit{Stabilization}}$$

$$c_* a^K(v_h, v_h) \leq \mathcal{S}^K(v_h, v_h) \leq c^* a^K(v_h, v_h) \qquad \forall v_h \in \ker(\Pi^a)$$

- Stability: there are α^* and α_* (depending only on ρ)

$$\alpha_* a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* a^K(v_h, v_h) \quad \forall v \in V_h^k(K).$$

  ▷ provide *continuity* and *coercivity* of $a_h(\cdot, \cdot)$

Construction of RHS

- $\mathcal{P}_K^{\ell} : L^2(K) \longrightarrow \mathbb{P}^{\ell}(K)$ $\quad$ L^2-projection

$$(f_h)|_K := \begin{cases} \mathcal{P}_K^{k-2}(f) & \text{for } k \geq 2 \\ \mathcal{P}_K^{0}(f) & \text{for } k = 1 \end{cases} \qquad \forall\, K \in \mathscr{T}_h$$

- $k \geq 2$ $\qquad \langle f_h, v_h \rangle := \sum_K \int_K \mathcal{P}_K^{k-2}(f) v_h d\mathbf{x}$ $\qquad$ *computable*
- $k = 1$ $\qquad \langle f_h, \tilde{v}_h \rangle := \sum_K \int_K \mathcal{P}_K^{0}(f) \tilde{v}_h d\mathbf{x} \;\approx\; \sum_K |K| \mathcal{P}_K^0(f) \mathcal{P}_K^0(v_h)\,.$

$\mathcal{P}_K^0(v_h)$ is computed using quadrature rule [Lipnikov, Manzini (14)]

$$\tilde{v}_h|_K := \frac{1}{n} \sum_{e \in \partial K} \frac{1}{|e|} \int_e v_h ds \;\approx\; \mathcal{P}_K^0(v_h)\,,$$

Nonconforming VEM: Recap general plan

Let $k \geq 1$ be fixed

$$\begin{cases} \text{Find } u_h \in V_h^k \qquad \text{such that:} \\ a_h(u_h, v_h) = < f_h, v_h > \qquad \forall\, v_h \in V_h^k \end{cases} \qquad (P)$$

Ingredients:

- Definition-Construction of $V_h^k \not\subset V$ ✓
- Definition-Construction of $a_h : V_h^k \times V_h^k \longrightarrow \mathbb{R}$
  - ▷ computable, stable & continuous.... ✓
- Definition-Construction of $f_h \in V_h'$ ✓

Lax Milgram $\longrightarrow$ (P) has unique solution u_h

▷ optimal convergence?

Measuring the Nonconformity

- Variational Formulation: $V = H_0^1(\Omega)$

$$\text{Find } u \in V = H_0^1(\Omega) \text{ s.t } \quad a(u,v) := \int_\Omega \nabla u \nabla v d\Omega = < f, v > \qquad \forall\, v \in V$$

For $v \in H^{1,nc}(\mathscr{T}_h; 1)$

$$a(u,v) = \sum_{K \in \mathscr{T}_h} \int_K -(\Delta u) v dx + \underbrace{\sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{\partial u}{\partial n_K} v \, ds}_{\neq 0} = < f, v > + \mathcal{N}_h(u,v)$$

- $$\mathcal{N}_h(u,v) = \sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{\partial u}{\partial n_K} v \, ds = \sum_{e \in \mathcal{E}_h} \int_e \nabla u \cdot [\![v]\!] \, ds$$

- If $u \in H^{s+1}(\Omega)$ with $s \geq 1/2$ and $v \in H^{1,nc}(\mathscr{T}_h; 1)$

$$|\mathcal{N}_h(u,v)| \leq C(\rho) h^{\min(s,k)} \|u\|_{s+1,\Omega} |v|_{1,h}$$

Error Analysis: *Strang-type Lemma*

approximations of u: $\begin{cases} \bullet \ u_\pi \in \mathbb{P}^k(\mathscr{T}_h) \\ \bullet \ u^I \in V_h^k \end{cases}$

$\exists\, C = C(\rho, \alpha^*, \alpha_*) > 0$ such that:

$$|u - u_h|_{1,h} \leq C \left(\underbrace{|u - u^I|_{1,h}} + \underbrace{|u - u_\pi|_{1,h}} + \boxed{\sup_{v_h \in V_h^k} \frac{|< f - f_h, v_h >|}{|v_h|_{1,h}}} + \boxed{\sup_{v_h \in V_h^k} \frac{\mathcal{N}_h(u, v_h)}{|v_h|_{1,h}}} \right)$$

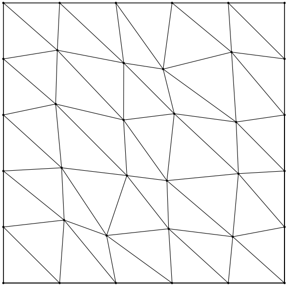
- if $f \in H^{s-1}(\Omega)$ with $s \geq 1$

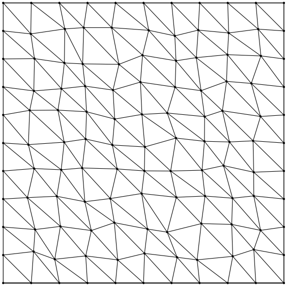
$$|u - u_h|_{1,h} \leq C h^{\min(k,s)} (\|u\|_{1+s,\Omega} + \|f\|_{s-1,\Omega}) \,.$$

- L^2-Optimal error estimates similar to [Beirao,Brezzi, Marini (13)]

Nc-VEM-k versus Nc-FEM-k

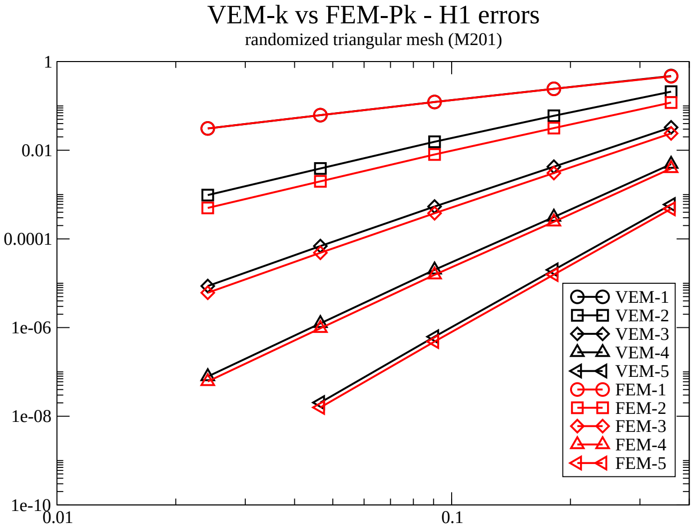
Randomized triangular mesh





Nc-VEM-k versus Nc-FEM-k

Randomized triangular mesh

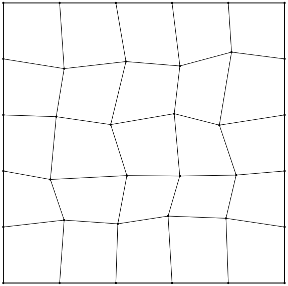


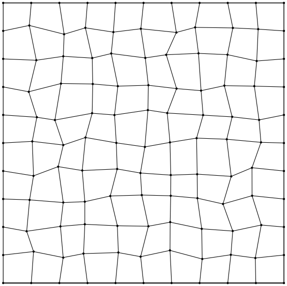
Further Bridges.. (to be built..)

- **Stokes, Elasticity,.. and many others**
- Analysis for low regularity
- Bridges with Mixed VEM [Arnold & Brezzi (82)]
- Bridges with HHO [Ern, Pietro (14–)]
- Bridges with DG..
- L^2-projections..?
- VEM for non-symmetric?
- ....

Nc-VEM-k versus FEM-k

Randomized quadrilateral mesh





Nc-VEM-k versus FEM-k

Randomized quadrilateral mesh