Improved stability estimates for the *hp*-Raviart-Thomas projection operator on quadrilaterals

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Building bridges: Connections and Challenges in Num. PDEs

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RT-elements are frequently used for discr. of problems in $H(\mathrm{div}, \Omega)$

- Mixed methods for the Poisson equation
- Mixed methods for incompressible flows (Stokes problem)
- etc.

Analysis of *hp*-methods → approx. & stability of Π_k^{RT} w.r.t. degree k

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In this talk we consider

$$|\Pi_k^{RT}\mathbf{w}|_{1,K} \leq C(k)|\mathbf{w}|_{1,K} \qquad \forall \mathbf{w} \in H^1(K)$$

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Outline

1. Explicit representation of Π_k^{RT} and review of Legendre polynomials
2. Existing stability estimates: proof technique
3. Partial improvenemt and . . .
4. Optimal stability estimates
5. Optimality: analytic examples and numerical tests

Polynomial spaces:

$$\mathbb{P}_n = \operatorname{span}\{\ x^i \ \mid 0 \le i \le m\},$$
$$\mathbb{Q}_{m,n} = \operatorname{span}\{x^i y^j \mid 0 \le i \le m, 0 \le j \le n\},$$

Reference elements:

$$I = (-1,1), \qquad K = (-1,1)^2.$$

The Raviart-Thomas space of order k is defined as

$$RT_k(K) := \mathbb{Q}_{k+1,k} \times \mathbb{Q}_{k,k+1}$$

There is a unique interpolation operator $\Pi_k^{RT} : H^1(K)^2 \to RT_k(K)$

$$\int_K (\Pi_k^{RT}\mathbf{w} - \mathbf{w}) \cdot \mathbf{v}_k \quad = 0 \qquad \forall \mathbf{v}_k \in \mathbb{Q}_{k-1,k} \times \mathbb{Q}_{k,k-1},$$
$$\int_E (\Pi_k^{RT}\mathbf{w} - \mathbf{w}) \cdot \mathbf{n}\,\varphi_k = 0 \qquad \forall \varphi_k \in \mathbb{P}_k(E) \text{ and faces } E \subset \partial K.$$

[Brezzi/Fortin,1991]

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Define:

- π_k is the one-dimensional L^2-projection onto $\mathbb{P}_k$
- $Q_{m,n}$ is the two-dimensional L^2-projection onto $\mathbb{Q}_{m,n}$

Notice: $$Q_{m,n} = \pi_m^x \circ \pi_n^y$$

Lemma [Ainsworth/Pinchedez, 2002]

$$(\Pi_k^{RT}\mathbf{w})_x = Q_{k-1,k}\mathbf{w}_x + \sum_{i=1}^{2} \mathcal{E}_k^{E_i}\left(\pi_k^y(\mathbf{w}_x - Q_{k-1,k}\mathbf{w}_x)\Big|_{E_i}\right)$$

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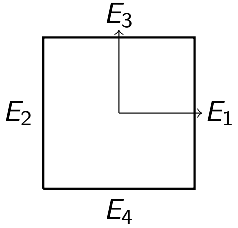

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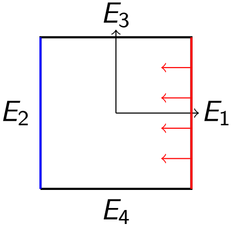

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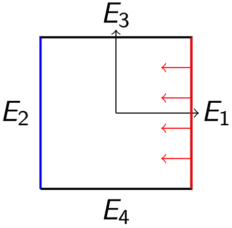

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Excurion to Legendge polynomials $L_k(x)$

- Orthogonal polynomial system in $L^2(-1,1)$

$$\int_{-1}^{1} L_k(x)L_j(x)dx = \|L_k\|^2\delta_{kj}$$

- Normalized such that $L_k(1) = 1$, $k \geq 0$.

Properties:

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Properties:

- $L_k(x)$ are even/odd functions for even/odd values k

$$\Rightarrow L_k(-1) = (-1)^k$$

$$\Rightarrow M_k(-1) = \frac{(-1)^k + (-1)^{k+1}}{2} = 0, \quad M_k(1) = \frac{1+1}{2} = 1$$

i.e. $M_k(x)$ is a proper extension from a side of K.

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Properties of the L^2-projection in 1d

$$u = \sum_{i=0}^{\infty} a_i L_i \qquad \pi_k u = \sum_{i=0}^{k} a_i L_i$$

- $\|\pi_k u\|_0 \le \|u\|_0$
- $\|u - \pi_k u\|_0 \lesssim k^{-1} \|u'\|_0$
- $\|(\pi_k u)'\|_0 \lesssim k^2 \|u\|_0$ (inv. ineq.)
- $|(u - \pi_k u)(\pm 1)|^2 \lesssim \|u\|_{0,I} \|u'\|_{0,I}$ [Georgoulis/Hall/Melenk,2010]

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Theorem [Schötzau/Schwab/Toselli, 2003]

$$|\Pi_k^{RT}\mathbf{w}|_{1,K}^2 \le Ck^2|\mathbf{w}|_{1,K}^2 \qquad \forall \mathbf{w} \in H^1(K)^2$$

Proof. For $u = \mathbf{w}_x$ it holds that

$$|Q_{k-1,k}u|_{1,K}^2 \le Ck|u|_{1,K}^2. \qquad \checkmark$$

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→ the "∂_x" part of the gradient converges at optimal rate k.

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Proceed first with the x-part rather than with y-part

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$$\|(Id - \pi_{k-1}^x) \circ (\partial_y \pi_k^y) u\big|_{x=1}\|_{0,I}^2 \leq C \underbrace{\|(\partial_y \pi_k^y) u\|_{0,K}}_{\lesssim k^{1/2}\|\partial_y u\|_{0,K}} \underbrace{\|\partial_x (\partial_y \pi_k^y) u\|_{0,K}}_{\lesssim k^2 \|\partial_x u\|_{0,K}}$$

1) Notice the *exact* relation (recall: $L'_{k+1} = (2k+1)L_k + (2k-1)L_{k-2} + \dots$)

$$\|M'_k\|^2 = \frac{1}{4}\|L'_{k+1} + L'_k\|^2 = \frac{1}{4}(\|L'_{k+1}\|^2 + \|L'_k\|^2) = \frac{(k+1)^2}{2}.$$

→ the "∂_x" part of the gradient converges at optimal rate k.

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→ the "∂_y" part of the gradient converges at the rate $k^{3/2}$.
Is this optimal?

Numerical experiment: estimate the smallest $C(k)$ s.t.

$$\|\partial_y \Pi_k^{RT,x} u\|_{0,K}^2 \leq C(k) \|\nabla u\|_{0,K}^2$$

Let

$$u = \sum_{i,j}^{\infty} u_{ij} \hat{L}_i(x) \hat{L}_j(y)$$

Then

$$\|\partial_y \Pi_k^{RT,x} u\|_{0,K}^2 = u^\top A_k u, \qquad A_k = \mathcal{M}_k \otimes \mathcal{H}_k$$

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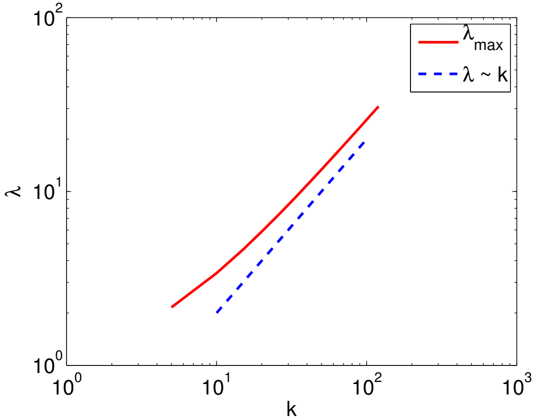
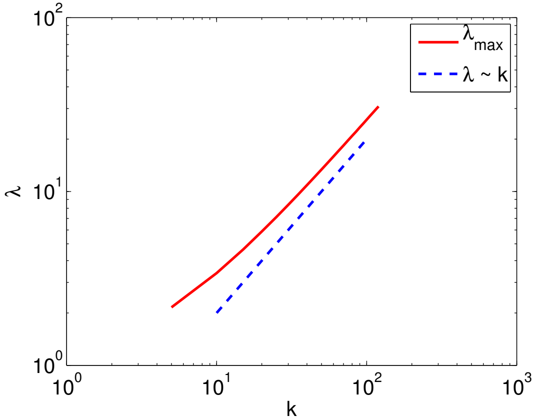


Potential for an improvement!
We need for an new tool, however...

Key $\rightarrow$ multiplicative stability estimates!



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Multiplicative stability estimates:

(a) stability of $\pi_k : H^1(I) \to H^1(I)$, [C/Egger, 2013]

$$\|(\pi_k w)'\|_{0,I}^2 \leq 4k^2 \|w\|_{0,I} \|w'\|_{0,I},$$

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Then (a) + (b) imply by interpolation, cf. [Tartar, 2007]

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Accounting for the extension $\|M_k\|_{0,I}^2 \sim k^{-1}$

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How to get to the $H^1(K)$-norm on the right-hand side?

One-dimensional results (inv. ineq., stability) imply

$$|\Pi_k^{RT,x} u|_{1,K}^2 \lesssim \|u\|_{0,K}^2 + k^4 \|\Pi_k^{RT,x} u\|_{0,K}^2$$

Theorem

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- A sharp stability estimate for $\Pi_k^{RT} : H^1(K) \to H^1(K)$
- Same stability estimate holds for $K = (-1,1)^3$
- The same result can be proven using the representation

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[Hientsch, 2001],[Schötzau et al., 2004]

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