Questions in Spectral Geometry

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Geometry and Spectral Geometry

- **C. F. Gauss** laid the foundations of modern geometry while he was working for the geodetic survey of Hanover around 1828. **Geometry** studies points and the distance function between points and length minimizing curves (geodesics).
- until the beginning of the 20th century it was thought that particles can be represented as points in space. It was also thought that light travels on geodesics (Fermat's principle). However, with the emerge of Quantum mechanics and Quantum Field Theory we know better:
- particles are represented by functions on space. A particle of energy E is represented by an **eigenfunction** of the Laplace operator with eigenvalue E (E. Schroedinger)

Geometry and Spectral Geometry

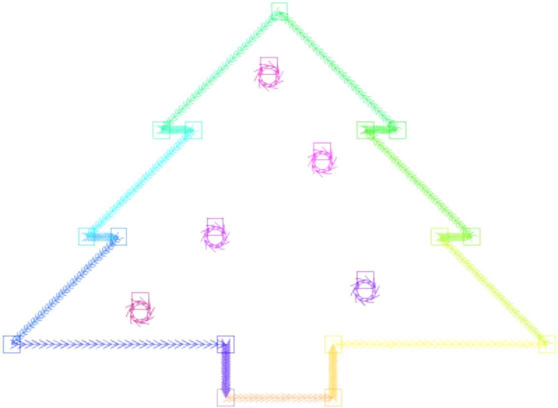
We therefore should study the geometry of eigenfunctions as well as the influence of the background geometry on the eigenvalues and eigenfunctions of the Laplace operator $\Delta = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a non-empty open bounded subset. Then there exists a sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$ with smooth eigenfunctions $\phi_j \in C^\infty(\Omega) \cap L^2(\Omega)$ that form an orthonormal basis in $L^2(\Omega)$ and vanish at the boundary of Ω (in the sense that they are in $H^1_0(\Omega)$).

The above eigenvalues are the so-called Dirichlet eigenvalues of the Laplace operator (or eigenvalues of the Dirichlet Laplace operator).

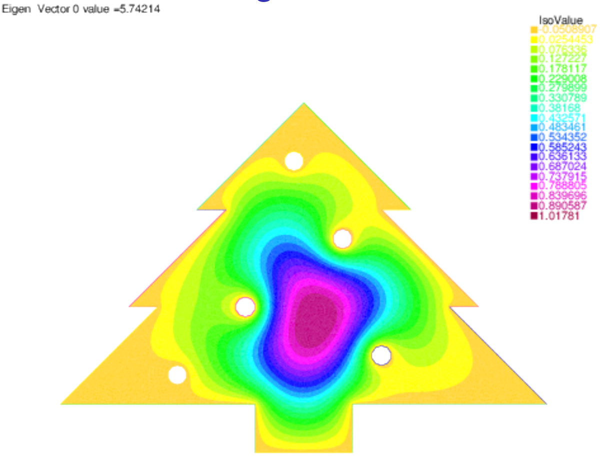
Example of Dirichlet eigenvalues



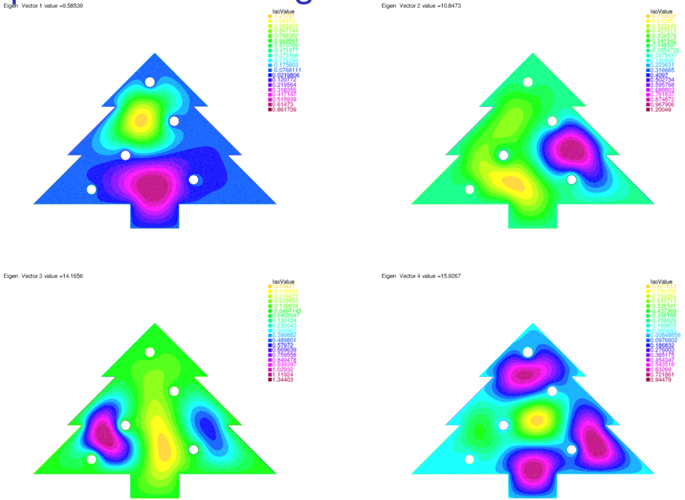
Dirichlet Eigenvalues of a Christmas Tree

j	λ_j
1	5.74214
2	9.58539
3	10.8473
4	14.1656
5	15.9267
6	18.7281
7	19.4242
8	21.6526
9	24.2471
10	26.1383
11	29.2705
12	30.843
13	32.4437
14	34.3834

Example of Dirichlet eigenvalues



Example of Dirichlet eigenvalues



Derived Spectral Quantities

From the sequence of eigenvalues one can derive the following important functions and quantities.

- The heat trace: $h(t) = \mathrm{tr}(e^{-t\Delta}) = \sum_j e^{-\lambda_j t}$,
- The spectral zeta function: $\zeta(s) = \mathrm{tr}(\Delta^{-s}) = \sum_j \frac{1}{\lambda_j^s}$.
- The spectral determinant: $\det_\zeta(\Delta) = e^{-\zeta'(0)}$.

Classical known theorems

Assuming that Ω has smooth (or piecewise smooth to cover the Christmas tree) boundary we have:

- Heat expansion:
  $\sum_j e^{-\lambda_j t} = a_0 t^{-n/2} + a_1 t^{(1-n)/2} + a_2 t^{(2-n)/2} + \ldots \text{as } t \to 0^+$
  and $a_0 = C_n \mathrm{Vol}_n(\Omega)$, $a_1 = \tilde{C}_n \mathrm{Vol}_{n-1}(\partial\Omega), \ldots$.
- Weyl's law: $\lambda_j \sim C\, j^{2/n}$.
- $\zeta(s)$ (originally defined only for $\mathrm{Re}(s) > n/2$) admits a meromorphic continuation to the complex plane and 0 is not a pole.

Manifolds

All these theorems generalize to the case of the Laplace operator on Riemannian manifolds with or without boundary. Therefore, volume, volume of the boundary and other quantities are spectrally determined (are equal for manifolds with the same eigenvalues). For completeness: the Dirichlet Laplace operator on a compact Riemannian manifold M with or without boundary can be defined by the so-called Dirichlet quadratic form on $C_0^\infty(M \backslash \partial M)$:

$$\langle \phi, \Delta \psi \rangle = \int_M d\phi \wedge * d\psi = \int_M g(\mathrm{grad}\phi, \mathrm{grad}\psi) \omega_g.$$

Manifolds and Vector Bundles

More generally, if E is a hermitian vector bundle over a Riemannian manifold then a differential operator $P : C^\infty(M; E) \to C^\infty(M; E)$ is called of Laplace type if in local coordinates

$$P = -g^{ik} \frac{\partial^2}{\partial x^i \partial x^k} + \text{lower order terms.}$$

Examples are squares of Dirac operators, the Hodge Laplace operator on the bundle of p-forms, the Dolbeault Laplace operator on Kähler manifolds and many more. Each of these operators has a heat kernel and an associated zeta functions as well as a spectral determinant associated with it. A lot of the geometry of the manifolds and the bundle is contained in these functions.

Osgood-Philipps-Sarnak's theorem

On a two dimensional closed manifold (surface) consider a conformal class of a metric g with fixed volume, i.e. all metrics of the form $g_h = e^h g$, where h is a smooth function such $\mathrm{Vol}_{g_h}(M) = \mathrm{Vol}_g(M)$. Then in each conformal class there is precisely one metric of constant curvature and this is also the unique maximum of the function $\det_\zeta(\Delta)$.

Questions:

- What are the maxima in Teichmüller space in fixed genus?
- What are the maxima of other values of the zeta function in the conformal class?
- What are the variations of the spectral determinant under boundary variations?

Shnirelman's theorem

For negatively curved manifolds eigenfunctions equidistribute, i.e. quantum ergodicity holds. For $f \in C(M)$:

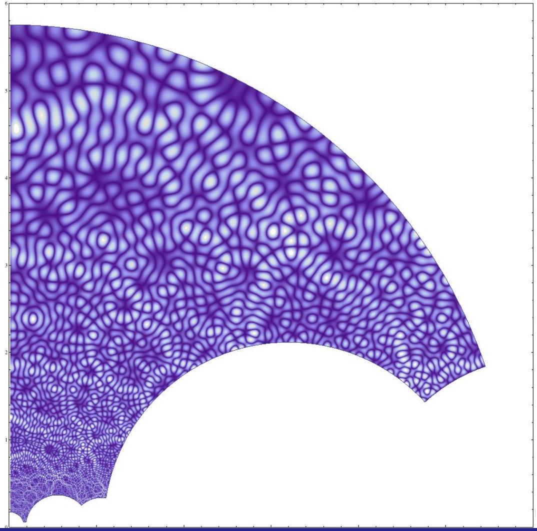
$$\int_M |\phi_j(x)|^2 f(x)dx \to \int_M f(x)dx \quad a.e.$$

for almost all eigenfunctions (apart from the sequence of counting density zero). For A a pseudodifferential operator

$$\langle \phi_j, A\phi_j \rangle \to \int_{SM} \sigma_A(\xi)d\xi \quad a.e.$$

This theorem was extended in suitable ways by myself and collaborators for Dirac operators, Laplace Beltrami operators and also manifolds with discontinuous metrics.

Shnirelman's theorem



QE and QUE conjecture

QUE conjecture (Rudnik and Sarnak): The square of eigenfunctions converge to the Liouville measure.
Conjecture proved by E. Lindenstrauss (fields medal in 2010) in the arithmetic case for a special basis of eigenfunctions.
QE is a weaker result of the type on the previous slide.
Questions:

- what are the geometric conditions implying QE in the semi-classical limit?
- in the case of discontinuous metrics one has to deal with ray-splitting billiard flows. What are these in the case of geometric operators on vector bundles such as for example Dirac operators?

Domain optimization

Questions:

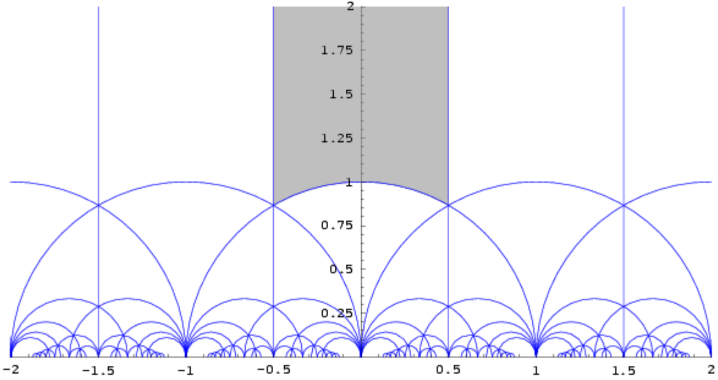
- What domains and geometries maximize, minimize λ_1 or λ_2 or the spectral determinant or other values of the spectral zeta function.
- are the minimizers always geometrically significant (regular polygons, systole maximizing surfaces, symmetry group maximizing surfaces).

Scattering Theory

For a bounded domain there was a discrete set of eigenvalues for the Dirichlet Laplace operator yielding an orthonormal basis of eigenfunctions. For the complement of the domain (the so called exterior problem) the opposite is true: there are no eigenfunctions at all.

One has to use the **spectral theorem for unbounded self-adjoint operators** that provides us in this case with a meromorphic family of generalized eigenfunctions. The method here is **scattering theory**. The eigenvalues are then replaced by the so called scattering matrix and its poles (the so called resonances).

Scattering theory on the modular domain and number theory



The modular domain

On this domain the spectrum consists of a discrete set of eigenvalues and eigenfunctions, the so called Maass-cusp-forms.
There is also an absolutely continuous part of the spectrum $[1/4, \infty)$ with generalized eigenfunctions of the form

$$E(z, s) = y^s + C(s)y^{1-s} + T(x, y, s),$$

where T is exponentially decaying in y as $y \to \infty$.
$C(s)$ is the scattering matrix and the eigenvalue is $s(1-s)$.

The modular domain

$C(s)$ is a meromorphic function and its poles (resonances) are the non-trivial zeros of the Riemann zeta function!!
Questions:

- How do the resonances behave as the surface is perturbed conformally?
- How do the embedded eigenvalues behave as the surface is perturbed (Sarnak-Philipps conjecture)?
- Is there Quantum ergodicity of even QUE for resonances (Zworski conjecture)?

Techniques

- spectral theory, meromorphic Fredholm theorem and abstract scattering theory
- pseudodifferential operators, parametrices, traces of pseudodifferential operators
- perturbation theory for eigenvalues and resonances
- Fourier integral operators
- differential geometry, topology: index theory, K-theory, cohomology
- representation theory of groups