Diophantine Approximation

Sanju Velani

Department of Mathematics
University of York

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What is Diophantine approximation?

Diophantine approximation is a branch of number theory that can loosely be described as a quantitative analysis of the property the rationals $\mathbb{Q}$ are dense in $\mathbb{R}$; i.e.

- For any $x \in \mathbb{R}$ and $\epsilon > 0$ there exists a rational p/q $(q > 0)$ such that
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In the case of $\pi = 3.142...$, the following rationals all lie within $1/(\text{denominator})^2$ of π:
$$\frac{3}{1}, \quad \frac{22}{7}, \quad \frac{333}{106}, \quad \frac{355}{113}, \quad \frac{103993}{33102}.$$

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The answer is given by Dirchlet's fundamental theorem from 1848 – a simple consequence of the powerful *Pigeon Hole Principle*:

> *If n objects are placed in m boxes and $n > m$, then some box will contain at least two objects.*

Dirichlet's Theorem

Theorem (1848) *For any* $x \in \mathbb{R}$ *and integer* $N \geq 1$, *there exists a rational* p/q *such that*

$$|x - p/q| \leq 1/qN \qquad \text{and} \qquad 1 \leq q \leq N\,.$$

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The theory of continued fraction provides a simple mechanism for finding these good 'Dirichlet' rational approximates.

Continued Fractions: finding the Dirichlet rationals

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Then

$$\frac{1}{(a_{n+1} + 2)q_n^2} \ \le \ \left| x - \frac{p_n}{q_n} \right| \ \le \ \frac{1}{a_{n+1} q_n^2} \, .$$

UPSHOT: the convergents provide explicit solutions to Dirichlet.

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Regarding $\pi = 3.142...$ the rationals

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are the first 5 convergents of π. Thus, for sure

$$|\pi - \tfrac{103993}{33102}| \leq (33102)^{-2} < 9.1 \times 10^{-10} .$$

Dirichlet and **Bad**

For $x \in \mathbb{R}$, let $\|x\| := \min\{|x-m| : m \in \mathbb{Z}\}$. Recall $\mathrm{I} := [0,1]$.

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UPSHOT: there exist numbers for which we can not improve Dirichlet by arbitrary $\epsilon > 0$ – *badly approximable numbers*

$$\mathbf{Bad} := \{x \in \mathbb{R} : \exists c(x) > 0 \ s.t.\ q\,\|qx\| > c(x) \quad \forall\ q \in \mathbb{N}\}\ .$$

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Bad has a lovely characterization via continued fractions

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- $m(\textbf{Bad}) = 0$ – Lebesgue measure zero (Khintchine (1924))
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Measure theoretically we can improve on Dirichlet by a log factor.

Khintchine's Theorem

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a real positive function and let

$$W(\psi) := \{x \in \mathrm{I} : \|qx\| \leq \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}$$

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Khintchine's Theorem (1924) *If ψ is monotonic, then*

$$m(W(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) \; < \infty \, , \\ \\ 1 & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) \; = \infty \, . \end{cases}$$

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• Removing monotonicity from Khintchine is a key open problem in metric number theory – the Duffin-Schaeffer Conjecture (1941).

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Overcome this by insisting that p, q are co-prime: let $W'(\psi)$ be the set of $x \in \mathrm{I}$ such that $|qx - p| \leq \psi(q)$ for infinitely many p/q with $(p, q) = 1$.

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The Duffin-Schaeffer Conjecture (1941) *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a real positive function. Then*

$$m(W'(\psi)) = 1 \quad \textit{if} \quad \sum_{q=1}^{\infty} \frac{\varphi(q)\phi(q)}{q} = \infty \, .$$

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True with extra divergence (Beresnevich-Harman-Haynes-V, 2013):

$$\sum_{q=16}^{\infty} \frac{\varphi(q)\psi(q)}{q\exp(c(\log\log q)(\log\log\log q))} = \infty \, .$$

Littlewood's Conjecture

Dirichlet $\Longrightarrow$ For every $(\alpha, \beta) \in \mathrm{I}^2 := [0, 1]$ there exist infintely many $q > 0$ such that $q\|q\alpha\|\|q\beta\| \leq 1$.

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Equivalently, $\exists$ infinitely many $(s/q, t/q)$ such that

$$\left|\alpha - \frac{s}{q}\right| \left|\beta - \frac{t}{q}\right| \ < \ \frac{\epsilon}{q^3} \quad (\epsilon > 0 \ \ \text{arbitrary})$$

" every point in the plane lies in infinitely many hyperbolic regions given by $|x| \cdot |y| < \epsilon/q^3$ centred at rational points"

Littlewood's Conjecture: metric results

Gallagher's Theorem (1961) *Let* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a real positive monotonic function. Then for almost all* $(\alpha, \beta) \in \mathrm{I}^2$ *there exist infintely many* $q > 0$ *such that*

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Claim: *For every* $(\alpha, \beta) \in \mathrm{I}^2$*:*

$$\liminf_{q \to \infty} q \, \log q \, \|q\alpha\| \, \|q\beta\| < \infty.$$

Where in the UK?

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- Other UK universities: Bristol, Durham, Liverpool, Oxford, Royal Holloway.