Geometry of Higher Yang-Mills Fields

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Symmetries and Geometry of Branes in String/M-Theory,
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Based on work with:

- S Palmer, D Harland, C Papageorgakis, F Sala (M-brane models)
- M Wolf (Twistor description)
- R Szabo (Geometric Quantization)

Motivation
There might be an effective descripion of M5-branes.

- Effective description of M2-branes proposed in 2007.
- This created lots of interest:
  BLG-model: >625 citations, ABJM-model: >917 citations

Question: Is there a similar description for M5-branes?

For cautious people:
Is there a a reasonably interesting superconformal field theory of a non-abelian tensor multiplet in six dimensions?
(The mysterious, long-sought $\mathcal{N} = (2, 0)$ SCFT in six dimensions)

A possible way to approach the problem: Look at BPS subsector

- This was how the M2-brane models were derived originally.
- BPS subsector is interesting itself: Integrability
- BPS subsector should be more accessible than full theory.

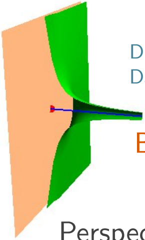
Results so far/Outline

Things do look very promising.

- Integrability found:
  Nahm construction for self-dual strings using loop space
  CS, S Palmer & CS
- Use of loop space justified:
  M-theory suggests this, e.g. Geometric quantization of S^3
  CS & R Szabo
- Integrability reasonable:
  Gauge structure of M2- and M5-brane models the same
  S Palmer & CS
- Integrability works even without loop space:
  Twistor constructions of self-dual strings and non-abelian tensor multiplets work
  CS & M Wolf
- On the way to Geometry of Higher Yang-Mills Fields:
  Explicit solutions to non-abelian tensor multiplet equations
  F Sala, S Palmer & CS

Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration

Perspective of D1:

Nahm eqn.

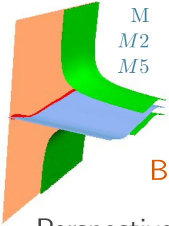
$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

↕ Nahm transform ↕

Perspective of D3:

Bogomolny monopole eqn.

$$F_{ij} = \varepsilon_{ijk}\nabla_k\Phi$$



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration

Perspective of M2:

Basu-Harvey eqn.

$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

↕ generalized Nahm transform ↕

Perspective of M5:

Self-dual string eqn.

$$H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\partial_\sigma\Phi$$

3-Lie Algebras

In analogy with Lie algebras, we can introduce 3-Lie algebras.

BH: $$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + [A_s, X^\mu] + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0\ , \quad X^\mu \in \mathcal{A}$$

3-Lie algebra

Obviously: $\mathcal{A}$ is a vector space, $[\cdot,\cdot,\cdot]$ trilinear+antisymmetric.

Satisfies a "3-Jacobi identity," the fundamental identity:

$$[A,B,[C,D,E]] = [[A,B,C],D,E] + [C,[A,B,D],E] + [C,D,[A,B,E]]$$

Filippov (1985)

Gauge transformations from Lie algebra of inner derivations:

$$D : \mathcal{A}\wedge\mathcal{A} \to \mathsf{Der}(\mathcal{A}) =: \mathfrak{g}_\mathcal{A} \quad D(A,B) \rhd C := [A,B,C]$$

Algebra of inner derivations closes due to fundamental identity.

Brief Remarks on 3-Lie Algebras

In analogy with Lie algebras, we can introduce 3-Lie algebras.

Examples:

Lie algebra	3-Lie algebra
Heisenberg-algebra: $[\tau_a, \tau_b] = \varepsilon_{ab}\mathbb{1}, \quad [\mathbb{1}, \cdot] = 0$	Nambu-Heisenberg 3-Lie Algebra: $[\tau_i, \tau_j, \tau_k] = \varepsilon_{ijk}\mathbb{1}, \quad [\mathbb{1}, \cdot, \cdot] = 0$
$\mathsf{SU}(2) \simeq \mathbb{R}^3$: $[\tau_i, \tau_j] = \varepsilon_{ijk}\tau_k$	$A_4 \simeq \mathbb{R}^4$: $[\tau_\mu, \tau_\nu, \tau_\kappa] = \varepsilon_{\mu\nu\kappa\lambda}\tau_\lambda$

Generalizations:

- Real 3-algebras: $[\cdot, \cdot, \cdot]$ antisymmetric only in first two slots
  S. Cherkis & CS, 0807.0808
- Hermitian 3-algebras: complex vector spaces, $\rightarrow$ ABJM
  Bagger & Lambert, 0807.0163

Generalizing the ADHMN construction to M-branes

That is, find solutions to $H = \star \mathrm{d}\Phi$
from solutions to the Basu-Harvey equation.

As M5-branes seem to require gerbes, let's start with them.

Dirac Monopoles and Principal $\mathsf{U}(1)$-bundles

Dirac monopoles are described by principal $\mathsf{U}(1)$-bundles over S^2.

Manifold M with cover $(U_i)_i$. Principal $\mathsf{U}(1)$-bundle over M:

$$F \in \Omega^2(M, \mathfrak{u}(1)) \text{ with } \mathrm{d}F = 0$$

$$A_{(i)} \in \Omega^1(U_i, \mathfrak{u}(1)) \text{ with } F = \mathrm{d}A_{(i)}$$

$$g_{ij} \in \Omega^0(U_i \cap U_j, \mathsf{U}(1)) \text{ with } A_{(i)} - A_{(j)} = \mathrm{d}\log g_{ij}$$

Consider monopole in $\mathbb{R}^3$, but describe it on S^2 around monopole:

S^2 with patches U_+, U_-, $U_+ \cap U_- \sim S^1$: $g_{+-} = \mathrm{e}^{-\mathrm{i}k\phi},\ k \in \mathbb{Z}$

$$c_1 = \frac{\mathrm{i}}{2\pi} \int_{S^2} F = \frac{\mathrm{i}}{2\pi} \int_{S^1} A^+ - A^- = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, k = k$$

Monopole charge: k

Manifold M with cover $(U_i)_i$. Abelian (local) gerbe over M:

$$H \in \Omega^3(M, \mathfrak{u}(1)) \text{ with } \mathrm{d}H = 0$$

$$B_{(i)} \in \Omega^2(U_i, \mathfrak{u}(1)) \text{ with } H = \mathrm{d}B_{(i)}$$

$$A_{(ij)} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1)) \text{ with } B_{(i)} - B_{(j)} = \mathrm{d}A_{ij}$$

$$h_{ijk} \in \Omega^0(U_i \cap U_j \cap U_k, \mathfrak{u}(1)) \text{ with } A_{(ij)} - A_{(ik)} + A_{(jk)} = \mathrm{d}h_{ijk}$$

Note: Local gerbe: principal $\mathsf{U}(1)$-bundles on intersections $U_i \cap U_j$.

Consider S^3, patches U_+, U_-, $U_+ \cap U_- \sim S^2$: bundle over S^2
Reflected in: $H^2(S^2, \mathbb{Z}) \cong H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$

$$\frac{\mathrm{i}}{2\pi} \int_{S^3} H = \frac{\mathrm{i}}{2\pi} \int_{S^2} B_+ - B_- = \ldots = k$$

Charge of self-dual string: k

Describe p-gerbes + connective structure $\rightarrow$ Deligne cohomology.

Gerbes are somewhat unfamiliar, difficult to work with.

Can we somehow avoid using gerbes?

Abelian Gerbes and Loop Space

By going to loop space, one can reduce differential forms by one degree.

Consider the following double fibration:

$$\begin{array}{ccccc} & & \mathcal{L}M \times S^1 & & \\ & \overset{ev}{\swarrow} & & \overset{pr}{\searrow} & \\ M & & & & \mathcal{L}M \end{array}$$

Identify $T\mathcal{L}M = \mathcal{L}TM$, then: $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in T\mathcal{L}M$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M)~, \quad v_i = \oint \mathrm{d}\tau\, v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \oint_{S^1} \mathrm{d}\tau\, \omega(x(\tau))(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

Nice properties: reparameterization invariant, chain map, ...

An abelian local gerbe over M is a principal $\mathsf{U}(1)$-bundle over $\mathcal{L}M$.

Recall the self-dual string equation on $\mathbb{R}^4$: $H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda}\frac{\partial}{\partial x^\lambda}\Phi$

Its transgressed form is an equation for a 2-form F on $\mathcal{L}\mathbb{R}^4$:

$$F_{(\mu\sigma)(\nu\rho)} = \delta(\sigma-\rho)\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\tau)\,\frac{\partial}{\partial y^\lambda}\Phi(y)\Big|_{y=x(\tau)}$$

Extend to full non-abelian loop space curvature:

$$\begin{aligned}F^\pm_{(\mu\sigma)(\nu\tau)} =& \big(\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\big)_{(\sigma\tau)}\\ &\mp\big(\dot{x}_\mu(\sigma)\nabla_{(\nu\tau)}\Phi+\dot{x}_\nu(\sigma)\nabla_{(\mu\tau)}\Phi-\delta_{\mu\nu}\dot{x}^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\big)_{[\sigma\tau]}\end{aligned}$$

where $\nabla_{(\mu\sigma)} := \oint \mathrm{d}\tau\,\delta x^\mu(\tau)\wedge\left(\frac{\delta}{\delta x^\mu(\tau)}+A_{(\mu\tau)}\right)$

Goal: Construct solutions to this equation.

The ADHMN Construction

The ADHMN construction nicely translates to self-dual strings on loop space.

Nahm transform: Instantons on $T^4 \mapsto$ instantons on $(T^4)^*$

Roughly here:

$$T^4\colon \begin{cases} 3 \text{ rad. } 0 & \\ 1 \text{ rad. } \infty & : \text{ D1 WV} \end{cases} \quad \text{and} \quad (T^4)^*\colon \begin{cases} 3 \text{ rad. } \infty & : \text{ D3 WV} \\ 1 \text{ rad. } 0 & \end{cases}$$

Dirac operators: X^i solve Nahm eqn., X^μ solve Basu-Harvey eqn.

$$\text{IIB}: \quad \nabla\!\!\!/ = -\mathbb{1}\frac{\mathrm{d}}{\mathrm{d}x^6} + \sigma^i(\mathrm{i}X^i + x^i\mathbb{1}_k)$$

$$\text{M}: \quad \nabla\!\!\!/ = -\gamma_5\frac{\mathrm{d}}{\mathrm{d}x^6} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - \mathrm{i}\oint \mathrm{d}\tau\, x^\mu(\tau)\dot{x}^\nu(\tau)\right)$$

normalized zero modes: $\bar{\nabla\!\!\!/}\psi = 0$ and $\mathbb{1} = \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\psi$

Solution to Bogomolny/self-dual string equations:

$$A := \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\, \mathrm{d}\, \psi \quad \text{and} \quad \Phi := -\mathrm{i}\int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\, s\, \psi$$

Remarks on The Construction

The construction is very natural and behaves as expected.

- Nahm eqn. and Basu-Harvey eqn. play analogous roles.
- Construction extends to general. Basu-Harvey eqn. (ABJM).
- One can construct many examples explicitly.
- It reduces nicely to ADHMN via the M2-Higgs mechanism.

CS, 1007.3301, S Palmer & CS, 1105.3904

More Motivation for Loop Spaces

3-Lie algebra valued tensor multiplet equations:

$$\nabla^2 X^I - \tfrac{\mathrm{i}}{2}[\bar\Psi, \Gamma_\nu \Gamma^I \Psi, C^\nu] - [X^J, C^\nu, [X^J, C_\nu, X^I]] = 0$$

$$\Gamma^\mu \nabla_\mu \Psi - [X^I, C^\nu, \Gamma_\nu \Gamma^I \Psi] = 0$$

$$\nabla_{[\mu} H_{\nu\lambda\rho]} + \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I, \nabla^\tau X^I, C^\sigma] + \tfrac{\mathrm{i}}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar\Psi, \Gamma^\tau \Psi, C^\sigma] = 0$$

$$F_{\mu\nu} - D(C^\lambda, H_{\mu\nu\lambda}) = 0$$

$$\nabla_\mu C^\nu = D(C^\mu, C^\nu) = 0$$

$$D(C^\rho, \nabla_\rho X^I) = D(C^\rho, \nabla_\rho \Psi) = D(C^\rho, \nabla_\rho H_{\mu\nu\lambda}) = 0$$

N Lambert & C Papageorgakis, 1007.2982

Factorization of $C^\rho = C\dot{x}^\rho$. Here, 3-Lie algebra transgression:

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\, D(\omega(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau)), C)$$

C Papageorgakis & CS, 1103.6192

Often: A vector short of happiness. Loop space has this vector.

Geometric quantization prescription: (e.g. fuzzy sphere)

Special symplectic manifold (M,ω) → line bundle L with (h,∇) over M → Hilbert space $\mathscr{H}$: global holomorphic sections of L

Quantization map: $[\hat{f},\hat{g}] = \mathrm{i}\hbar\widehat{\{f,g\}} + \mathcal{O}(\hbar^2)$

M-theory: 2-plectic manifold (M,ϖ), $\varpi \in \Omega^3(M)$

- hol. secs. of gerbe?, quantization of one-forms? Rogers, ...
- Solution: ω on $\mathcal{L}M$ as $\omega := \mathcal{T}\varpi$, then proceed as above
- Example: $\mathbb{R}^3$ with 2-plectic form $\varpi = \varepsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$:

$$[x^i(\tau), x^j(\sigma)] = \varepsilon^{ijk}\frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2}\delta(\tau-\sigma) + \mathcal{O}(\theta^2)$$

CS & R Szabo, 1211.0395

- Cf. Kawamoto & Sasakura, Bergshoeff, Berman et al. [2000]

The duality D1 $\leftrightarrow$ D3 is a duality between Yang-Mills theories.

Question: In what sense are M2- and M5-brane models related?

Start by looking at gauge structure

Parallel transport of particles in representation of gauge group G:

- holonomy functor: hol : path $p \mapsto \mathsf{hol}(p) \in G$
- $\mathsf{hol}(p) = P\exp(\int_p A)$, P: path ordering, trivial for $\mathsf{U}(1)$.

Parallel transport of strings with gauge group $\mathsf{U}(1)$:

- 2-holonomy functor: hol_2 : surface $s \mapsto \mathsf{hol}_2(s) \in \mathsf{U}(1)$
- $\mathsf{hol}_2(s) = \exp(\int_s B)$, B: connective structure on gerbe.

Nonabelian case:

- much more involved!
- no straightforward definition of surface ordering
- solution: Categorification!

see Baez, Huerta, 1003.4485

Warning: Categorification neither unique nor straightforward.

Lie 2-group

A Lie 2-group is a

- monoidal category, morph. invertible, obj. weakly invertible.
- Lie groupoid + product $\otimes$ obeying weakly the group axioms.

Simplification: use strict Lie 2-groups $\xleftrightarrow{1:1}$ Lie crossed modules

Lie crossed modules

Pair of Lie groups (G,H), written as $(\mathsf{H}\xrightarrow{\mathsf{t}}\mathsf{G})$ with:

- left automorphism action $\rhd: \mathsf{G}\times\mathsf{H}\to\mathsf{H}$
- group homomorphism $\mathsf{t}:\mathsf{H}\to\mathsf{G}$ such that

$$\mathsf{t}(g\rhd h)=g\mathsf{t}(h)g^{-1}\quad\text{and}\quad\mathsf{t}(h_1)\rhd h_2=h_1h_2h_1^{-1}$$

Also: strict Lie 2-algebras $\xleftrightarrow{1:1}$ differential crossed modules

Lie crossed modules

Pair of Lie groups (G,H), written as $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$ with:

- left automorphism action $\rhd \colon \mathsf{G}\times\mathsf{H}\rightarrow\mathsf{H}$
- group homomorphism $\mathsf{t}:\mathsf{H}\rightarrow\mathsf{G}$

$$\mathsf{t}(g\rhd h)=g\mathsf{t}(h)g^{-1}\quad\text{and}\quad\mathsf{t}(h_1)\rhd h_2=h_1h_2h_1^{-1}$$

Simplest examples:

- Lie group G, Lie crossed module: $(1 \xrightarrow{\mathsf{t}} G)$.
- Abelian Lie group G, Lie crossed module: $BG=(G \xrightarrow{\mathsf{t}} 1)$.

More involved:

- Automorphism 2-group of Lie group G: $(G \xrightarrow{\mathsf{t}} \mathsf{Aut}(G))$

Principal 2-Bundles

Higher gauge theory is the dynamical theory of principal 2-bundles.

Consider a manifold M with cover (U_a)

Object	Principal G-bundle	Principal $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle
Cochains	(g_{ab}) valued in G	(g_{ab}) valued in G, (h_{abc}) valued in H
Cocycle	$g_{ab}g_{bc} = g_{ac}$	$\mathsf{t}(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(g_{ab} \rhd h_{bcd})$
Coboundary	$g_a g'_{ab} = g_{ab}g_b$	$g_a g'_{ab} = \mathsf{t}(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \rhd h'_{abc})h_{ab}(g_{ab} \rhd h_{bc})$
gauge pot.	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$, $B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$
Curvature	$F_a = \mathrm{d}A_a + A_a \wedge A_a$	$F_a = \mathrm{d}A_a + A_a \wedge A_a$, $F_a = \mathsf{t}(B_a)$ $H_a = \mathrm{d}B_a + A_a \rhd B_a$
Gauge trafos	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a$	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a + \mathsf{t}(\Lambda_a)$ $\tilde{B}_a := g_a^{-1} \rhd B_a + \tilde{A}_a \rhd \Lambda_a + \mathrm{d}\Lambda_a - \Lambda_a \wedge \Lambda_a$

Remarks:

- A principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is a principal G-bundle.
- A principal $(\mathsf{U}(1) \xrightarrow{\mathsf{t}} 1) = B\mathsf{U}(1)$-bundle is an abelian gerbe.
- Gauge part of (2,0)-theory: $H = \star H$, $F = \mathsf{t}(B)$.

Is all this machinery really useful/necessary?

Recall the definition of a 3-algebra $\mathcal{A}$:

- $[\cdot,\cdot,\cdot] : \mathcal{A}^{\otimes 3} \to \mathcal{A}$
- Fundamental identity says that $[a,b,\cdot] \in \mathsf{Der}(\mathcal{A})$, $a,b \in \mathcal{A}$.

Theorem

3-algebras $\xleftrightarrow{1:1}$ metric Lie algebras $\mathfrak{g} \cong \mathsf{Der}(\mathcal{A})$
faithful orthog. representations $V \cong \mathcal{A}$

J Figueroa-O'Farrill et al., 0809.1086

Observations

- $\mathfrak{g} \xrightarrow{\mathrm{t}} V$ is a simple differential crossed modules
- M2- and M5-brane models have the same gauge structure.
- Via Faulkner construction, all DCMs come with $[\cdot,\cdot,\cdot]$
- Application of this to M2- and M5-models looks promising.

S Palmer & CS, 1203.5757

3-Lie algebra valued tensor multiplet equations:

$$\nabla^2 X^I - \tfrac{\mathrm{i}}{2}[\bar\Psi, \Gamma_\nu \Gamma^I \Psi, C^\nu] - [X^J, C^\nu, [X^J, C_\nu, X^I]] = 0$$
$$\Gamma^\mu \nabla_\mu \Psi - [X^I, C^\nu, \Gamma_\nu \Gamma^I \Psi] = 0$$
$$\nabla_{[\mu} H_{\nu\lambda\rho]} + \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I, \nabla^\tau X^I, C^\sigma] + \tfrac{\mathrm{i}}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar\Psi, \Gamma^\tau \Psi, C^\sigma] = 0$$
$$F_{\mu\nu} - D(C^\lambda, H_{\mu\nu\lambda}) = 0$$
$$\nabla_\mu C^\nu = D(C^\mu, C^\nu) = 0$$
$$D(C^\rho, \nabla_\rho X^I) = D(C^\rho, \nabla_\rho \Psi) = D(C^\rho, \nabla_\rho H_{\mu\nu\lambda}) = 0$$

N Lambert & C Papageorgakis, 1007.2982

Factorization of $C^\rho = C\dot{x}^\rho$. Here, fake curvature equation:

$$\mathsf{t} : \mathcal{A} \to \mathsf{Der}(\mathcal{A}) , \; a \mapsto D(C, a) , \quad F_{\mu\nu} = \mathsf{t}(H_{\mu\nu\lambda} x^\lambda) =: \mathsf{t}(B)$$

$\Rightarrow$ More natural interpretation as higher gauge theory.

S Palmer & CS, 1203.5757

Division algebras, spheres and groups:

$\mathcal{A}$	$\mathcal{A}P^1$	$\lvert a\rvert = 1$	$\mathsf{Aut}(\mathcal{A})$	Physics
$\mathbb{R}$	$\mathbb{R}P^1 \cong S^1$	$\mathbb{Z}_2 \cong S^0$	$\mathsf{Aut}(\mathbb{R}) \cong 1$	Vortex?
$\mathbb{C}$	$\mathbb{C}P^1 \cong S^2$	$\mathsf{U}(1) \cong S^1$	$\mathsf{Aut}(\mathsf{U}(1)) \cong \mathbb{Z}_2$	Monopole
$\mathbb{H}$	$\mathbb{H}P^1 \cong S^4$	$\mathsf{SU}(2) \cong S^3$	$\mathsf{Aut}(\mathsf{SU}(2)) \cong \mathsf{SU}(2)$	Instanton
$\mathbb{O}$	$\mathbb{O}P^1 \cong S^8$	S^7	$\mathsf{Aut}(\mathbb{O}) \cong \mathsf{G}_2$	?

How should we regard the unit octonions?

- By themselves, they form a Moufang loop ☹
- Better: Use Faulkner construction to get a 3-algebra

  Nambu, Yamazaki, Figueroa-O'Farrill et al.
- Therefore, we have a DCM $(\mathfrak{g}_2 \xrightarrow{\mathsf{t}} \mathbb{R}^8 \cong \mathbb{O})$
- This suggests sequence: $\mathbb{Z}_2$, $\mathsf{U}(1)$, $\mathsf{SU}(2)$, a Lie 2-group ☺
- Not (yet) clear how useful this actually is.

Drop loop spaces: Principal 2-bundles over Twistor Spaces

Now that we saw the power of non-abelian gerbes, let's use them!

Twistor Description of Higher Yang-Mills Fields

Using twistor spaces, one can map holomorphic data to solutions to field equations.

Recall the principle of the Penrose-Ward transform:

- Interested in field equations that are equivalent to integrability of connections along subspaces of spacetime M
- Establish a double fibration

$$
\begin{array}{ccc}
 & F & \\
\swarrow & & \searrow \\
P & & M
\end{array}
$$

  P: twistor space, moduli space of subspaces in M
  F: correspondence space

- $H^n(P, \mathfrak{S})$ (e.g. vector bundles) $\xleftrightarrow{1:1}$ sols. to field equations.
- Explicitly appearing: gauge transformations, moduli, symmetries of the equations, etc.
- BTW: here, $\xleftrightarrow{1:1}$ is actually a "holomorphic transgression".

Known Examples of Twistor Descriptions

For Yang-Mills theories and its BPS subsectors, there is a wealth of twistor descriptions.

$$\begin{array}{ccc} & \mathbb{C}^4\times\mathbb{C}P^1 & \\ \swarrow & & \searrow \\ \mathbb{C}P^3_\circ & & \mathbb{C}^4 \end{array}$$

Instantons
hol. vector bundle

$$\begin{array}{ccc} & \mathbb{C}^3\times\mathbb{C}P^1 & \\ \swarrow & & \searrow \\ T\mathbb{C}P^1 & & \mathbb{C}^3 \end{array}$$

Monopoles
hol. vector bundle

$$\begin{array}{ccc} & \mathbb{C}^{4|12}\times\mathbb{C}P^1\times\mathbb{C}P^1 & \\ \swarrow & & \searrow \\ P^{5|6} & & \mathbb{C}^{4|12} \end{array}$$

(Super) Yang-Mills
hol. vector bundle

$$\begin{array}{ccc} & \mathbb{C}^6\times\mathbb{C}P^3 & \\ \swarrow & & \searrow \\ P^6 & & \mathbb{C}^6 \end{array}$$

abelian $H=\star H$
hol. gerbe

Hughston, Murray, Eastwood, CS & M.Wolf, Mason et al.

Note: last twistor space reduces nicely to the above ones.

New Results

New: Penrose-Ward transform for self-dual strings.

New twistor space parameterizing hyperplanes in $\mathbb{C}^4$:

$$\begin{array}{ccc} & \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1 & \\ \swarrow & & \searrow \\ P^3 & & \mathbb{C}^4 \end{array}$$

self-dual strings
hol. principal 2-bundle

CS & M Wolf, 1111.2539, 1205.3108

Note:

- The Hyperplane twistor space P^3 is the total space of the line bundle $\mathcal{O}(1,1) \to \mathbb{C}P^1 \times \mathbb{C}P^1$.
- The spheres $\mathbb{C}P^1 \times \mathbb{C}P^1$ parameterize an α- and a β-plane.
- The span of both is a hyperplane.
- Nonabelian self-dual string equations: $H = \star \mathrm{d}_A \Phi$, $F = \mathsf{t}(B)$.
- Reduces nicely to the monopole twistor space: $\mathcal{O}(2) \to \mathbb{C}P^1$.

New Results

New: Penrose-Ward transform for self-dual tensor multiplet.

$$\begin{array}{ccc} & \mathbb{C}^{6|16} \times \mathbb{C}P^3 & \\ \swarrow & & \searrow \\ P^{6|4} & & \mathbb{C}^{6|16} \end{array}$$

non-abelian self-dual tensor multiplet

hol. principal 2-bundle

CS & M Wolf, 1205.3108

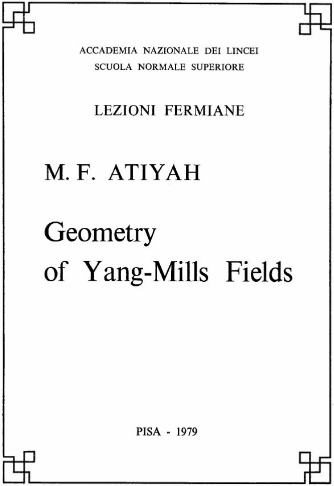
Note:

- $P^{6|4}$ is a straightforward SUSY generalization of P^6
- EOMs, abelian: $H = \star H$, $F = \mathsf{t}(B)$, $\nabla\!\!\!/\,\psi = 0$, $\Box\phi = 0$
- $\mathcal{N} = (2,0)$ SC non-abelian tensor multiplet EOMs!
- EOMs on superspace, remain to be boiled down (expected).
- Non-gerby Alternatives: Chu, Samtleben et al., ...

Higher ADHM construction

Recall that the conventional ADHM and ADHMN constructions exist due to a twistor construction in the background.

Thus, there should be a direct ADHM-like construction here, too.



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LEZIONI FERMIANE

M. F. ATIYAH

Geometry of Yang-Mills Fields

PISA - 1979

Translate this to higher gauge theory:

- Find elementary solutions
- Identify moduli
- Identify topological charges
- Higher Serre-Swan theorem
- Higher ADHM construction

Work in progress

F Sala & S Palmer & CS

Elementary Solution: The Higher Instanton

The quaternionic form of Belavin et al.'s solution almost translates perfectly.

Recall the quaternionic form of the elementary instanton on S^4:

Conformal geometry of S^4

Describe S^4 by $\mathbb{H} \cup \{\infty\}$. Coordinates: $x = x^1 + \mathrm{i}x^2 + \mathrm{j}x^3 + \mathrm{k}x^4$.
Conformal transformations:

$$x \mapsto (ax+b)(cx+d)^{-1}\ , \quad a,b,c,d \in \mathbb{H}$$

$\mathsf{SU}(2)$-Instanton:

$$A = \mathrm{im}\left(\frac{\bar{x}\mathrm{d}x}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}\bar{x}\wedge\mathrm{d}x}{(1+|x|^2)^2}\right)$$

$\mathsf{SU}(2)$-Anti-Instanton:

$$A = \mathrm{im}\left(\frac{x\mathrm{d}\bar{x}}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}x\wedge\mathrm{d}\bar{x}}{(1+|x|^2)^2}\right)$$

Belavin et al. 1975, Atiyah 1979

Elementary Solution: The Higher Instanton

The quaternionic form of Belavin et al.'s solution almost translates perfectly.

Issue: $H = \pm \star H$ is sensible only on Minkowski space $\mathbb{R}^{1,5}$.

Recall:

- conformally compactify $\mathbb{R}^4$, $\mathbb{R}^{1,3}$ yields S^4, $M^c \cong S^1 \times S^3$.
- Both S^4 and M^c real slices of $G_{2;4}$, a quadric in $\mathbb{C}P^5$.

General pattern:

Conf. compact. of $\mathbb{R}^{i,n-i} \to \mathbb{C}^n$: real slice of quadric in $\mathbb{C}P^{n+1}$
This illuminates also the conformal transformations:

$$x = x^\mu \gamma_\mu \mapsto (ax+b)(cx+d)^{-1}$$

For certain elements $a, d \in \mathcal{C}\ell_{\rm even}(\mathbb{C}^n)$, $b, c \in \mathcal{C}\ell_{\rm odd}(\mathbb{C}^n)$.

Solution: Quaternions have to be regarded as blocks of $\mathcal{C}\ell(\mathbb{C}^4)$
Work with blocks of the Clifford algebra $\mathcal{C}\ell(\mathbb{C}^6)$.

Elementary Solution: The Higher Instanton

The quaternionic form of Belavin et al.'s solution almost translates perfectly.

Solution to the higher instanton equations $H = \star H$, $F = \mathsf{t}(B)$:

- Gauge structure: $(\mathbb{C}^3 \otimes \mathfrak{sl}(4,\mathbb{C}) \xrightarrow{\mathsf{t}} \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(4,\mathbb{C}))$

$$\mathsf{t} : h = \left(\begin{array}{c|c} h_1 & h_3 \\ \hline 0 & h_2 \end{array}\right) \mapsto \left(\begin{array}{c|c} h_1 & 0 \\ \hline 0 & h_2 \end{array}\right) \in \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(4,\mathbb{C}) ,$$

  $h_1, h_2, h_3 \in \mathfrak{sl}(4,\mathbb{C})$, $\triangleright$: the usual commutator.

- Solution in coordinates $x = x^M \sigma_M$, $\hat{x} = x^M \bar{\sigma}_M$

$$A = \begin{pmatrix} \frac{\hat{x}\,\mathrm{d}x}{1+|x|^2} & 0 \\ 0 & \frac{\mathrm{d}x\,\hat{x}}{1+|x|^2} \end{pmatrix} \qquad B = F + \begin{pmatrix} 0 & \frac{\hat{x}\,\mathrm{d}x \wedge \mathrm{d}\hat{x}}{(1+|x|^2)^2} \\ 0 & 0 \end{pmatrix}$$

$$F := \mathrm{d}A + A \wedge A = \begin{pmatrix} \frac{\mathrm{d}\hat{x} \wedge \mathrm{d}x}{(1+|x|^2)^2} + \frac{2\,\mathrm{d}\hat{x}\,x \wedge \mathrm{d}\hat{x}\,x}{(1+|x|^2)^2} & 0 \\ 0 & -\frac{\mathrm{d}x \wedge \mathrm{d}\hat{x}}{(1+|x|^2)^2} \end{pmatrix}$$

$$H := \mathrm{d}B + A \triangleright B = \begin{pmatrix} 0 & \frac{\mathrm{d}\hat{x} \wedge \mathrm{d}x \wedge \mathrm{d}\hat{x}}{(1+|x|^2)^3} \\ 0 & 0 \end{pmatrix}$$

but: Peiffer violated

F Sala & S Palmer & CS

Summary:

- ✓ Generalized ADHMN-like construction on loop space
- ✓ Geometric quantization using loop space
- ✓ Gauge structures in M2- and M5-brane models similar
- ✓ Twistor construction of self-dual tensor fields
- ✓ 6d superconformal tensor multiplet equations
- ✓ On our way to develop Geometry of Higher Yang-Mills Fields

Future directions:

- ▷ More general higher bundles and twistors — with M Wolf
- ▷ Continue translation of ADHM — with S Palmer, F Sala
- ▷ Geometric Quant. with higher Hilbert spaces — with R Szabo

Geometry of Higher Yang-Mills Fields

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Symmetries and Geometry of Branes in String/M-Theory,
1.2.2013