Chain Spaces Beyond $\mathbb{R}^d$

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Definition

Let $\ell \in \mathbb{R}^n_{>0}$ (i.e. $\ell = (\ell_1, \ldots, \ell_n)$ with $\ell_i > 0$ for $i = 1, \ldots, n$). A *chain space* for ℓ in $\mathbb{R}^2$ can be defined as

$$\mathcal{C}_2^n(\ell) = \{(u_1, \ldots, u_n) \in (S^1)^{n-1} : \sum_{i=1}^{n-1} \ell_i u_i = \ell_n\}$$

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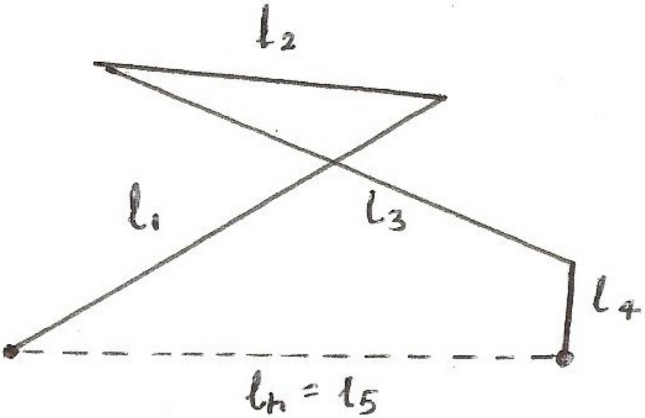
A definition for a chain space in $\mathbb{R}^d$ can be given completely analogously as:

$$\mathcal{C}_d^n(\ell) = \{(u_1, \ldots, u_n) \in (S^{d-1})^{n-1} : \sum_{i=1}^{n-1} \ell_i u_i = \ell_n \mathbf{e_1}\}$$



l_2
l_1
l_3
l_4
$l_n = l_5$

Properties of $\mathcal{C}_d^n(\ell)$

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1. $\mathcal{C}_d^n(\ell)$ is a manifold of dimension $(n-2)(d-1)-1$, provided that ℓ is *generic* (i.e. $\sum \epsilon_i \ell_i \neq 0$ for $\epsilon_i = \pm 1$).

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2. If σ is a permutation of $\{1, \ldots, n-1\}$, then $\mathcal{C}_d^n(\ell)$ and $\mathcal{C}_d^n(\sigma\ell)$ are diffeomorphic, where $\sigma\ell = (\ell_{\sigma(1)}, \ldots \ell_{\sigma(n-1)}, \ell_n)$.

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3. Each $\mathcal{C}_d^n(\ell)$ is determined (up to diffeomorphism) by subsets $J \subset \{1, \dots, n\}$ with the property that:

$$\sum_{i \in J} \ell_i < \sum_{i \notin J} \ell_i$$

Such subsets J are called *short*.

Moduli Space (Or Polygon Space)

The special orthogonal group $SO(d-1)$ acts on $\mathcal{C}_d^n(\ell)$ diagonally, leaving the first coordinate fixed. Thus, we can define the *moduli space* of linkages (or chains) as

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The topology of $\mathcal{M}_d(\ell)$?: Quite a lot is known for the case $d = 2, 3$. The cases $d > 3$ are (mostly) still unknown.

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It can be shown $N_G(\ell)$ is a manifold. The proof of this relies on a natural map from $N_G(\ell)$ to the space

$$N_{G(v)}(\ell) = \{(y_1, \ldots, y_n) \in G(v)^n : \sum_{i=1}^{n} \ell_i y_i = 0 \in V\}.$$

The (compact) Lie group G acts naturally on $N_G(\ell)$. Therefore we can define the moduli space $M_G(\ell) = N_G(\ell)/G$ as

$$M_G(\ell) = \{(x_1, \ldots, x_{n-1}) \in G^{n-1} : \sum_{i=1}^{n-1} \ell_i v^{x_i} = \ell_n v\}.$$

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3. If $G = U(d)$ acts on $V = \mathbb{C}^d$, then $H^*(M_G(\ell))$ can be expressed in terms of $H^*(\mathcal{C}_{2d}^n)$ and $H^*(U(n-1))$.