1-form Laplacian on a Graph-like Manifold

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Metric Graph

Given a discrete graph $G = (V, E, \partial)$ where $\partial : E \to V \times V$ such that $e \mapsto (\partial^+ e, \partial^- e)$ and a length function $\ell : E \to (0, +\infty)$, we identify $e \sim I_e := [0, \ell e]$. A metric graph $X_0$ is $X_0 := \dot{\bigcup}_{e \in E} I_e / \sim$ where $\sim$ identifies end points of $I_e$. 
Given a discrete graph $G = (V, E, \partial)$ where $\partial : E \rightarrow V \times V$ such that $e \mapsto (\partial_+ e, \partial_- e)$ and a length function $\ell : E \rightarrow (0, +\infty)$, $e \mapsto \ell_e > 0$, we identify $e \sim I_e := [0, \ell_e]$. A *metric graph* $X_0$ is

$$X_0 := \bigcup_{e \in E} I_e / \sim$$

where $\sim$ identifies end points of $I_e$. 
Metric Graph and metric Laplacian

Metric Laplacian

1-form \( \alpha \) on \( X \): 
\[
\alpha = (\alpha_e)_e \in E \quad \text{with} \quad \alpha_e = f_e ds
\]

\( L^2 \)-space 
\[
L^2(\Lambda^1(X_0)) = \bigoplus_e L^2(\Lambda^1(I_e))
\]

\[
\|\alpha\|_{L^2(\Lambda^1(X_0))} := \sum_e \|\alpha_e\|_{L^2(\Lambda^1(I_e))}
\]

Exterior derivative: 
\[
d: \text{dom} d \rightarrow L^2(\Lambda^1(X_0)), \quad (f_e)_e \in E = (f'_e ds)_e \in E
\]

\( \text{dom} d = H^1(X_0) \cap C(X_0) \)
Metric Laplacian

- 1-form $\alpha$ on $X_0$: $\alpha = (\alpha_e)_{e \in E}$ with $\alpha_e = f_e \, ds$
Metric Laplacian

- **1-form $\alpha$ on $X_0$:** $\alpha = (\alpha_e)_{e \in E}$ with $\alpha_e = f_e \, ds$
- **$L^2$-space $L^2(\Lambda^1(X_0))$:**

\[
L^2(\Lambda^1(X_0)) = \bigoplus_{e \in E} L^2(\Lambda^1(I_e))
\]

\[
\|\alpha\|^2_{L^2(\Lambda^1(X_0))} := \sum_{e \in E} \|\alpha_e\|^2_{L^2(\Lambda^1(I_e))}
\]
Metric Laplacian

- **1-form \( \alpha \) on \( X_0 \):** \( \alpha = (\alpha_e)_{e \in E} \) with \( \alpha_e = f_e \, ds \)

- **\( L^2 \)-space \( L^2(\Lambda^1(X_0)) \):**
  \[
  L^2(\Lambda^1(X_0)) = \bigoplus_{e \in E} L^2(\Lambda^1(I_e))
  \]

- **\( \| \alpha \|_{L^2(\Lambda^1(X_0))}^2 := \sum_{e \in E} \| \alpha_e \|_{L^2(\Lambda^1(I_e))}^2 \)**

- **exterior derivative:**
  \[
  d : \text{dom } d \rightarrow L^2(\Lambda^1(X_0)), \quad (f_e)_{e \in E} = (f'_e \, ds)_{e \in E}
  \]
  \[
  \text{dom } d = H^1(X_0) \cap C(X_0)
  \]
adjoint operator:

\[
d^* \alpha = -(f_e)'_{e \in E} \quad \text{dom } d^* = \{ \alpha \in H^1(\Lambda^1(X_0)) \mid \sum_{e \in E} \vec{\alpha}_e(v) = 0 \}
\]

\[
\vec{\alpha}_e(v) = \begin{cases} 
-f_e(0) \ ds & v = \partial_- e \\
f_e(l_e) \ ds & v = \partial_+ e 
\end{cases}
\]
adjoint operator:

\[ d^* \alpha = - (f_e)'_{e \in E} \quad \text{dom } d^* = \{ \alpha \in H^1(\Lambda^1(X_0)) \mid \sum_{e \in E} \alpha_e(v) = 0 \} \]

\[ \alpha_e(v) = \begin{cases} -f_e(0) \ ds & v = \partial_- e \\ f_e(l_e) \ ds & v = \partial_+ e \end{cases} \]

Laplacian on 1-forms on \( X_0 \):

\[ \Delta_{X_0}^1 \alpha = dd^* \alpha = -\alpha'' \]

\[ \text{dom } \Delta_{X_0} = \{ \alpha \in \text{dom } d^* \mid d^* \alpha \in \text{dom } d \} \]
Given a metric graph $X_0$, we associate a family of compact and connected manifolds $X_{\varepsilon} = (X_{\varepsilon}, g_{\varepsilon})$, $\varepsilon > 0$, of dimension $n \geq 2$ built as

$$X_{\varepsilon} = \bigcup_{e \in E} X_{\varepsilon}, e \cup \bigcup_{v \in V} X_{\varepsilon}, v \cap X_{\varepsilon}, e = \{\emptyset e / \in \{e \in E | v = \partial \pm e\}\} Y_{\varepsilon}, e$$

Here:

$X_{\varepsilon}, v = (X_v, \varepsilon^2 g_v)$ and $Y_{\varepsilon}, e = (Y_e, \varepsilon^2 h_e)$ and $X_{\varepsilon}, e = (X_e, g_{\varepsilon}, e) = (I_e \times Y_e, ds^2 + \varepsilon^2 h_e)$ with $X_v, X_e$ compact Riemannian manifolds with boundary of dimension $n$ and $Y_e$ a closed manifold of dimension $n - 1$. 
Graph-like Manifold

Given a metric graph $X_0$, we associate a family of compact and connected manifolds $X_\varepsilon = (X, g_\varepsilon)$, $\varepsilon > 0$, of dimension $n \geq 2$ built as

$$X_\varepsilon = \bigcup_{e \in E} X_{\varepsilon,e} \cup \bigcup_{v \in V} X_{\varepsilon,v}$$

$$X_{\varepsilon,v} \cap X_{\varepsilon,e} = \begin{cases} \emptyset & e \notin \{ e \in E \mid v = \partial_{\pm} e \} \\ Y_{\varepsilon,e} & e \in \{ e \in E \mid v = \partial_{\pm} e \} \end{cases} \quad (1)$$

Here: $X_{\varepsilon,v} = (X_v, \varepsilon^2 g_v)$, $Y_{\varepsilon,e} = (Y_e, \varepsilon^2 h_e)$ and $X_{\varepsilon,e} = (X_e, g_{\varepsilon,e}) = (I_e \times Y_e, ds^2 + \varepsilon^2 h_e)$ with $X_v, X_e$ compact Riemannian manifolds with boundary of dimension $n$ and $Y_e$ a closed manifold of dimension $n - 1$. 

Figure: An example of a 2-dimensional graph-like manifold
Hodge Laplacian

Hodge Laplacian on 1-forms on $X_{\varepsilon}$:

$$\Delta^1_{X_{\varepsilon}} = \text{dd}^* + \text{d}^*\text{d}$$

with $\text{d}$ and $\text{d}^*$ are the classical exterior derivative and co-derivative on a manifold.
Hodge Laplacian

- \( L^2 \)-space:

\[
L^2(\Lambda^1(X_\varepsilon)) = \left\{ w \in \Lambda^1(X_\varepsilon) \mid \| w \|_{L^2(\Lambda^1(X_\varepsilon))}^2 := \int_{X_\varepsilon} w \wedge *w < \infty \right\}
\]
Hodge Laplacian

- **$L^2$-space:**

  $$L^2(\Lambda^1(X_\varepsilon)) = \left\{ w \in \Lambda^1(X_\varepsilon) \mid \|w\|_{L^2(\Lambda^1(X_\varepsilon))}^2 := \int_{X_\varepsilon} w \wedge *w < \infty \right\}$$

- **Hodge Laplacian on 1-forms on $X_\varepsilon$:**

  $$\Delta^1_{X_\varepsilon} = dd^* + d^*d$$

  with $d$ and $d^*$ are the classical exterior derivative and co-derivative on a manifold.
Aim and Strategy

We want to relate $\sigma(\Delta_1 X \varepsilon)$ with $\sigma(\Delta_1 X 0)$. By Hodge Theorem, any 1-form splits as $\alpha = df + d^* \beta + h$ where $f$, $\beta$ and $h$ are a function, a 2-form and a harmonic 1-form respectively. Then, we have $\Delta_1 X \varepsilon \alpha = \dd d^* (df) + d^* \dd (d^* \beta) = \Delta_1, \text{ex} X \varepsilon (df) + \Delta_1, \text{co-ex} X \varepsilon (d^* \beta)$. Hence, we study the two new Laplacian separately.
We want to relate $\sigma(\Delta^{1}_{X_\varepsilon})$ with $\sigma(\Delta^{1}_{X_0})$.
Aim and Strategy

- We want to relate $\sigma(\Delta^1_{X_\varepsilon})$ with $\sigma(\Delta^1_{X_0})$
- By Hodge Theorem, any 1-form splits as

$$\alpha = df + d^*\beta + h$$

where $f$, $\beta$ and $h$ are a function, a 2-form and a harmonic 1-form respectively.
Aim and Strategy

- We want to relate $\sigma(\Delta_{X_\varepsilon}^1)$ with $\sigma(\Delta_{X_0}^1)$

- By Hodge Theorem, any 1-form splits as

$$\alpha = df + d^*\beta + h$$

where $f$, $\beta$ and $h$ are a function, a 2-form and a harmonic 1-form respectively.

Then, we have

$$\Delta_{X_\varepsilon}^1 \alpha = dd^*(df) + d^*d(d^*\beta) = \Delta_{X_\varepsilon}^{1,ex}(df) + \Delta_{X_\varepsilon}^{1,co-ex}(d^*\beta)$$
Aim and Strategy

- We want to relate $\sigma(\Delta^1_{X_\varepsilon})$ with $\sigma(\Delta^1_{X_0})$
- By Hodge Theorem, any 1-form splits as

$$\alpha = df + d^* \beta + h$$

where $f$, $\beta$ and $h$ are a function, a 2-form and a harmonic 1-form respectively.

Then, we have

$$\Delta^1_{X_\varepsilon} \alpha = dd^*(df) + d^* d(d^* \beta) = \Delta^1_{X_\varepsilon} (df) + \Delta^1_{X_\varepsilon} (d^* \beta)$$

Hence, we study the two new Laplacian separately.
Theorem 1

Let \( X_\varepsilon \) be a graph-like manifold with underlined graph \( X_0 \). Let \( \lambda_{ex,1}(X_\varepsilon) \) and \( \lambda_{1}(X_0) \) be the \( j \)-th eigenvalue on exact 1-forms on \( X_\varepsilon \) and on 1-forms on \( X_0 \), respectively. Then, \( \lambda_{ex,1}(X_\varepsilon) \rightarrow \varepsilon \rightarrow 0 \lambda_{1}(X_0) \) for all \( j > 0 \).
Theorem 1

Let $X_\varepsilon$ be a graph-like manifold with underlined graph $X_0$. Let $\lambda^{\text{ex},1}_j(X_\varepsilon)$, $\lambda^1_j(X_0)$ be the $j$-th eigenvalue on exact 1-forms on $X_\varepsilon$ and on 1-forms on $X_0$, respectively. Then,

$$\lambda^{\text{ex},1}_j(X_\varepsilon) \xrightarrow{\varepsilon \to 0} \lambda^1_j(X_0) \quad \text{for all } j > 0$$
Theorem 2

Let $X_\varepsilon$ be a graph-like manifold of dimension $n \geq 3$ shrinking to a metric graph $X_0$ as $\varepsilon \to 0$. Then, the first eigenvalue of the Laplacian defined on co-exact 1-forms on $X_\varepsilon$ satisfies

$$\lambda_{1}^{\text{co-ex},1}(X_\varepsilon) \xrightarrow{\varepsilon \to 0} \infty$$

Therefore, in the limit, the spectrum of the Laplacian on co-exact 1-forms contains just 0 and all the other eigenvalues tend to $\infty$ as $\varepsilon \to 0$. 
Exact 1-forms: convergence

\[ \lambda_j^{(X \varepsilon)} = j \text{-th eigenvalue of } \Delta_{X \varepsilon} \]

We have:

\[ \lambda^{ex,1}_{j}(X \varepsilon) \mathop{\rightarrow}^{\varepsilon \to 0} \lambda_j^{(X \varepsilon)} = \lambda_j^{(X_0)} \]

(a) is given applying \( d^* \) to the eigen-exact 1-forms on \( X \varepsilon \) and \( d \) to the eigenfunctions on \( X \varepsilon \).

(b) is given applying \( d^* \) to the eigen-1-forms on \( X_0 \) and \( d \) to the eigenfunctions on \( X_0 \).

(c) is due to convergence results by Exner and Post in [2].
Exact 1-forms: convergence

\( \lambda_j^0(\cdot) = j\)-th eigenvalue of \( \Delta^0 \)
Exact 1-forms: convergence

\[ \lambda_j^0(\cdot) = j\text{-th eigenvalue of } \Delta^0 \]
\[ \lambda_j^1(X_0) = j\text{-th eigenvalue of } \Delta_1^{X_0} \]
Case of exact 1-forms

Exact 1-forms: convergence

\[ \lambda^0_j(\cdot) = j\text{-th eigenvalue of } \Delta^0 \]
\[ \lambda^1_j(X_0) = j\text{-th eigenvalue of } \Delta^1_{X_0} \]
\[ \lambda^{ex,1}_j(X_\varepsilon) = j\text{-th eigenvalue of } \Delta^{ex,1}_{X_\varepsilon} \]
Case of exact 1-forms

Exact 1-forms: convergence

\[ \lambda_j^0(\cdot) = j\text{-th eigenvalue of } \Delta^0 \]
\[ \lambda_j^1(X_0) = j\text{-th eigenvalue of } \Delta^1_{X_0} \]
\[ \lambda_j^{\text{ex},1}(X_\varepsilon) = j\text{-th eigenvalue of } \Delta^{\text{ex},1}_{X_\varepsilon} \]

We have:

\[ \lambda_j^{\text{ex},1}(X_\varepsilon) \overset{(a)}{=} \lambda_j^0(X_\varepsilon) \implies \lambda_j^{\text{ex},1}(X_\varepsilon) \xrightarrow{\varepsilon \to 0} \lambda_j^1(X_0) \]

\[ \lambda_j^1(X_0) \overset{(b)}{=} \lambda_j^0(X_0) \]

(a) is given applying \( d^* \) to the eigen-exact 1-forms on \( X_\varepsilon \) and \( d \) to the eigenfunctions on \( X_\varepsilon \).

(b) is given applying \( d^* \) to the eigen-1-forms on \( X_0 \) and \( d \) to the eigenfunctions on \( X_0 \).

(c) is due to convergence results by Exner and Post in [2].
Exact 1-forms: convergence

\[ \lambda_j^0(\cdot) = j\text{-th eigenvalue of } \Delta^0 \]
\[ \lambda_j^1(X_0) = j\text{-th eigenvalue of } \Delta^X_0 \]
\[ \lambda_j^{\text{ex}, 1}(X_\varepsilon) = j\text{-th eigenvalue of } \Delta^{\text{ex}, 1}_X \]

We have:

\[ \lambda_j^{\text{ex}, 1}(X_\varepsilon) \stackrel{(a)}{=} \lambda_j^0(X_\varepsilon) \quad \Longrightarrow \quad \lambda_j^{\text{ex}, 1}(X_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \lambda_j^1(X_0) \]

\[ \lambda_j^1(X_0) \stackrel{(b)}{=} \lambda_j^0(X_0) \]

\[ \lambda_j^{\text{ex}, 1}(X_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \lambda_j^1(X_0) \]

(a) is given applying \( d^* \) to the eigen-exact 1-forms on \( X_\varepsilon \) and \( d \) to the eigenfunctions on \( X_\varepsilon \).
Exact 1-forms: convergence

\[ \lambda^0_j(\cdot) = j\text{-th eigenvalue of } \Delta^0 \]
\[ \lambda^1_j(X_0) = j\text{-th eigenvalue of } \Delta^1_{X_0} \]
\[ \lambda^{\text{ex},1}_j(X_\varepsilon) = j\text{-th eigenvalue of } \Delta^{\text{ex},1}_{X_\varepsilon} \]

We have:

\[ \lambda^{\text{ex},1}_j(X_\varepsilon) \overset{(a)}{=} \lambda^0_j(X_\varepsilon) \implies \lambda^{\text{ex},1}_j(X_\varepsilon) \xrightarrow{\varepsilon \to 0} \lambda^1_j(X_0) \]
\[ \lambda^1_j(X_0) \overset{(b)}{=} \lambda^0_j(X_0) \quad \varepsilon \to 0 \]

(a) is given applying \( d^* \) to the eigen-exact 1-forms on \( X_\varepsilon \) and \( d \) to the eigenfunctions on \( X_\varepsilon \).

(b) is given applying \( d^* \) to the eigen-1-forms on \( X_0 \) and \( d \) to the eigenfunctions on \( X_0 \).
Case of exact 1-forms

Exact 1-forms: convergence

\[ \lambda^0_j(\cdot) = \text{j-th eigenvalue of } \Delta^0 \]
\[ \lambda^1_j(X_0) = \text{j-th eigenvalue of } \Delta^1_{X_0} \]
\[ \lambda^{\text{ex},1}_j(X_\varepsilon) = \text{j-th eigenvalue of } \Delta^{\text{ex},1}_{X_\varepsilon} \]

We have:

\[ \lambda^{\text{ex},1}_j(X_\varepsilon) \overset{(a)}{=} \lambda^0_j(X_\varepsilon) \quad \implies \quad \lambda^{\text{ex},1}_j(X_\varepsilon) \xrightarrow{\varepsilon \to 0} \lambda^1_j(X_0) \]

\[ \lambda^1_j(X_0) \overset{(b)}{=} \lambda^0_j(X_0) \]

(a) is given applying \( d^* \) to the eigen-exact 1-forms on \( X_\varepsilon \) and \( d \) to the eigenfunctions on \( X_\varepsilon \).

(b) is given applying \( d^* \) to the eigen-1-forms on \( X_0 \) and \( d \) to the eigenfunctions on \( X_0 \).

(c) is due to convergence results by Exner and Post in [2].
Co-exact 1-forms: divergence

Let $\lambda_{co-ex, 1}^j(X) = \lambda_{ex, 2}^j(X)$ be the $j$-th eigenvalues, $j > 0$, on co-exact 1-forms and exact 2-forms on $X$. We have $\lambda_{co-ex, 1}^j(X) = \lambda_{ex, 2}^j(X)$.

Hence, we choose to work with 2-exact forms.
Co-exact 1-forms: divergence

Let $\lambda_j^{\text{co-ex},1}(X_\varepsilon), \lambda_j^{\text{ex},2}(X_\varepsilon)$ be the $j$-th eigenvalues, $j > 0$, on co-exact 1-forms and exact 2-forms on $X_\varepsilon$. 
Co-exact 1-forms: divergence

Let $\lambda_j^{co-ex,1}(X_\varepsilon), \lambda_j^{ex,2}(X_\varepsilon)$ be the $j$-th eigenvalues, $j > 0$, on co-exact 1-forms and exact 2-forms on $X_\varepsilon$.

We have

$$\lambda_j^{co-ex,1}(X_\varepsilon) = \lambda_j^{ex,2}(X_\varepsilon)$$
Co-exact 1-forms: divergence

Let $\lambda_{j}^{\text{co-ex},1}(X_{\varepsilon})$, $\lambda_{j}^{\text{ex},2}(X_{\varepsilon})$ be the $j$-th eigenvalues, $j > 0$, on co-exact 1-forms and exact 2-forms on $X_{\varepsilon}$. We have

$$\lambda_{j}^{\text{co-ex},1}(X_{\varepsilon}) = \lambda_{j}^{\text{ex},2}(X_{\varepsilon})$$

Hence, we choose to work with 2-exact forms.
Theorem 3

Let $M$ be a $n$-dimensional compact Riemannian manifold without boundary and let $\{U_i\}_{i=1}^m$ be an open cover of $M$. Let $U_{ij} = U_i \cap U_j$ and let $\mu(U_i)$, resp. $\mu(U_{ij})$, be the smallest positive eigenvalue of the Laplacian acting on exact 2-forms on $U_i$, resp. on 1-forms on $U_{ij}$, satisfying absolute boundary conditions. Further, assume $H^1(U_{ij}) = 0$ for all $i,j$. Then, the first eigenvalue of the Laplacian on exact 2-forms on $M$ satisfies

$$\lambda_{1,^\text{ex},2}^\text{ex}(M) \geq \frac{2^{-3}}{\sum_{i=1}^m \left( \frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} \left( \frac{w_{n,2}c_\rho}{\mu(U_{ij})} + 1 \right) \left( \frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)} \right) \right)}$$

where $m_i$ is the number of $j, j \neq i$ for which $U_i \cap U_j \neq \emptyset$ and $c_\rho, w_{n,2}$ are constants.

See [3, Lemma 2.3]
Assume $n \geq 3$ and choose the open cover $\{ U_{\varepsilon,i} \}_{i \in E, V}$ of $X_\varepsilon$ as the disjoint union of edge and vertex neighbourhoods in (1) in such a way that they overlap a bit.
Assume $n \geq 3$ and choose the open cover $\{U_{\varepsilon,i}\}_{i \in E, V}$ of $X_{\varepsilon}$ as the disjoint union of edge and vertex neighbourhoods in (1) in such a way that they overlap a bit.

Figure: The open cover of $X_{\varepsilon}$
Assume $H^1(Y_\varepsilon,e) = 0$ for all $e \in E$. 
Case of co-exact 1-forms

Assume $H^1(Y_\varepsilon,e) = 0$ for all $e \in E$.

Then,

$$
\lambda_1^{\text{ex},2}(X_\varepsilon) \geq \frac{2^{-3}}{\sum_{v \in V} \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} \right) + \sum_{e \in E_v} \left( \frac{w_{n,2}c_\rho}{\mu_{1,1}(X_\varepsilon,e)} + 1 \right) \left( \frac{1}{\mu_{1,1}(X_\varepsilon,v)} + \frac{1}{\mu_{1,2}(X_\varepsilon,e)} \right)}
$$
Assume $H^1(Y_\varepsilon, e) = 0$ for all $e \in E$.

Then,

$$\lambda_{1}^{\text{ex},2}(X_\varepsilon) \geq \frac{2^{-3}}{\sum_{v \in V} \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} + \sum_{e \in E_v} \left( \frac{w_{n,2} c_\rho}{\mu_{1,1}(X_\varepsilon,e)} + 1 \right) \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} + \frac{1}{\mu_{1,2}(X_\varepsilon,e)} \right) \right)}$$

where $\mu_{k,p}(\cdot)$ denotes the $k$-eigenvalue on $p$-forms satisfying absolute boundary conditions and $w_{n,2}$ and $c_\rho$ are constants.
Assume $H^1(Y_\varepsilon,e) = 0$ for all $e \in E$.

Then,

$$\lambda_{ex,2}^1(X_\varepsilon) \geq \frac{2^{-3}}{\sum_{v \in V} \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} + \sum_{e \in E_v} \left( \frac{w_{n,2} c_\rho}{\mu_{1,1}(X_\varepsilon,e)} + 1 \right) \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} + \frac{1}{\mu_{1,2}(X_\varepsilon,e)} \right) \right)}$$

where $\mu_{k,p}(\cdot)$ denotes the $k$-eigenvalue on $p$-forms satisfying absolute boundary conditions and $w_{n,2}$ and $c_\rho$ are constants.

Using a Rayleigh question argument, we have
Assume $H^1(Y_{\varepsilon,e}) = 0$ for all $e \in E$.

Then,

$$\lambda_{1}^{ex,2}(X_{\varepsilon}) \geq \frac{2^{-3}}{\sum_{v \in V} \left( \frac{1}{\mu_{1,2}(X_{\varepsilon,v})} + \sum_{e \in E_v} \left( \frac{w_{n,2}c_\rho}{\mu_{1,1}(X_{\varepsilon,e})} + 1 \right) \left( \frac{1}{\mu_{1,2}(X_{\varepsilon,v})} + \frac{1}{\mu_{1,2}(X_{\varepsilon,e})} \right) \right)}$$

where $\mu_{k,p}(\cdot)$ denotes the $k$-eigenvalue on $p$-forms satisfying absolute boundary conditions and $w_{n,2}$ and $c_\rho$ are constants.

Using a Rayleigh question argument, we have

$$\mu_{1,2}(X_{\varepsilon,v}) = \frac{\mu_{1,2}(X_v)}{\varepsilon^2} \quad \mu_{1,i}(X_{\varepsilon,e}) = \frac{c_i(\varepsilon)}{\varepsilon^2}, \quad c_i(\varepsilon) \xrightarrow{\varepsilon \to 0} \lambda_{1,i}(Y_e) \quad i = 1, 2$$
Assume \( H^1(\mathcal{Y}_\varepsilon,e) = 0 \) for all \( e \in E \).

Then,

\[
\lambda_1^{ex,2}(X_\varepsilon) \geq \frac{2^{-3}}{\sum_{v \in V} \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} + \sum_{e \in E_v} \left( \frac{w_{n,2}c_\rho}{\mu_{1,1}(X_\varepsilon,e)} + 1 \right) \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} + \frac{1}{\mu_{1,2}(X_\varepsilon,e)} \right) \right)}
\]

where \( \mu_{k,p}(\cdot) \) denotes the \( k \)-eigenvalue on \( p \)-forms satisfying absolute boundary conditions and \( w_{n,2} \) and \( c_\rho \) are constants.

Using a Rayleigh question argument, we have

\[
\mu_{1,2}(X_\varepsilon,v) = \frac{\mu_{1,2}(X_v)}{\varepsilon^2}, \quad \mu_{1,i}(X_\varepsilon,e) = \frac{c_i(\varepsilon)}{\varepsilon^2}, \quad c_i(\varepsilon) \xrightarrow{\varepsilon \to 0} \lambda_{1,i}(\mathcal{Y}_e) \quad i = 1, 2
\]

Therefore,

\[
\lambda_1^{co-ex,1}(X_\varepsilon) = \lambda_1^{ex,2}(X_\varepsilon) \geq \frac{\text{const}}{\varepsilon^2} \xrightarrow{\varepsilon \to 0} \infty
\]
Assume $H^1(Y_\varepsilon,e) = 0$ for all $e \in E$.

Then,

$$\lambda_1^{ex,2}(X_\varepsilon) \geq \frac{2^{-3}}{\sum_{v \in V} \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} + \sum_{e \in E_v} \left( \frac{w_{n,2}c_\rho}{\mu_{1,1}(X_\varepsilon,e)} + 1 \right) \left( \frac{1}{\mu_{1,2}(X_\varepsilon,v)} + \frac{1}{\mu_{1,2}(X_\varepsilon,e)} \right) \right)}$$

where $\mu_{k,p}(\cdot)$ denotes the $k$-eigenvalue on $p$-forms satisfying absolute boundary conditions and $w_{n,2}$ and $c_\rho$ are constants.

Using a Rayleigh question argument, we have

$$\mu_{1,2}(X_\varepsilon,v) = \frac{\mu_{1,2}(X_v)}{\varepsilon^2}, \quad \mu_{1,i}(X_\varepsilon,e) = \frac{c_i(\varepsilon)}{\varepsilon^2}, \quad c_i(\varepsilon) \xrightarrow{\varepsilon \to 0} \lambda_{1,i}(Y_e) \quad i = 1, 2$$

Therefore,

$$\lambda_1^{co-ex,1}(X_\varepsilon) = \lambda_1^{ex,2}(X_\varepsilon) \geq \frac{const}{\varepsilon^2} \xrightarrow{\varepsilon \to 0} \infty$$

Hence, the whole spectrum escapes to infinity.
Remarks

The hypothesis $\dim X > 2$ is crucial. In fact, if $\dim X = 2$, then $\dim Y$, $e = 1$ and this would not allow the cohomology to vanish since $Y$, $e$ is a finite union of circles.

If $H_1(Y$, $e) \neq 0$ for some $e \in E$, we obtain the same result proving $\lambda_{ex}$, $2j(X$, $e)$ $\rightarrow \infty$ as $\varepsilon \rightarrow 0$ for $j \geq N > 1$ and $\lambda_{ex}$, $2j(X$, $e)$ $\rightarrow 0$ for smaller $j$'s (again [3, Lemma 2.3]).

The bound for $\lambda_{ex}$, $21(X$, $e)$ extends to any exact $p$-forms assuming that $H_{p-1}(Y$, $e) = 0$.

Therefore, we have a complete description of the spectrum of the Laplacian on any degree forms.
Remarks

- The hypothesis $\dim X_\varepsilon > 2$ is crucial. In fact, if $\dim X_\varepsilon = 2$, then $\dim Y_{\varepsilon,e} = 1$ and this would not allow the cohomology to vanish since $Y_{\varepsilon,e}$ is a finite union of circles.
Remarks

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- If \( H^1(Y_{\varepsilon,e}) \neq 0 \) for some \( e \in E \), we obtain the same result proving \( \lambda_{j}^{\text{ex},2}(X_\varepsilon) \to \infty \) as \( \varepsilon \to 0 \) for \( j \geq N > 1 \) and \( \lambda_{j}^{\text{ex},2}(X_\varepsilon) \to 0 \) for smaller \( j \)'s (again [3, Lemma 2.3]).
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- The bound for \( \lambda_{1,2}^{\text{ex}}(X) \) extends to any exact \( p \)-forms assuming that \( H^{p-1}(Y, e) = 0 \). Therefore, we have a complete description of the spectrum of the Laplacian on any degree forms.
Given a metric graph $X_0$ with a spectral gap $(a, b)$ in its 1-form Laplacian, i.e., $(a, b)$ does not belong to the spectrum, then the associated graph-like manifold $X_\varepsilon$ has a spectral gap close to $(a, b)$ in its 1-form Laplacian for $\varepsilon$ small enough.
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Let $X_0$ and $	ilde{X}_0$ be two finite metric graphs. Attach the graph $	ilde{X}_0$ to each vertex of $X_0$. This process opens up gaps in the spectrum of its 1-form Laplacian, and so in the spectrum of the 1-form Laplacian of its associated graph-like manifold.
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Example 2: Ramanujan graphs, part 1

Let $G$ be a Ramanujan graph, i.e., $G$ is a $k$-regular graph with $n$ vertices such that

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\mu_1(G) = \max |\mu_i(G)| \leq k \mu_1(G) \leq 2 \sqrt{k - 1}/k
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Consider the associated metric graph and associated graph-like manifold $X_0$, $X_\varepsilon$. The discrete (normalized) Laplacian $\Delta_G$ on functions has a spectral gap $\mu_0(G) - \mu_1(G) \geq 1 - 2 \sqrt{k - 1}/k > 0$.

Due to the relation $\mu = 1 - \cos(\sqrt{\lambda}) \in \Delta_G \Leftrightarrow \lambda \in \Delta_{X_0}$ between discrete and metric graph (see [1]), it follows that $\sigma(\Delta_{X_0}) = \sigma(\Delta_1(X_0)$ and therefore $\sigma(\Delta_1(X_{\varepsilon}))$ has a spectral gap given by $I = (0, b_{\varepsilon})$ with $b_{\varepsilon}$ close to $\arccos(2(1 - 2 \sqrt{k - 1}/k))$. 

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Consider a family of $k$-regular bipartite Ramanujan graphs with increasing number of vertices $n$ and construct the associated family of metric graphs and graph-like manifolds.

Let the length of the edge neighborhoods shrink according to $n^{-\frac{1}{2\alpha}}$, $\alpha \geq 1$. Then, the volume grows according to some power of $n$. It is possible to estimate the rate of convergence/divergence of the eigenvalues. There is an increasing spectral gap that grows according to a power of $n$. 

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THANK YOU FOR YOUR ATTENTION!