FROM TWISTORS TO AMPLITUDES

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- BRANDHUBER, SPENCE, GT hep-th/0407214
- BEDFORD, BRANDHUBER, SPENCE, GT hep-th/0410280
- BEDFORD, BRANDHUBER, SPENCE, GT hep-th/0412108
- BEDFORD, BRANDHUBER, SPENCE, GT hep-th/0502146
- BRANDHUBER, MCNAMARA, SPENCE, GT hep-th/0506068

MOTIVATIONS

- DYNAMICS OF PARTICLES

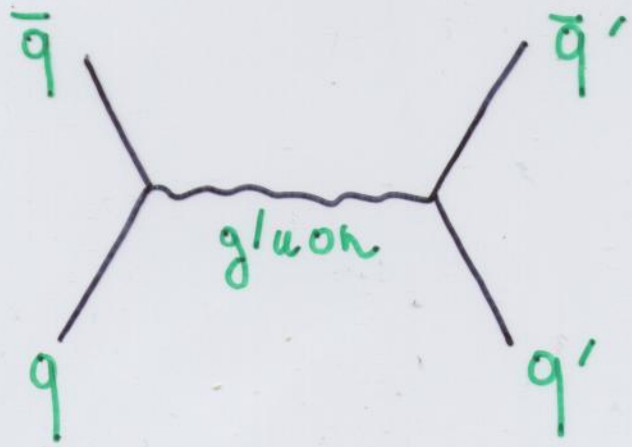
ELECTRONS
QUARKS
GLUONS

GAUGE THEORY

QUANTUM CHROMODYNAMICS, OR QCD

- AT WEAK COUPLING, WE CAN USE PERTURBATION THEORY

A SCATTERING AMPLITUDE



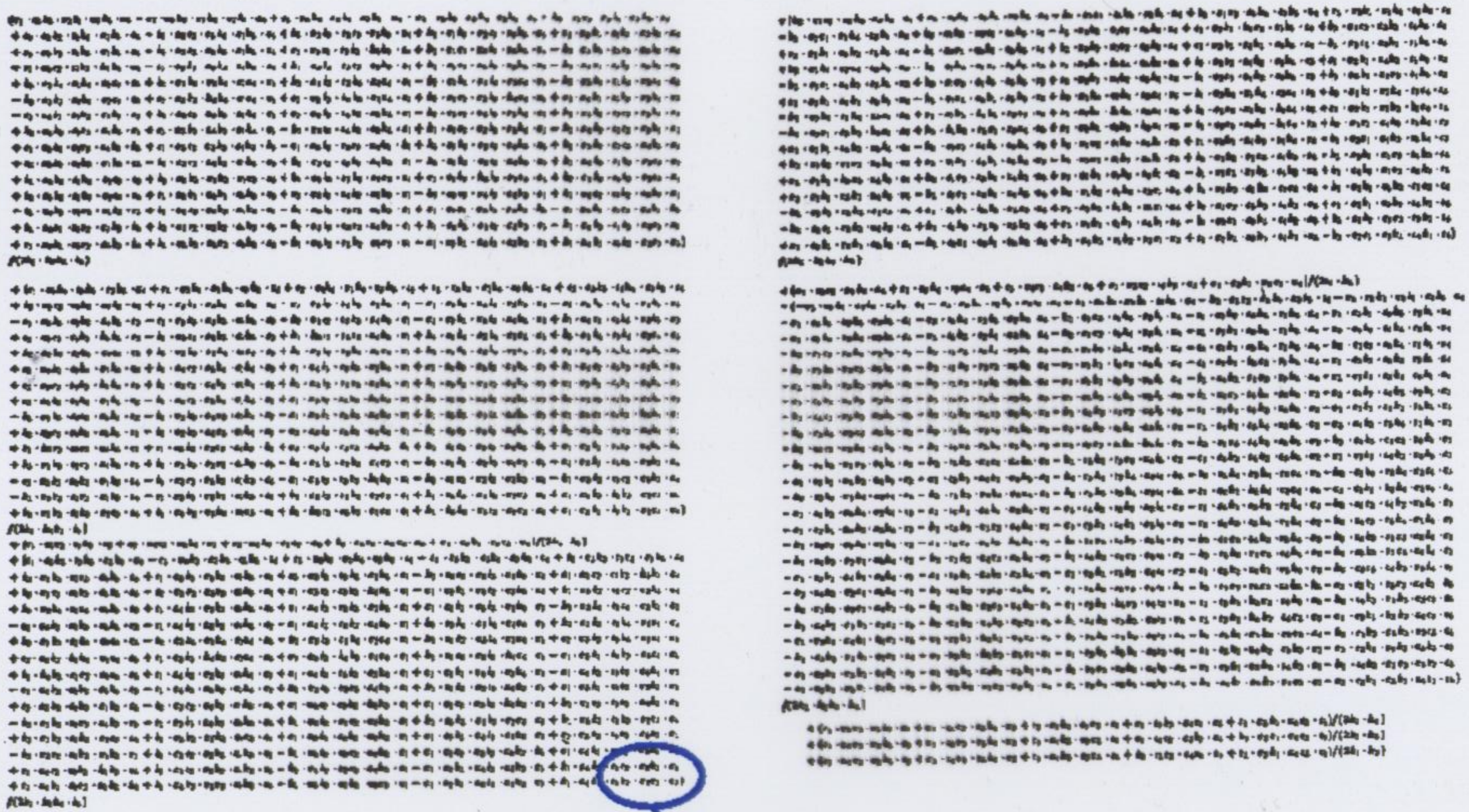
TEXTBOOK APPROACH:

- PROPAGATORS 
- INTERACTION VERTICES 

FEYNMAN DIAGRAMS

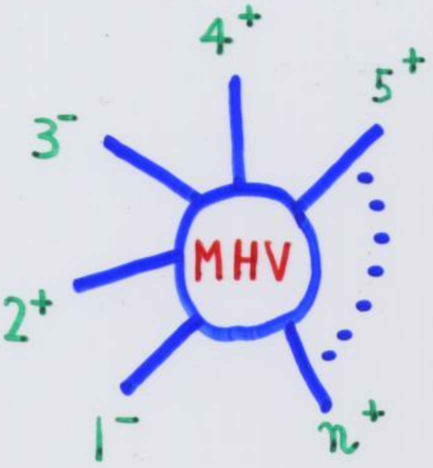
- LEADS TO EXTREMELY CUMBERSOME EXPRESSIONS

Result of a brute force calculation:



$$k_1 \cdot k_4\,\varepsilon_2 \cdot k_1\,\varepsilon_1 \cdot \varepsilon_3\,\varepsilon_4 \cdot \varepsilon_5$$

- LARGE NUMBERS OF FEYNMAN DIAGRAMS COMBINE TO PRODUCE UNEXPECTEDLY AND MYSTERIOUSLY SIMPLE EXPRESSIONS



$$g^{n-2} \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots\cdots \langle n1 \rangle}$$

MAXIMALLY
HELICITY
VIOLATING
AMPLITUDE

n	9	10	11
# DIAGRAMS	559,405	10,525,900	224,449,225

FACTORIAL GROWTH

- FEYNMAN DIAGRAMS DO NOT CAPTURE THE SIMPLICITY OF PERTURBATIVE GAUGE THEORY

7 + 7 TeV proton – proton collider

TWO KEY QUESTIONS :

- CAN WE EXPLAIN THIS SIMPLICITY ?

- CAN WE IMPROVE OUR CALCULATIONAL ABILITY ?

YES!

WITTEN DEC '03

- PERTURBATIVE N=4 SUPER YANG-MILLS
  DUAL TO
  TOPOLOGICAL B MODEL ON
  TWISTOR SPACE

  - WEAK-TO-WEAK DUALITY
  - CLASSICAL LEVEL : OK
    QUANTUM LEVEL : ✗

- AMPLITUDES HAVE SIMPLE GEOMETRIC STRUCTURE IN TWISTOR SPACE

  - ALGEBRAIC CURVES

  - NOVEL DIFFERENTIAL EQUATIONS FOR AMPLITUDES

NEW TECHNIQUES

≥ MARCH '04

- MHV DIAGRAMS

  CACHAZO
  SVRČEK
  WITTEN

  - CLASSICAL LEVEL
  - QUANTUM LEVEL
    (UNEXPECTED...)

  BRANDHUBER
  SPENCE
  GT

- GENERALISED UNITARITY
  (LOOPS)

  BRITTO
  CACHAZO
  FENG

  - UNITARITY: RECONSTRUCT A FUNCTION FROM ITS CUTS
  - GENERALISED UNITARITY: MULTIPLE CUTS
  - LOOPS FROM TREES

- RECURSION RELATIONS FOR GLUON SCATTERING AT TREE LEVEL

  BRITTO, CACHAZO, FENG + WITTEN

  - TREE AMPLITUDES HAVE ONLY SIMPLE POLES
  - RECONSTRUCT (COMPLEX) FUNCTION FROM POLES + RESIDUES
  - AMPLITUDES BUILT OUT OF

IN COMPLEXIFIED MINKOWSKI

- RECENT APPLICATIONS TO LOOPS IN YANG-MILLS

  BERN, DIXON, KOSOWER

- RECURSION RELATIONS IN GENERAL RELATIVITY

  BEDFORD, BRANDHUBER, SPENCE, GT
  CACHAZO, SVRČEK

BRYCE S. DEWITT

corrections. For the Yang-Mills field it takes the form

$$V_{(\alpha\gamma''\sigma'')\beta'} \to -ic_{\alpha\beta\gamma}p'^{\sigma} = -ic_{\alpha\gamma\beta}(p^{\sigma}+p''^{\sigma}). \tag{2.3}$$

The propagators for the normal and fictitious quanta are, respectively,

$$G \to \gamma^{\alpha\beta}\eta_{\mu\nu}/p^2, \tag{2.4}$$

$$\hat{G} \to \gamma^{\alpha\beta}/p^2, \tag{2.5}$$

with p^2 being understood to have the usual small negative imaginary part.

The corresponding quantities for the gravitational field are much more complicated. In this case we shall employ the momentum-index combinations $p\mu\nu$, $p'\sigma'\tau'$, $p''\rho''\lambda''$, $p'''\iota'''\kappa'''$. The vertices must not only be symmetric in each index pair but must also remain unchanged under arbitrary permutations of the momentum-index triplets. At least 171 separate terms are required in the complete expression for S_3 in order to exhibit this full symmetry, and for S_4 the number is 2850. However, these numbers can be greatly reduced by counting only the combinatorially distinct terms[2] and leaving it understood that the appropriate symmetrizations are to be carried out. In this way S_3 is reduced to 11 terms and S_4 to 28 terms, as follows:

3-PT VERTEX

$$\begin{aligned}
&\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \to \\
&\mathrm{Sym}\Big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^{\sigma}p^{\lambda}\eta^{\mu\nu}\eta^{\tau\rho}) \\
&-\tfrac{1}{2}P_3(p^{\tau}p'^{\mu}\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^{\rho}p^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^{\sigma}p'^{\lambda}\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu}) \\
&\qquad -P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\Big],
\end{aligned} \tag{2.6}$$

4-PT VERTEX

$$\begin{aligned}
&\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \to \\
&\mathrm{Sym}\Big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa}) \\
&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa}) \\
&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^{\sigma}p^{\tau}\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa}) \\
&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma}) \\
&-P_{12}(p^{\sigma}p^{\tau}\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^{\rho}p'^{\lambda}\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_{\sigma}p'^{\rho}\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^{\rho}p'^{\iota}\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa}) \\
&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^{\sigma}p^{\rho}\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa}) \\
&\qquad -P_6(p^{\rho}p'^{\iota}\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\Big].
\end{aligned} \tag{2.7}$$

The "Sym" standing in front of these expressions indicates that a symmetrization is to be performed on each index pair $\mu\nu$, $\sigma\tau$, etc. The symbol P indicates that a summation is to be carried out over all distinct permutations of the momentum-index triplets, and the subscript gives the number of permutations required in each case.

Expressions (2.6) and (2.7) can be obtained in a straightforward manner by repeated functional differentiation of the Einstein action. This procedure, however, is exceedingly laborious. A more efficient (but still lengthy) method is to make use of the hierarchy of identities (II, 17.31). It is a remarkable fact that once $S_2{}^0$ is known all the higher vertex functions, and hence the complete action functional itself, are determined by the general coordinate invariance of the theory. It is convenient, in the actual computation of the vertices via (II, 17.31), to invent diagrammatic schemes for displaying the combinatorics of indices. Since each reader will devise the scheme which suits him best we shall not shackle him by describing one here. We also make no attempt to display S_5 or any higher vertices.

The vertex $V_{(\alpha i)\beta}$ has the following form for the gravitational field:

$$\begin{aligned}
&V_{(\mu}{}^{\sigma''\tau''}{}_{)\nu'} \to \\
&\tfrac{1}{2}\mathrm{Sym}\big[2p''{}_{\mu}p'^{\sigma}\delta_{\nu}{}^{\tau}-p''{}_{\mu}p'{}_{\nu}\eta^{\sigma\tau} \\
&\qquad +(p_{\nu}p'^{\sigma}-p'{}_{\nu}p^{\sigma})\delta_{\mu}{}^{\tau}+p\cdot p'\delta_{\mu}{}^{\sigma}\delta_{\nu}{}^{\tau}\big],
\end{aligned} \tag{2.8}$$

where the momentum-index combinations are $p\mu$, $p'\nu'$, $p''\sigma''\tau''$, and the symmetrization is to be performed on the index pair $\sigma\tau$. The propagators for the normal and fictitious quanta are given by

$$G \to (\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\sigma\tau})/p^2, \tag{2.9}$$

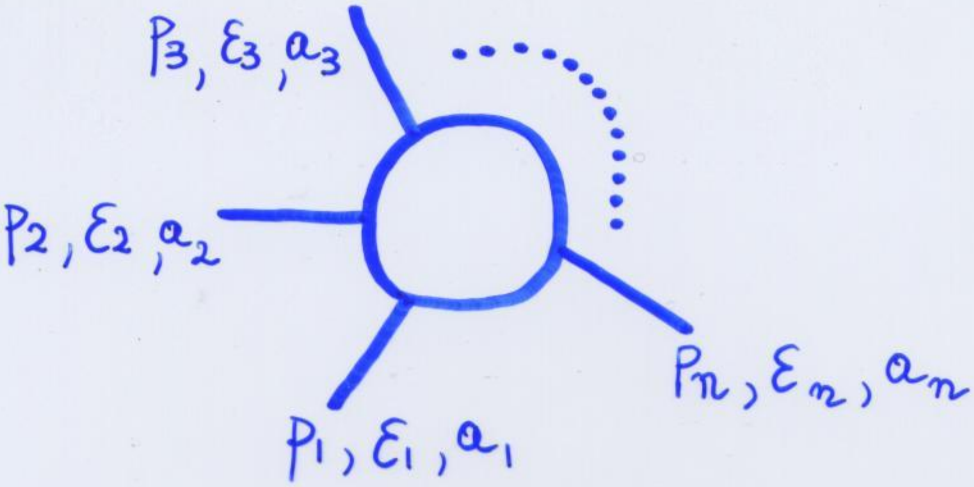
$$\hat{G} \to \eta^{\mu\nu}/p^2. \tag{2.10}$$

[2] The choice of terms is not completely unique since momentum conservation may be used to replace a given term by other terms. We give here what we believe (but have not proved) to be the expressions containing the smallest number of terms.

OUTLINE

- SCATTERING AMPLITUDES
- GEOMETRICAL STRUCTURE OF AMPLITUDES IN TWISTOR SPACE
- MHV DIAGRAMS
- GENERALISED UNITARITY
- RECURSION RELATIONS

A SCATTERING PROCESS



GLUON SCATTERING

- TO DESCRIBE A SCATTERING PROCESS, WE NEED, FOR EACH GLUON:

  1) MOMENTUM, p_i^μ
  2) POLARISATION VECTOR, ε_i^μ
  3) COLOUR, a_i

- $\mathcal{A} = \mathcal{A}(\{p_i^\mu, \varepsilon_i^\mu, a_i\})$

  IS, IN GENERAL, A VERY COMPLICATED EXPRESSION

THE ROAD TO SIMPLICITY

- **COLOUR DECOMPOSITION**

  BERENDS, GIELE ((1987)
  MANGANO, PARKE, XU (1988)
  MANGANO ((1988)
  BERN, KOSOWER (1991)

- **SPINOR HELICITY FORMALISM**

  BERENDS, KLEISS, DE CAUSMAECKER, GASTMANS, WU ((1981)
  DE CAUSMAECKER, GASTMANS, TROOST, WU (1982)
  KLEISS, STIRLING (1985)
  XU, ZHANG, CHANG ((1987)
  GUNION, KONSZT ((1985)

COLOUR DECOMPOSITION

SEPARATE COLOUR STRUCTURE

- AT TREE LEVEL, YANG-MILLS INTERACTIONS ARE PLANAR

$$A^{\text{tree}}\left(\{p_i,\varepsilon_i,a_i\}\right)=\sum_{\sigma\in S_n/Z_n}\mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right)$$

$$\cdot\, A\left(\sigma(p_1,\varepsilon_1),\dots,\sigma(p_n,\varepsilon_n)\right)$$

COLOUR-ORDERED PARTIAL AMPLITUDE

- INCLUDE ONLY DIAGRAMS WITH A FIXED CYCLIC ORDERING OF GLUONS
- ANALYTIC STRUCTURE IS SIMPLER

- AT LOOP LEVEL, MULTI-TRACE STRUCTURES APPEAR
  - SUBLEADING IN $1/N$

SPINOR HELICITY FORMALISM

- DESCRIPTION IN TERMS OF p^μ, ε^μ IS REDUNDANT

- COMPLEXIFIED LORENTZ GROUP IN 4D

$$SO(3,1,\mathbb{C}) \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$$

- p^μ, 4-VECTOR

$$p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}} \qquad a, \dot{a} = 1,2$$

$\sigma^0 = \mathbb{1}_2$ $\qquad$ σ^i = PAULI MATRICES

- MASSLESS PARTICLES, $p_\mu p^\mu = 0$

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

- LORENTZ-INVARIANT PRODUCTS

$$\langle \lambda, \lambda' \rangle \equiv \varepsilon_{ab} \lambda^a \lambda'^b \qquad [\tilde{\lambda}, \tilde{\lambda}'] = \varepsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{\dot{a}} \tilde{\lambda}'^{\dot{b}}$$

$$2(p_1 \cdot p_2) = \langle 12 \rangle [21]$$

- MOMENTA
- WAVEFUNCTIONS

} FUNCTIONS OF $\lambda, \tilde{\lambda}$

↳ e.g. $h = \frac{1}{2}$

- $\psi_{\dot{a}} \sim \tilde{\lambda}_{\dot{a}} \; e^{i \lambda_a \tilde{\lambda}_{\dot{a}} x^{\dot{a}a}}$

$h = \pm 1$

- $\varepsilon^{(+)}_{a\dot{a}} = \dfrac{\tilde{\lambda}_{\dot{a}} \eta_a}{\langle \lambda \, \eta \rangle}$

- $\varepsilon^{(-)}_{a\dot{a}} = \dfrac{\lambda_a \tilde{\eta}_{\dot{a}}}{[\tilde{\lambda} \, \tilde{\eta}]}$

- $\lambda, \tilde{\lambda}$ RELATED TO TWISTOR COORDINATES

- $A = A\left(\{\lambda_i, \tilde{\lambda}_i; h_i\}\right)$

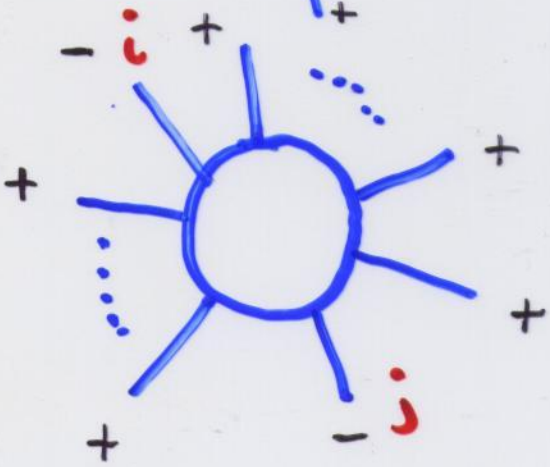
SIMPLE !

n-GLUON TREE AMPLITUDES

[CONTRIBUTION PROPORTIONAL TO $Tr(T_1 \cdots T_n)$]

$$= \quad 0$$

$$= \quad 0$$



$$= \quad A^{(n)}_{MHV}$$

PARKE-TAYLOR

- $$A^{(n)}_{MHV} = i\, g^{n-2} (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{n} p_i\right) \cdot \frac{\langle \lambda_i \lambda_j \rangle^4}{\prod_{k=1}^{n} \langle \lambda_k \lambda_{k+1} \rangle}$$
- $\langle \lambda_i \lambda_j \rangle = \varepsilon_{ab} \lambda_i^a \lambda_j^b$
- "HOLOMORPHIC"

Amplitude for n-Gluon Scattering

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(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

PACS numbers: 12.38.Bx

Computations of the scattering amplitudes for the vector gauge bosons of non-Abelian gauge theories, besides being interesting from a purely quantum-field-theoretical point of view (determination of the S matrix), have a wide range of important applications. In particular, within the framework of quantum chromodynamics (QCD), the scattering of vector gauge bosons (gluons) gives rise to experimentally observable multijet production at high-energy hadron colliders. The knowledge of cross sections for the gluon scattering is crucial for any reliable phenomenology of jet physics, which holds great promise for testing QCD as well as for the discovery of new physics at present (CERN $Sp\bar{p}S$ and Fermilab Tevatron) and future (Superconducting Super Collider) hadron colliders.[1]

In this short Letter, we give a nontrivial squared helicity amplitude for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors. To our knowledge this is the first time in a non-Abelian gauge theory that a nontrivial, on-mass-shell, squared Green's function has been written down for an arbitrary number of external points. Our result can be used to improve the existing numerical programs for the QCD jet production, and in particular for the studies of the four-jet production for which no analytic results have been available so far. Before presenting the helicity amplitude, let us make it clear that this result is an educated guess which we have compared to the existing computations and verified by a series of highly nontrivial and nonlinear consistency checks.

For the n-gluon scattering amplitude, there are $(n+2)/2$ independent helicity amplitudes. At the tree level, the two helicity amplitudes which most violate the conservation of helicity are zero. This is easily seen by the embedding of the Yang-Mills theory in a supersymmetric theory.[2,3] Here we give an expression for the next helicity amplitude, also at tree level, to leading order in the number of colors in SU(N) Yang-Mills theory.

If the helicity amplitude for gluons $1, \ldots, n$, of momenta $p_1, \ldots, p_n$ and helicities $\lambda_1, \ldots, \lambda_n$, is $\mathcal{M}_n(\lambda_1, \ldots, \lambda_n)$, where the momenta and helicities are labeled as though all particles are outgoing, then the three helicity amplitudes of interest, squared and summed over color, are

$$|\mathcal{M}_n(+++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{1}$$

$$|\mathcal{M}_n(-++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{2}$$

MHV AMPLITUDE

$$|\mathcal{M}_n(--+++\cdots)|^2 = c_n(g,N)[(p_1\cdot p_2)^4 \times \textstyle\sum_P[(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4)\cdots(p_n\cdot p_1)]^{-1} + O(N^{-2}) + O(g^2)], \tag{3}$$

where $c_n(g,N) = g^{2n-4}N^{n-2}(N^2-1)/2^{n-4}n$. The sum is over all permutations P of $1, \ldots, n$.

Equation (3) has the correct dimensions and symmetry properties for this n-particle scattering amplitude squared. Also it agrees with the known results[4,5] for $n=4$, 5, and 6. The agreement for $n=6$ is numerical.[5,6] More importantly, this set of amplitudes is consistent with the Altarelli and Parisi[7] relationship for all n, when two of the gluons are made parallel. This is trivial for the first two helicity amplitudes but is a highly nontrivial statement for the last amplitude, as shown here:

$$|\mathcal{M}_n(--+++\cdots)|^2 \xrightarrow[1\parallel 2]{} 0, \tag{4}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \xrightarrow[2\parallel 3]{} 2g^2N\frac{z^4}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{5}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \xrightarrow[3\parallel 4]{} 2g^2N\frac{1}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{6}$$

where s is the corresponding pole and z is the momentum fraction. The result for particles 2 and 3 nearly parallel, Eq. (5), is only simple because $\mathcal{M}_{n-1}(-+++\cdots)$ is zero to this order in g so that there is no interference term and therefore azimuthal averaging is not required.

The surprise about this result is that all denominators are simple dot products of two external momenta. The Feynman diagrams for n-gluon $(n>5)$ scattering contain propagators $(p_i+p_j+p_k)^2$, $(p_i+p_j+p_k+p_m)^2$, ... These propagators must cancel for Eq. (3) to be correct; this occurs for $n=6$. Of course, Altarelli and Parisi have taught us that many cancellations are expected.

We do not expect such a simple expression for the other helicity amplitudes. Also, we challenge the string theorists to prove more rigorously that Eq. (3) is correct.

!

Fermilab is operated by the Universities Research Association Inc. under contract with the United States Department of Energy.

[1]E. Eichten, I. Hinchliffe, K. Lane, and C. Quigg, Rev. Mod. Phys. **56**, 579 (1984).

[2]M. T. Grisaru, H. N. Pendleton, and P. van Nieuwenhuizen, Phys. Rev. D **15**, 997 (1977); M. T. Grisaru and H. N. Pendleton, Nucl. Phys. **B124**, 81 (1977).

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[4]T. Gottschalk and D. Sivers, Phys. Rev. D **21**, 102 (1980); F. A. Berends, R. Kleiss, P. de Causmacker, R. Gastmans, and T. T. Wu, Phys. Lett. **103B**, 124 (1981).

[5]S. J. Parke and T. R. Taylor, Fermilab Report No. Pub-85/118-T, 1985 (to be published); Z. Kunszt, CERN Report No. TH-4319, 1985 (to be published).

[6]Another numerical fact worth mentioning is that to leading order in g but to all orders in N, the amplitude $|\mathcal{M}_{n=6}(--++++)|^2$ is permutation symmetric apart from the factor $(p_1\cdot p_2)^4$. This allows all permutations of this amplitude to be trivially calculated from one such permutation.

[7]G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).

- COLOUR DECOMPOSITION AND SPINOR HELICITY FORMALISM MAKE SIMPLICITY MANIFEST...

- ... BUT WE STILL HAVE TO EXPLAIN IT !

① SIMPLE GEOMETRIC STRUCTURE IN TWISTOR SPACE

→ MHV DIAGRAMS

② RECURSIVE STRUCTURES IN SCATTERING AMPLITUDES

- CLASSICAL
- QUANTUM

TWISTOR SPACE

- $A = A(\{\lambda_i, \tilde{\lambda}_i, h_i\})$

$$(\lambda_a, \tilde{\lambda}_{\dot{a}}) \qquad a, \dot{a} = 1,2$$

"HALF FOURIER TRANSFORM"

$$\downarrow$$

$$Z_I \equiv (\lambda_a, \mu_{\dot{a}}) \qquad \mu \to -i\frac{\partial}{\partial \tilde{\lambda}} \qquad \tilde{\lambda} \to i\frac{\partial}{\partial \mu}$$

- FOUR-DIMENSIONAL COMPLEX VECTOR SPACE $\mathbb{C}^4$ OR TWISTOR SPACE

- $(\lambda, \tilde{\lambda}) \sim (t\lambda, t^{-1}\tilde{\lambda}) \qquad t \in \mathbb{C}^*$

$$(\lambda, \mu) \sim (t\lambda, t\mu) \Rightarrow$$

PROJECTIVE TWISTOR SPACE $\mathbb{CP}^3$

* $\mathbb{CP}^{3|4}$ IN THE OPEN B MODEL

REMARK ON SIGNATURES

- $+++-$
  - $\lambda, \tilde{\lambda}$ COMPLEX, $\tilde{\lambda} = \pm \lambda^*$
  - TWISTOR SPACE: $\mathbb{CP}^3$

- $++--$
  - $\lambda, \tilde{\lambda}$ REAL AND INDEPENDENT
  - TWISTOR SPACE: $\mathbb{RP}^3$

- FOURIER TRANSFORM:
  WORK WITH REAL TWISTOR SPACE

$\frac{1}{2}$- FOURIER TRANSFORM

$$A(\lambda,\tilde{\lambda}) \longrightarrow \tilde{A}(\lambda,\mu)$$

- $A_{MHV}(1^+2^+ \cdots i^- \cdots j^- \cdots n^+)$

$$= i\, g^{n-2}\, \delta^{(4)}\Big(\sum_{p=1}^{n} \lambda_p \tilde{\lambda}_p\Big) \cdot f(\lambda_1, \dots, \lambda_n)$$

- $f(\lambda_1, \dots, \lambda_n) = \dfrac{\langle \lambda_i\, \lambda_j \rangle^4}{\langle \lambda_1 \lambda_2 \rangle \langle \lambda_2 \lambda_3 \rangle \cdots \langle \lambda_n \lambda_1 \rangle}$

  - HOLOMORPHIC
  - SIMPLE AND ELEGANT

- $\tilde{A}(\lambda,\mu) = \displaystyle\int \prod_p \frac{d^2 \tilde{\lambda}_p}{(2\pi)^2}\, e^{i \sum \tilde{\lambda}^{\dot{a}}_p \mu^p_{\dot{a}}}\, A(\lambda,\tilde{\lambda})$

- $\tilde{A}_{MHV}(\lambda,\mu) = i\, g^{n-2}\, f(\lambda_1, \dots, \lambda_n)$

$$\cdot \int d^4x \int \prod \frac{d^2\tilde{\lambda}_p}{(2\pi)^2}\, e^{i \sum_p \tilde{\lambda}^p_{\dot{a}} \mu^{\dot{a}}_p}\, e^{i \sum_p x^{\dot{a}a} \lambda^p_a \tilde{\lambda}^{\dot{a}}_p}$$

WHERE

$$(2\pi)^4 \delta^{(4)}(\sum \lambda \tilde{\lambda}) = \int d^4x\, e^{i x^{\dot{a}a} \sum \lambda_a \tilde{\lambda}_{\dot{a}}}$$

- $\tilde{\lambda}$ INTEGRATION IS TRIVIAL!

$$\tilde{A}_{MHV} = \int d^4x \prod_i \delta^{(2)}(\mu^{\dot{a}}_i + x^{\dot{a}a} \lambda_{ai})$$

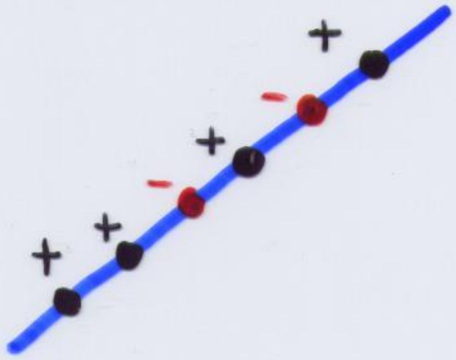
$$i\, g^{n-2} \cdot f(\lambda_1, \dots, \lambda_n)$$

FOR TREE-LEVEL MHV AMPLITUDE

- MHV AMPLITUDES AT TREE LEVEL HAVE SUPPORT ON A COMPLEX LINE IN TWISTOR SPACE [WITTEN'03]

$$\mu^{\dot{a}} + x^{\dot{a}a}\lambda_a = 0$$

PENROSE'S INCIDENCE RELATION



$\mathbb{CP}^1 \subset \mathbb{CP}^3$

- GIVEN A POINT x IN $\mathbb{C}M$, THE INCIDENCE RELATION DEFINES A $\mathbb{CP}^1$ IN TWISTOR SPACE

ALGEBRAIC CURVE OF $d=1$, $g=0$

- $\mathbb{C}M$ = MODULI SPACE OF THESE CURVES

CONJECTURE

[WITTEN, 2003]

- SCATTERING AMPLITUDES OF n GLUONS, q OF WHICH HAVE NEGATIVE HELICITY, HAVE SUPPORT ON ALGEBRAIC CURVES OF

$$d = q - 1 + \ell$$

$$g \leq \ell \qquad \ell = \# \text{LOOPS}$$

- MHV AMPLITUDE, TREE LEVEL

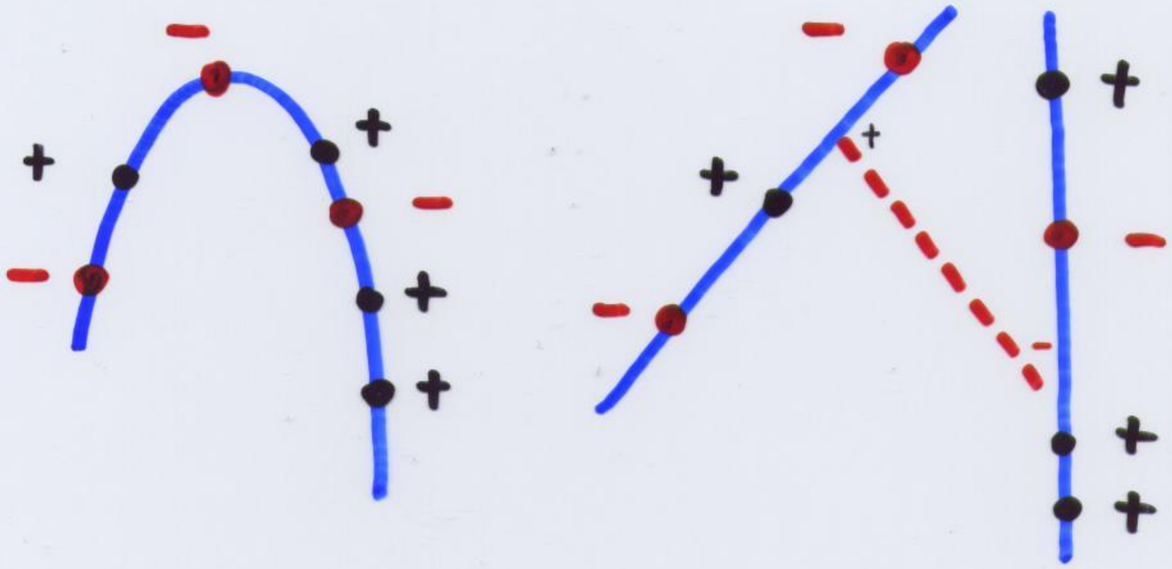
$$q = 2, \ \ell = 0 \quad \Rightarrow \quad \begin{array}{l} d = 1 \\ g = 0 \end{array}$$

BEYOND MHV

- NEXT-TO-MHV TREE LEVEL

e.g. $- + - + - + +$

$q = 3 \quad \ell = 0 \Rightarrow d = 2 \quad g = 0$



- STUDY LOCALISATION PROPERTIES IN TWISTOR SPACE OF TREE-LEVEL AMPLITUDES IN YM

  - COLLINEAR OPERATOR

$$F_I \equiv \varepsilon_{IJKL} Z^J_{(1)} Z^K_{(2)} Z^L_{(3)}$$

  - COPLANAR OPERATOR

$$K \equiv \varepsilon_{IJKL} Z^I_{(1)} Z^J_{(2)} Z^K_{(3)} Z^L_{(4)}$$

- IN SPINOR SPACE

$$(\lambda, \mu) \rightarrow \left(\lambda, -i \frac{\partial}{\partial \tilde{\lambda}}\right)$$

$F, K \rightarrow$ DIFFERENTIAL OPERATORS

e.g. $F_I^{(1,2,3)}$ FOR $I = \dot{a}$ $\longrightarrow$

$$\langle \lambda_1 \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_3^{\dot{a}}} + \langle \lambda_2 \lambda_3 \rangle \frac{\partial}{\partial \tilde{\lambda}_1^{\dot{a}}} + \langle \lambda_3 \lambda_1 \rangle \frac{\partial}{\partial \tilde{\lambda}_2^{\dot{a}}}$$

- TREE-LEVEL MHV AMPLITUDE IS INDEPENDENT OF $\tilde{\lambda}$

$$F_I^{(i,j,k)} A_{MHV}^{tree} (\cdots i \cdots j \cdots k \cdots) = 0$$

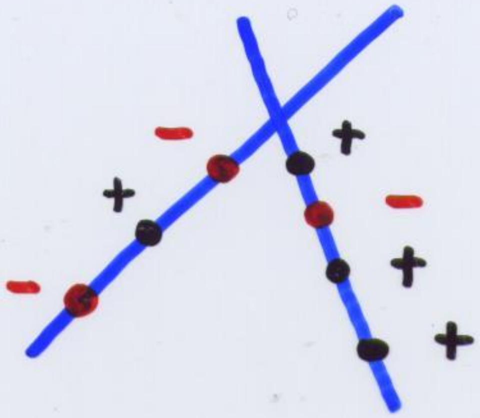
i, j, k ARE COLLINEAR

- NOVEL DIFFERENTIAL EQUATIONS FOR AMPLITUDES

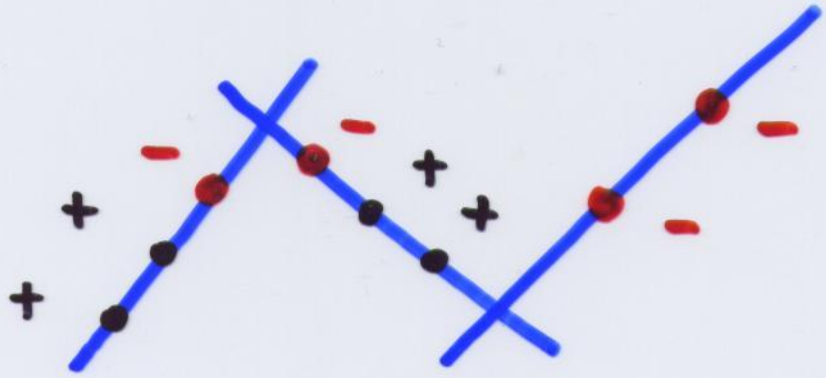
WITTEN, DEC '03

- YM AMPLITUDES TREE LEVEL
LOCALISE ON UNION OF LINES

[CACHAZO, SVRČEK, WITTEN]



$q = 3$



$q = 4$

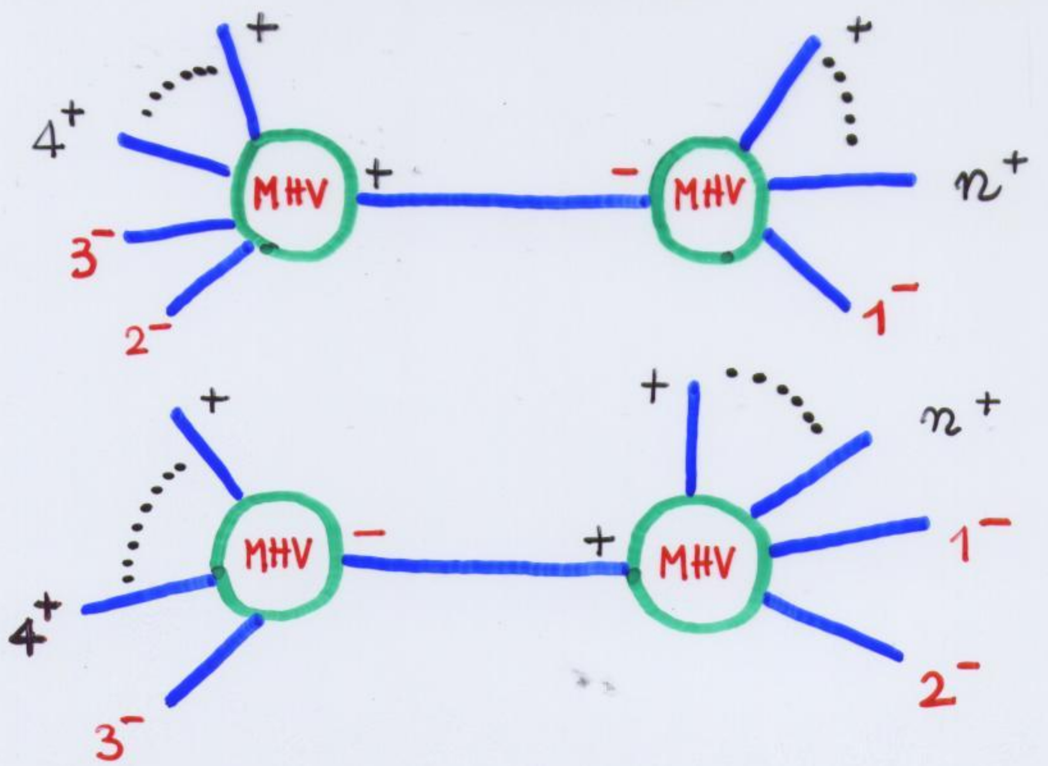
- IN A CERTAIN SENSE,
MHV AMPLITUDES CORRESPOND
TO A LOCAL INTERACTION
IN MINKOWSKI SPACE ...

– MHV DIAGRAMS –
(TREE LEVEL)

CACHAZO
SVRČEK
WITTEN

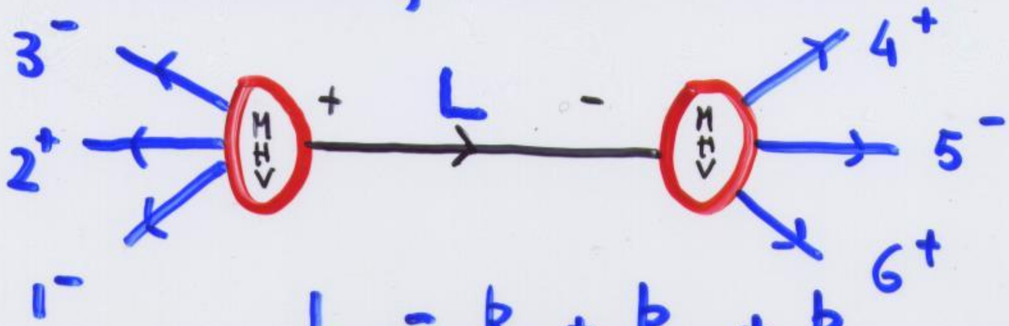
- LIFT MHV SCATTERING AMPLITUDES TO EFFECTIVE VERTICES
  - REQUIRES AN OFF-SHELL PRESCRIPTION
  - SCALAR PROPAGATORS
- OBTAIN PREVIOUSLY KNOWN AND UNKWNOWN AMPLITUDES WITH DRAMATIC SIMPLIFICATIONS

EX. NEXT-TO-MHV, $1^- 2^- 3^- 4^+ \ldots n^+$



OFF-SHELL PRESCRIPTION

CONSIDER, FOR EXAMPLE $1^- 2^+ 3^- 4^+ 5^- 6^+$



$$L = k_4 + k_5 + k_6 \qquad L^2 \neq 0$$

- WHAT ARE $\langle 1 L \rangle, [2 L\rangle, \ldots$?

- $$L_{a\dot{a}} = l_a \tilde{l}_{\dot{a}} + z \underbrace{\eta_a \tilde{\eta}_{\dot{a}}}_{\text{REFERENCE NULL VECTOR}}$$

$$l_a = \frac{L_{a\dot{a}} \tilde{\eta}^{\dot{a}}}{[\tilde{l}\, \tilde{\eta}]}$$

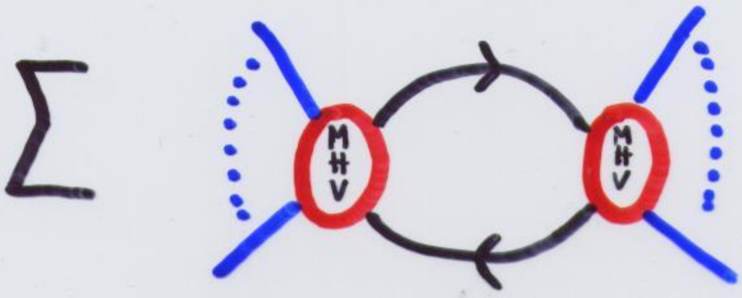
- $$\langle 1 L \rangle \longrightarrow k_{1a} L^{\dot{a}a} \tilde{\eta}_{\dot{a}}$$

- $$z = \frac{L^2}{2 L \cdot \eta}$$

- NEXT STEP: LOOP DIAGRAMS

BRANDHUBER, SPENCE, GT

MHV, 1-LOOP



[SCHEMATICALLY]

$d = q - 1 + \ell$

$q = 1 \quad \ell = 1 \Rightarrow$

$d = 2$

- INITIAL PROGNOSIS VERY POOR...
  - TWISTOR STRING THEORY AT LOOP LEVEL DUAL TO CONFORMAL SUPERGRAVITY
    BERKOVITS, WITTEN
  - TWISTOR SPACE STRUCTURE OF KNOWN ONE-LOOP AMPLITUDES PUZZLING...

- TRY ANYWAY !

MHV DIAGRAMS FOR LOOPS

- SEW V MHV VERTICES

  $V = q - 1 + \ell$, $q = \#$ NEGATIVE HELICITIES

  $\ell = \#$ LOOPS

  - SCALAR PROPAGATORS
  - CSW OFF-SHELL PRESCRIPTION
  - SUM OVER DIAGRAMS WITH FIXED CYCLIC ORDERING OF EXTERNAL GLUONS

  (DIFFERENT FROM CUT-CONSTRUCTIBILITY APPROACH OF BDDK)

- SIMPLEST EXAMPLE :

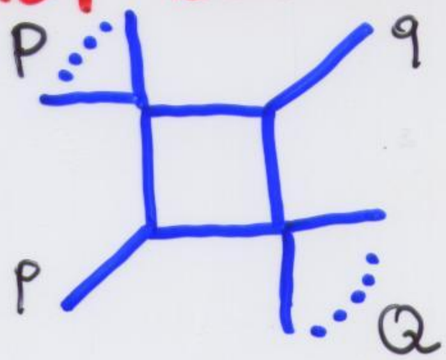
ONE-LOOP MHV AMPLITUDE IN $N=4$ SUPER YANG-MILLS

- COMPUTED BY

  BERN, DIXON, DUNBAR, KOSOWER (1994)

  USING CUT-CONSTRUCTIBILITY

- EXPRESSED IN TERMS OF "2-MASS EASY" BOX FUNCTIONS



* $F(s,t,P^2,Q^2) =$

$$s = (P+p)^2$$
$$t = (P+q)^2$$

* $$A^{1loop}_{MHV} = \left[\sum_{p,q} F(s,t,P^2,Q^2) \right] \cdot A^{tree}_{MHV}$$

EXPLICIT CALCULATION GIVES

$$A^{MHV}_{1\text{-loop}} = A^{MHV}_{tree} \sum_{p,q} \mathcal{F}(s,t,P^2,Q^2)$$

$$\mathcal{F}(s,t,P^2,Q^2) = -\frac{1}{\varepsilon^2}\left[(-s)^{-\varepsilon} + (-t)^{-\varepsilon} - (-P^2)^{-\varepsilon} - (-Q^2)^{-\varepsilon}\right]$$
$$+ B(s,t,P^2,Q^2)$$

- $$B(s,t,P^2,Q^2) = \mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at)$$

$$a = \frac{u}{P^2Q^2 - st} \qquad s+t+u = P^2+Q^2$$

- $$B'(s,t,P^2,Q^2) = \mathrm{Li}_2\left(1-\frac{P^2}{s}\right) + \mathrm{Li}_2\left(1-\frac{P^2}{t}\right) + \mathrm{Li}_2\left(1-\frac{Q^2}{s}\right) + \mathrm{Li}_2\left(1-\frac{Q^2}{t}\right) - \mathrm{Li}_2\left(1-\frac{P^2Q^2}{st}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right)$$

- IF $B = B'$, THEN

AGREEMENT WITH BDDK (1994)

- $B - B' \ni$ 9 DILOGARITHMS

$\Rightarrow$ MANTEL'S IDENTITY (1898)

$$
\begin{aligned}
Li_2\left(\frac{vw}{xy}\right) = {} & Li_2\left(\frac{v}{x}\right) + Li_2\left(\frac{w}{x}\right) + Li_2\left(\frac{v}{y}\right) + Li_2\left(\frac{w}{y}\right) + \\
& + Li_2(x) + Li_2(y) - Li_2(v) - Li_2(w) + \\
& + \frac{1}{2}\log^2\left(-\frac{x}{y}\right)
\end{aligned}
$$

$$x, y, v, w \in (0,1)$$

$$(1-v)(1-w) = (1-x)(1-y)$$

$x = as \qquad y = at$

$v = aP^2 \qquad w = aQ^2$

- $P^2 = 0,\ Q^2 \neq 0$ OR $P^2 \neq 0,\ Q^2 = 0$ (5-POINT CASE): HILL'S IDENTITY
- $P^2 = Q^2 = 0$ (4-POINT CASE): EULER'S IDENTITY

OUR RESULT :

- INCORPORATES LARGE NUMBERS OF FEYNMAN DIAGRAMS
- IS (SLIGHTLY) SIMPLER
  - NEW EXPRESSION FOR THE 2-MASS EASY BOX FUNCTION (CONTAINS 4 DILOGS)
- MHV DIAGRAMS WORK AT QUANTUM LEVEL

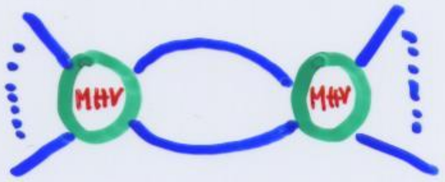
- WITH FEYNMAN DIAGRAMS :

$n = 7$: 227,585 DIAGRAMS

DAVID KOSOWER HAS ESTIMATED THAT ONE WOULD NEED

- ~ 3 VOLUMES OF Phys. Rev. D TO DRAW THEM, IN
- 22 MONTHS (FULL-TIME) AT 1 diagram / minute

- WITH MHV DIAGRAMS :

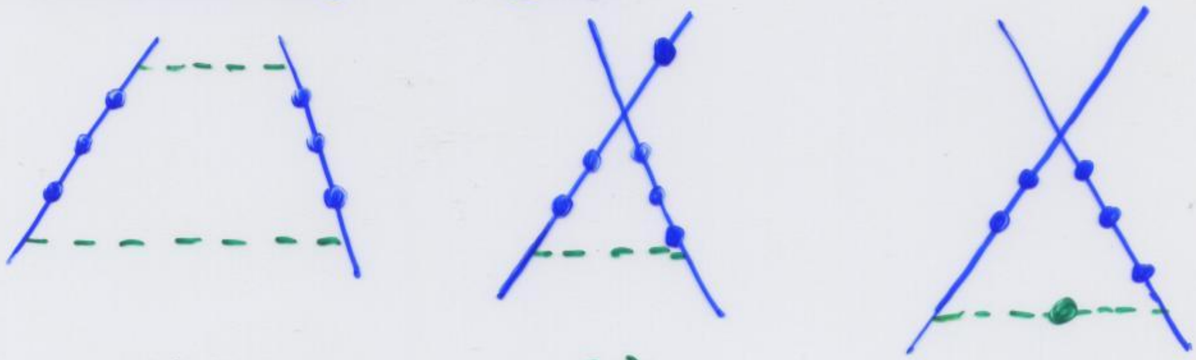
$\sum$ 

BASICALLY, 1 (SUPER) DIAGRAM

COMMENT ON TWISTOR SPACE PICTURE

- CACHAZO, SVRČEK and WITTEN HAVE STUDIED LOCALISATION PROPERTIES IN TWISTOR SPACE OF ONE-LOOP AMPLITUDES, USING DIFFERENTIAL OPERATORS IN SPINOR SPACE.

- CSW FIND THAT 1-LOOP AMPLITUDES LOCALISE ON



a) b) c)

- a) AND b) CONSISTENT WITH OUR RESULT

- c) GENERATED BY A **HOLOMORPHIC ANOMALY** [CACHAZO, SVRČEK, WITTEN]

- **VERY USEFUL !**

BEYOND N=4 SUPER YANG-MILLS ?

- 1-LOOP MHV AMPLITUDES IN N=1 SUPER YANG-MILLS

  BEDFORD, BRANDHUBER, SPENCE, GT
  QUIGLEY, ROZALI

  - RESULT AGREES WITH
    BERN, DIXON, DUNBAR, KOSOWER

- 1-LOOP MHV AMPLITUDES IN NON-SUPERSYMMETRIC YANG-MILLS

  BEDFORD, BRANDHUBER, SPENCE, GT

  - AGREES WITH 5-GLUON CASE
    BERN, DIXON, KOSOWER
  - ADJACENT NEGATIVE HELICITY GLUONS
    BERN, DIXON, DUNBAR, KOSOWER
  - NEW RESULTS FOR NEGATIVE HELICITY GLUONS IN ARBITRARY POSITIONS
  - RATIONAL TERMS

GENERALISED UNITARITY

BRITTO, CACHAZO, FENG

- ACTUALLY, AN OLD IDEA !

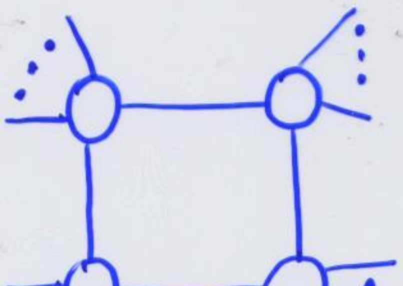
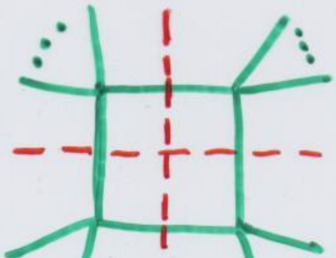
  EDEN, LANDSHOFF, OLIVE, POLKINGHORNE

- SUPERSYMMETRIC AMPLITUDES ARE 4-D CUT-CONSTRUCTIBLE

- AMPLITUDES IN N=4 SYM EXPRESSED ONLY IN TERMS OF BOX FUNCTIONS

$$A = \sum c_i$$ 

- USE QUADRUPLE CUTS :

 $= c_i$ 

— EACH CUT PICKS ONLY ONE COEFFICIENT

— LOOP INTEGRATION FROZEN

IN GENERIC SUPERSYMMETRIC THEORIES:

$A = \sum c_i$  $+ \sum d_i$  $+ \sum e_i$ 

- USE QUADRUPLE CUTS FIRST TO COMPUTE ALL BOX COEFFICIENTS
- THEN USE TRIPLE CUTS FOR THE TRIANGLES

$\sum c_j$  $+ \; d_i$ 

- MORE THAN ONE BOX SHARE THE SAME TRIPLE CUT

- NEW ONE-LOOP AMPLITUDES COMPUTED!
  - ALL NMHV AMPLITUDES IN $N=4$

    BERN, DIXON, KOSOWER
  - $1^- 2^- 3^- 4^+ \cdots n^+$ IN $N=1$

    BIDDER, BJERRUM-BOHR, DUNBAR, PERKINS

- NON-SUPERSYMMETRIC AMPLITUDES ARE CUT-CONSTRUCTIBLE IN $D=4-2\epsilon$

$$R \longrightarrow R(-s)^{-\epsilon} = R\left[1-\epsilon\log(-s)+\cdots\right]$$

- USE SUPERSYMMETRIC DECOMPOSITION

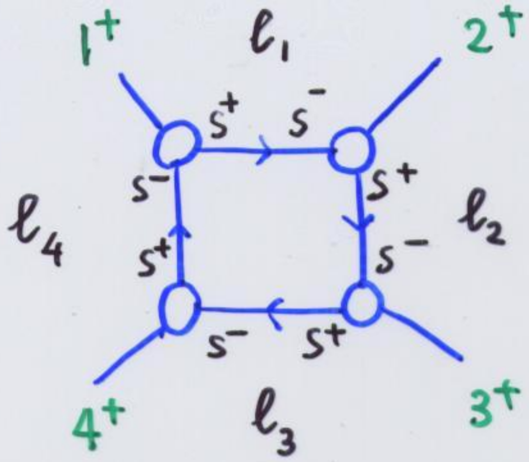
$$A_g = (A_g+4A_f+3A_s)-4(A_s+A_f)+A_s$$

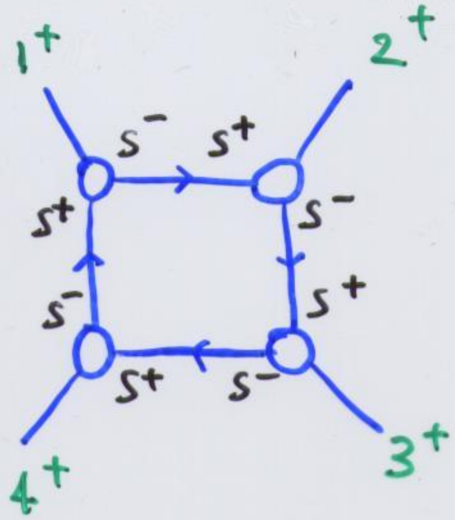
- COMPUTE ONE-LOOP AMPLITUDES WITH
  - SCALAR RONNING IN THE LOOP ($4-2\epsilon$ DIMENSIONS)
  - EXTERNAL PARTICLES IN 4D
- $$L^2_{4-2\epsilon} = \ell^2_4 + \ell^2_{-2\epsilon} = \ell^2_4 - \mu^2$$

EQUIVALENT TO HAVING A MASSIVE SCALAR IN 4D!

EXAMPLE : $1^+ 2^+ 3^+ 4^+$, GLUONS

BRANDHUBER, MCNAMARA, SPENCE, GT





$$A(\ell_{s^+}, k^+, \ell_{s^-}) = \frac{\langle q|\ell_{s^+}|k]}{\langle q k\rangle}$$

$$A^{1\text{-loop}}(1^+2^+3^+4^+) = \frac{[12]}{\langle 12\rangle}\frac{[34]}{\langle 34\rangle} K$$

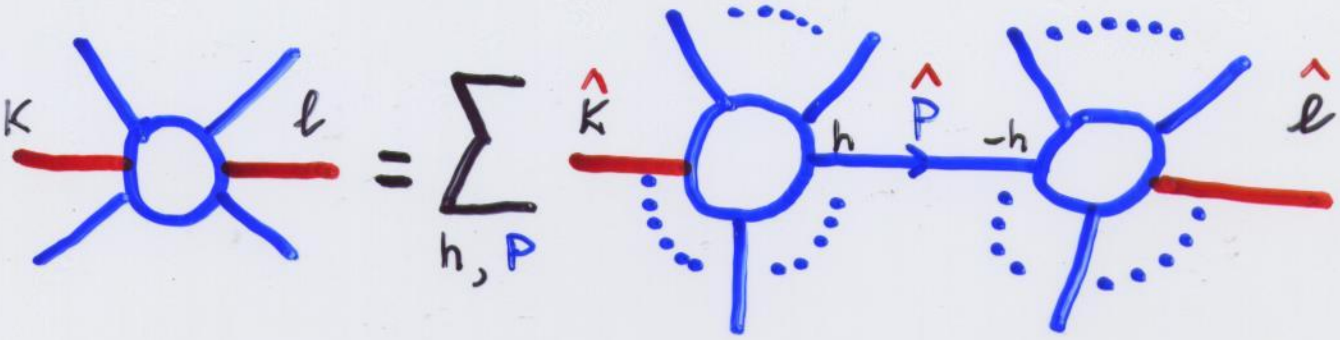
$$K = -i(4\pi)^{\frac{D}{2}} \int \frac{d^4\ell}{(2\pi)^4} \int \frac{d^{-2\varepsilon}\mu}{(2\pi)^{-2\varepsilon}} \frac{\mu^4}{(\ell_1^2-\mu^2)(\ell_2^2-\mu^2)(\ell_3^2-\mu^2)(\ell_4^2-\mu^2)}$$

$$K = -\varepsilon(1-\varepsilon)\, I_4^{D=8-2\varepsilon} \xrightarrow[\varepsilon\to 0]{} -\frac{1}{6}$$

$$A^{1\text{-loop}}(1^+2^+3^+4^+) \longrightarrow -\frac{i}{48\pi^2}\frac{[12]}{\langle 12\rangle}\frac{[34]}{\langle 34\rangle}$$

AGREES WITH BERN, KOSOWER

RECURSION RELATIONS FOR TREE-LEVEL AMPLITUDES



- SUM OVER ALL ALLOWED PARTITIONS OF PARTICLES WITH $K \in L$, $l \in R$
- AMPLITUDES EXPRESSED IN TERMS OF AMPLITUDES WITH FEWER LEGS
  - * ON SHELL
- BILINEAR STRUCTURE FROM FACTORISATION ON MULTIPARTICLE POLES
  - * GENERIC IN FIELD THEORY
  - * SCATTERING OF GLUONS, GRAVITONS...

THE IDEA

- SCATTERING AMPLITUDES AT TREE LEVEL HAVE ONLY SIMPLE POLES IN MULTIPARTICLE CHANNELS
- RESIDUES KNOWN FROM FACTORISATION

$$\Downarrow$$

RECONSTRUCT THE AMPLITUDE !

- DEFINE A ONE-PARAMETER FAMILY OF AMPLITUDES, $M(z)$ $z \in \mathbb{C}$
  - $M(0)$ IS THE AMPLITUDE WE WISH TO COMPUTE
  - SIMPLE POLES IN z
  - $M(z) \to 0$ AS $z \to \infty$

$$\boxed{M(z) = \sum_p \frac{z_p}{z - z_p}}$$

- VANISHING OF M AT LARGE z DEPENDS ON THE THEORY

  * YANG-MILLS: YES
  * GRAVITY: LIKELY (YES FOR MHV, NMHV)
  * ϕ^4: YES

NEXT, DEFINE $M(z)$

→ SINGLE OUT TWO EXTERNAL LEGS, K AND ℓ

- $p_K(z) = p_K + z\eta\ ,\ p_\ell(z) = p_\ell - z\eta$
  - OVERALL MOMENTUM CONSERVED
  - $\eta^2 = 0$

- ON-SHELL CONDITION $p_K^2(z) = p_\ell^2(z) = 0 \quad \forall z$

$$\eta = \begin{cases} \lambda_K \tilde{\lambda}_\ell & (1) \\ \lambda_\ell \tilde{\lambda}_K & (2) \end{cases} \quad \text{IN } \mathbb{C}M$$

FROM (1)

$$\boxed{\begin{array}{ll} \lambda_K(z) = \lambda_K & \lambda_\ell(z) = \lambda_\ell - z\lambda_K \\ \tilde{\lambda}_K(z) = \tilde{\lambda}_K + z\tilde{\lambda}_\ell & \tilde{\lambda}_\ell(z) = \tilde{\lambda}_\ell \end{array}}$$

- $M(z) = M(p_1, \ldots p_K(z) \ldots p_\ell(z) \ldots p_n)$

- IN A CHANNEL CONTAINING p_K (BUT NOT p_ℓ):

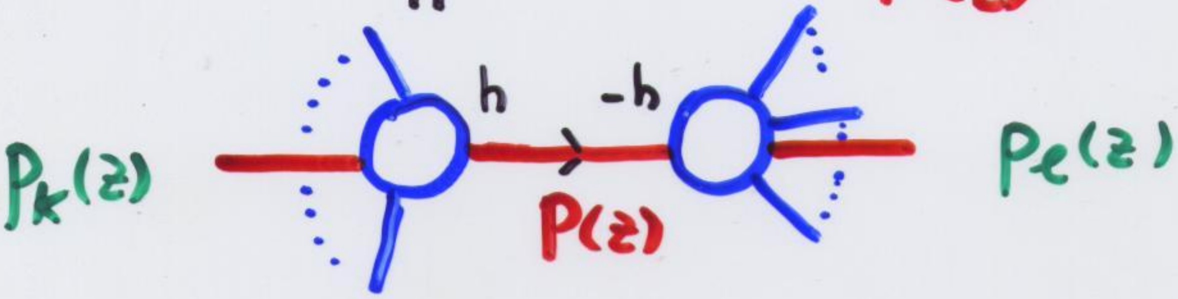
$$P(z) = P + z\eta$$

- $\dfrac{1}{P^2(z)} = -\dfrac{z_P}{P^2}\,\dfrac{1}{z - z_P}$ SIMPLE POLES IN z

- $z_P = -P^2 / \langle K | P | \ell]$

- FACTORISATION : AS $P^2(z) \to 0$

$$M(z) \longrightarrow \sum_h M_L^h(\ldots,\ldots,P(z)) \frac{1}{P^2(z)} M_R^{-h}(-P(z),\ldots\ldots)$$



- $$\mathcal{R}_p = \lim_{z \to z_p} (z - z_p)\, M(z) =$$
$$= \sum_h M_L^h(z_p) \frac{-z_p}{P^2} M_R^{-h}(z_p)$$

$$M(z) = \sum_{P,h} \frac{M_L^h(z_p)\, M_R^{-h}(z_p)}{P^2(z)}$$

$$\boxed{M(0) = \sum_{P,h} \frac{M_L^h(z_p)\, M_R^{-h}(z_p)}{P^2}}$$

COMMENTS

1. ON-SHELL AMPLITUDES

2. MOMENTUM IS CONSERVED AT EACH "VERTEX" (NOT IN CSW)

3. MULTIPLE APPLICATIONS OF THE RECURSION RELATIONS →

REQUIRES $\mathbb{C}$M

$$A^{YM}(1^-2^-3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \quad A^{YM}(1^+2^+3^-) = \frac{[12]^3}{[23][31]}$$

4. $$M^{GR}(1,2,3) = \left[A^{YM}(1,2,3)\right]^2$$

Ⓐ WHAT ARE THE ALLOWED P's ?

Ⓑ WHAT ABOUT $M(z)$, $z \to \infty$?

A. YANG-MILLS AMPLITUDES ARE

COLOUR-ORDERED

$$P \to P_{ij} = p_i + p_{i+1} + \cdots + p_{j-1} + p_j$$

CYCLICALLY ADJACENT

GENERAL RELATIVITY :

NO ORDERING $\Rightarrow$

RECURSION RELATION HAS

A SUM OVER ALL POSSIBLE

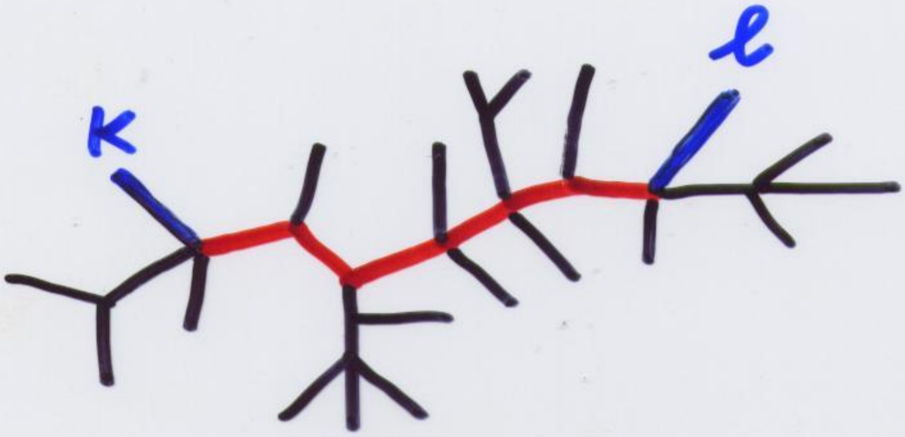
PARTITIONS SUCH THAT

$p_k \in P$, $p_\ell \notin P$ (k, ℓ ON OPPOSITE SIDES)

P : NOT NECESSARILY ADJACENT GRAVITONS

B. LARGE - Z BEHAVIOUR OF $M(z)$

FROM FEYNMAN DIAGRAMS



m PROPAGATORS

$m+1$ VERTICES

[$m = 6$ IN FIGURE]

YM : $V^{(3)} \sim P \quad , \; V^{(4)} \sim O(1)$

EH : $V^{(9)} \sim PP$

YM : $z^{-m+m+1} = z$

EH : $z^{-m+2(m+1)} = z^{m+2}$

INCLUDING EXTERNAL POLARISATION VECTORS / TENSORS

YANG - MILLS

- $\varepsilon^{(-)}_{a\dot{a}} = \dfrac{\lambda_a \tilde{\eta}_{\dot{a}}}{[\tilde{\lambda}\tilde{\eta}]}$ $\qquad$ $\varepsilon^{(+)}_{a\dot{a}} = \dfrac{\tilde{\lambda}_{\dot{a}} \eta_a}{\langle \lambda \eta \rangle}$

  $\eta_a, \tilde{\eta}_{\dot{a}}$ = REFERENCE SPINORS

- AT BEST, $\varepsilon_k \otimes \varepsilon_\ell \sim 1/z^2$

- $M^{YM}(z) \sim z\, z^{-2} = z^{-1} \xrightarrow[z\to\infty]{} 0$ (BCFW)

GRAVITY

$$\varepsilon^{(\mp)}_{\mu\nu} = \varepsilon^{(\mp)}_{\mu}\, \varepsilon^{(\mp)}_{\nu}$$

- AT BEST, $\varepsilon_k \otimes \varepsilon_\ell \sim 1/z^4$

- $M^{GR}(z) \sim z^{m+2}\, z^{-4} = z^{m-2}$

HOWEVER ...

- QUANTUM GRAVITY IS FULL OF SURPRISES...
  * BETTER UV BEHAVIOOR THAN EXPECTED
    (N = 8 SUGRA DIVERGES AT 6 LOOPS ...)

- USE KLT RELATIONS

Kawai
Lewellen
Tye

$$V^{\text{closed}} = V^{\text{open}}_{\text{left}} \, \bar{V}^{\text{open}}_{\text{right}}$$

IN THE FIELD THEORY LIMIT

$$* \; M(1,2,3) = [A^{YM}(1,2,3)]^2$$

$$* \; M(1,2,3,4) = s_{12} \, A^{YM}(1,2,3,4) \, A^{YM}(1,2,4,3)$$

- WE FIND $M(z) \to 0$ AS $z \to \infty$

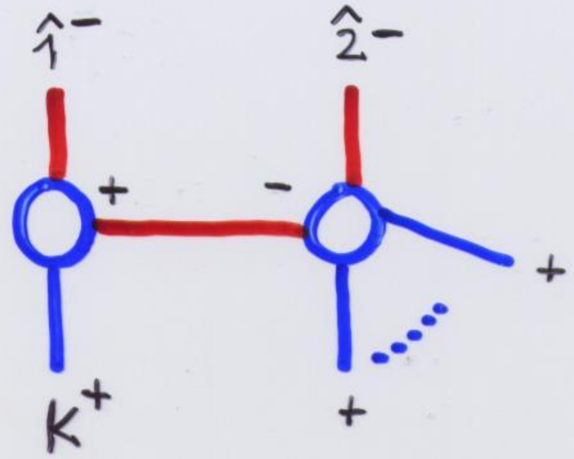
FOR MHV AND NMHV

(MHV: USE ALSO Berends, Giele, Kuijf)

APPLICATION : MHV , n

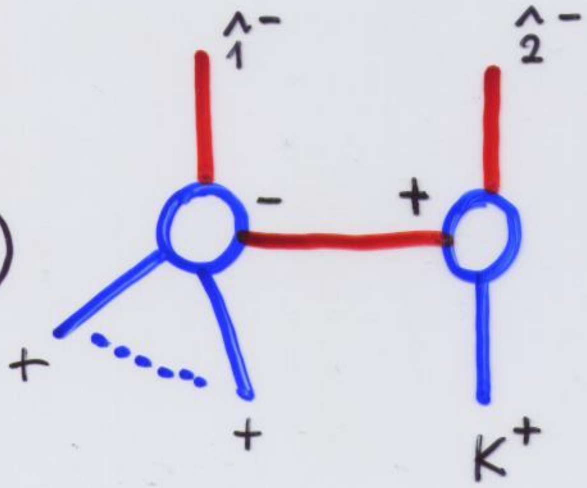
$M(1^-, 2^-, 3^+ \cdots n^+)$

(A)



$K = 3 \ldots n \Rightarrow n-2$ DIAGRAMS

(B)



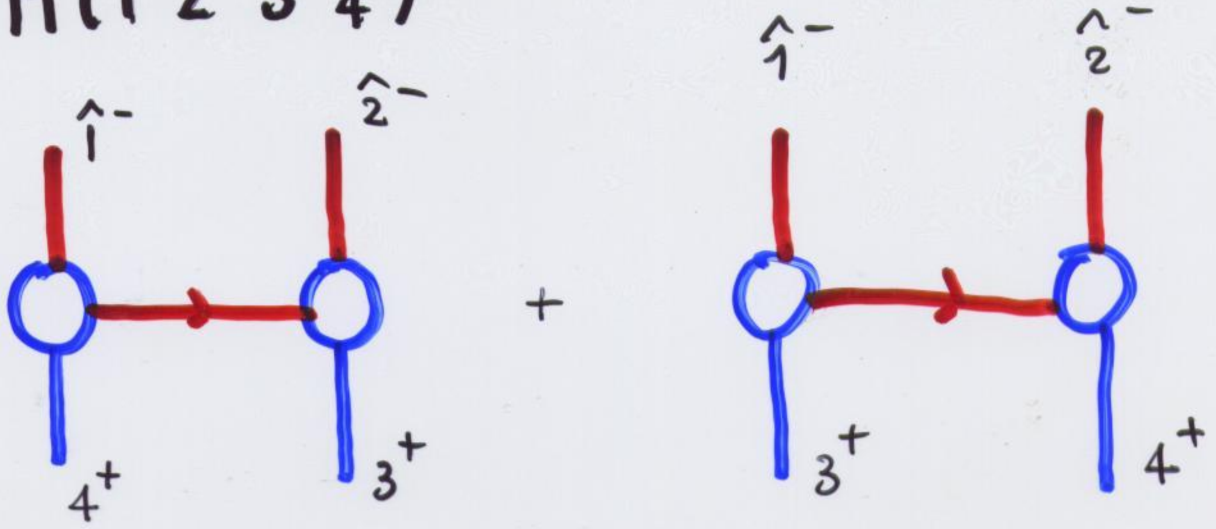
$K = 3 \ldots n \Rightarrow n-2$ DIAGRAMS

CHOOSING

$$\lambda_1(z) = \lambda_1 + z\lambda_2$$
$$\tilde{\lambda}_2(z) = \tilde{\lambda}_2 - z\tilde{\lambda}_1$$

ONE FINDS (B) $= 0$

$M(1^- 2^- 3^+ 4^+)$



$$M(1^- 2^- 3^+ 4^+) = \frac{\langle 12\rangle^6 \, [14]}{\langle 23\rangle^2 \langle 34\rangle^2 \langle 14\rangle} + 3 \leftrightarrow 4$$

* AGREES WITH KLT and BGK

- $$M(1^- 2^- 3^+ 4^+ 5^+) = \frac{\langle 12\rangle^6 [34]\,[15]}{\langle 23\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle\langle 51\rangle} + \, + P^c(3,4,5)$$

- SOLVE THE RECURSION RELATION FOR GENERAL n

* AGREES WITH BGK

SUMMARY

- EXCITING PROGRESS IN CALCULATING AMPLITUDES IN GAUGE THEORY AND IN GRAVITY
  - TWISTOR SPACE
  - MHV DIAGRAMS
  - RECURSION RELATIONS
  - GENERALISED UNITARITY

- FUNCTIONAL INTEGRAL
- GREEN'S FUNCTIONS
- ACTION
- OFF SHELL

- S-MATRIX ELEMENTS
- ON SHELL
- ANALYTICITY

THE ANALYTIC S-MATRIX

IS BACK !