Moments of spectral determinants of complex random matrices

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Spectral determinants: $|\det(zI-A)|^2 = \det(zI-A)(zI-A)^*$

Keating-Snaith conjecture: $\langle|\det(I-zU)|^s\rangle_{U(n)}$ predicts moments of $\zeta(1/2$

Averages of products and ratios of characteristic polynomials for ran
Hermitian and unitary matrices - extensively studied during last decade

Our motivation is different: distribution of complex eigenvalues.

Complex eigenvalues: why interesting?

Ginibre ensemble: $(A_{jk})_{1\le j,k\le n}$ are independent standard complex norma

have matrix distribution with density $c_n^{-1}\exp(-\operatorname{tr} AA^*)$

Ginibre's way: $A = SZS^{-1} \rightsquigarrow \operatorname{tr}(AA^*) = \operatorname{tr}(Z^*RZR^{-1})$, $R = S^*S$

Dyson's way: $A = U(Z+T)U^* \rightsquigarrow \operatorname{tr}(AA^*) = \operatorname{tr}(Z^*Z) + \operatorname{tr}(T^*T)$

j.p.d.f. $\tilde{c}_n^{-1}\prod_{j<k}|z_j-z_k|^2\exp(-\sum|z_j|^2)$; Circular Law, etc. (1

reliance on the specifics of gaussian weight, fail for other weights,
$\exp(-\operatorname{tr}[(AA^*)^2])$

Gaussian real matrices: Lehmann-Sommers (1991), Edelman (1994)

Why moments of $|\det(zI - A)|^2$ are relevant?

Poisson equation: $d\mu_n = \frac{1}{2\pi}\Delta p_n$;

$d\mu_n$ - eigenvalue counting measure

$p_n(z) = \int \log|z - w| d\mu_n(z)$ - log. potential

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ - the distributional Laplacian in Re z, Im z.

Under appropriate conditions, if $p_n \to p$ then $d\mu_n \Rightarrow \frac{1}{2\pi}\Delta p$.

Note: $p_n(z) = \frac{1}{2n}\log|\det(zI - A)|^2$,

$\langle|\det(zI - A)|^{2s}\rangle_A$ is a generating fnc for $\langle\frac{1}{2n}\log|\det(zI - A)|^2\rangle$

Unfortunately, our approach can handle integer moments only.

However,

$$\left\langle \frac{1}{\det[\varepsilon^2 I + (zI - A)(zI - A)^*]} \right\rangle_A,\ \varepsilon \to 0, \text{ indicator of domain of eigv.c}$$

and

for some RM ensembles, the mean eigv. density is $f(z)\langle|\det(zI - A)$

Mean eigenvalue density via matrix dimension reduction:

· Gaussian matrices A_n of dimension n (Edelman, Edelman-Kostlan-Sh

$$e^{-|z|^2} \left\langle |\det(zI_{n-1} - A_{n-1})|^2 \right\rangle_{A_{n-1}}$$ (complex matrices, com

$$ye^{-(x^2-y^2)} \operatorname{erfc}(y) \left\langle |\det(zI_{n-2} - A_{n-2})|^2 \right\rangle_{A_{n-2}}$$ (real matrices, complex

but mean density of real eigvs of real matrices is $e^{-x^2} \langle \det(xI_{n-1} - A_n$

· Rank 1 deviations from hermiticity: $A_n(\gamma) = \mathrm{GUE}_n + i\gamma\,\mathrm{diag}(1, 0, \ldots, ($

Eigv. density of $A_n(\gamma) = f_n(\gamma, z, z^*) \left\langle |\det(zI_{n-1} - A_{n-1})|^2 \right\rangle_{A_{n-1}(\gamma}$

· Rank 1 deviations from unitarity $A_n(\gamma)A_n(\gamma)^* = I_n - \gamma\,\mathrm{diag}(1, 0, \ldots, 0)$

Eigv. density of $A_n(\gamma) = g_n(\gamma, z, z^*) \left\langle |\det(zI_{n-1} - A_{n-1})|^2 \right\rangle_{A_{n-1}(\gamma}$

weakly non-Hermitian matrices (Fyodorov, K., Sommers)

Rank one deviations from unitarity: $A^*A = I -$ (rank one matrix)

$A_n(\gamma) = \sqrt{G_n}\,\mathcal{U}_n$, $n\times n$, $G_n = \mathrm{diag}(1-\gamma, 1, \dots, 1)$

$\mathcal{U}_n$ taken at random from $\mathcal{U}(n)$

$\gamma = 1$ corresponds to subunitary matrices (delete 1^{st} row & 1^{st} col

Mean density of eigenvalues of $A_n(\gamma)$ =

$$\frac{n-1}{n}\,\frac{1}{\pi|\gamma|}\left(\frac{1}{|z|^2}\right)^{n-1}\left(\frac{\gamma+|z|^2-1}{\gamma}\right)^{n-2}\int_{\mathcal{U}(n-1)}\left|\det\left(zI_{n-1}-\sqrt{\tilde{G}_{n-1}}\,\mathcal{U}_{n-1}\right)\right|^2 d\mathcal{U}_{n-1}$$

where $\tilde{G}_{n-1} = \mathrm{diag}(1-\tilde{\gamma}, 1, \dots, 1)$; $\tilde{\gamma} = \frac{\gamma+|z|^2-1}{|z|^2}$

and

$$1-\gamma \le |z|^2 \le 1 \qquad \text{if } \gamma > 0$$

$$1 \le |z|^2 \le 1-\gamma \qquad \text{if } \gamma < 0$$

For unitary invariant matrix distributions, like $e^{-\operatorname{tr} V(AA^*)}$ d

$$\langle F(A) \rangle_A = \langle F(A\mathcal{U}) \rangle_A \qquad \forall\, \mathcal{U} \in \mathcal{U}(n)$$

$$= \Big\langle \int_{\mathcal{U}(n)} d\mathcal{U}\; F(A\mathcal{U}) \Big\rangle_A$$

Polar coordinates analogy:

$$M = (MM^*)^{1/2}\, \mathcal{U}$$

angular component

radial component

Integrate over "angular variables" first

Our setup:

Consider matrices $A\mathcal{U}$, A fixed, $\mathcal{U}$ is Haar u

finite-n results.

$$\int_{\mathcal{U}(n)} \det[(I_n - AU)(I_n - BU)^*]\, dU = (n+1) \int_0^\infty \frac{\det(I_n + t AB^*)}{(1+t)^{n+2}}\, dt$$

$$\int_{\mathcal{U}(n)} \frac{dU}{\det[(I_n - AU)(I_n - BU)^*]} = (n-1) \int_0^1 \frac{(1-t)^{n-2}\, dt}{\det(I_n - tAB^*)} \qquad \text{for } n \geq 2$$

$$\int_{\mathcal{U}(n)} |\det(z - AU)|^2\, dU = \int_0^\infty \frac{\det(I_n + tAA^*)}{(1+t)^{n+2}}\, dt$$

$$= \int_0^\infty \frac{e^{n \int \ln(|z|^2 + t\lambda)\, d\sigma_{AA^*}(\lambda)}}{(1+t)^{n+2}}\, d$$

$\boxed{m=1}$ $(n+1)\int_0^\infty \frac{\det(I \ldots}{(1+\ldots}$

$\boxed{m=-1}$ $(n-1)\int_0^1 \frac{(1-\ldots}{\det(\ldots}$

Main results

Let A and B be two $n \times n$ matrices.

Thm 1 *For any positive integer m:*

$$\int\limits_{U(n)} \det{}^m\left[(I_n - AU)(I_n - BU)^*\right] dU = Q \int\limits_0^\infty \cdots \int\limits_0^\infty \prod_{i<j}(t_i - t_j)^2 \prod_{j=1}^m \frac{\det(I_n \ldots}{(1 + \ldots}$$

The integral on the rhs is a Hankel determinant

$$m! \left| \int_0^{+\infty} \frac{\det(I_n + tAB^*)}{(1+t)^{n+2m}}\, t^{i+j}\, dt \right|_0^{m-1}$$

Thm 2 *If $AA^* < I_n$, $BB^* < I_n$ and $2m \leq n$ then*

$$\int\limits_{U(n)} \frac{dU}{\det{}^m\left[(I_n - AU)(I_n - BU)^*\right]} = \tilde{Q} \int\limits_0^1 \cdots \int\limits_0^1 \prod_{i<j}(t_i - t_j)^2 \prod_{j=1}^m \frac{(1 - t \ldots}{\det(I_n \ldots}$$

Similar formula holds if $AA^* > I_n$, $BB^* > I_n$.

Integrals on the rhs are RMT integrals (Jacobi) $m \times m$

Thm 3 *For $n \geq 2$,*

$$\int_{U(n)} \frac{dU}{\det[\varepsilon^2 I_n + (I_n - AU)(I_n - AU)^*]}$$

$$= \frac{(n-1)}{2\pi i} \int_0^1 (1-t)^{n-2} dt \int_{-\infty}^{+\infty} \frac{dy}{y} \frac{1}{\det\left[AA^* + (\varepsilon^2 - t) I_n - i\varepsilon\sqrt{t}\left(y + \frac{1}{y}\right)\right.}$$

If the eigenvalues a_j^2 of AA^* are distinct, in the limit $\varepsilon \to 0$, the rhs is

$$-\alpha(AA^*) \log \varepsilon^2 + \beta(AA^*) + O(\varepsilon)$$

where

$$\begin{aligned}
\alpha(AA^*) &= (n-1) \sum_{j=1}^{n} (1 - a_j^2)^{n-2}\, \Theta(1 - a_j^2) \prod_{k \neq j} \frac{1}{a_k^2 - a_j^2} \\
\beta(AA^*) &= (n-1) \sum_{j=1}^{n} (1 - a_j^2)^{n-2}\, \Psi(a_j^2) \prod_{k \neq j} \frac{1}{a_k^2 - a_j^2}
\end{aligned}$$

where Θ is the step fnc and

$$\Psi(a^2) = \begin{cases} \gamma_{n-2} + \ln a^2 - \ln(a^2 - 1) & \text{if } a^2 > 1, \\ -\gamma_{n-2} + \ln(1 - a^2) & \text{if } a^2 < 1 \\ \ln 2 & \text{if } a^2 = 1. \end{cases}$$

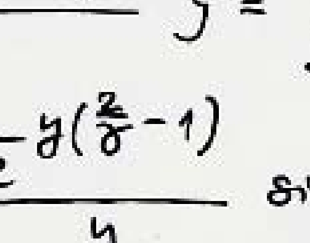
Rank one deviations from unitarity : $A^*A = \mathrm{diag}(1-\gamma, 1, \ldots, 1)$, $\gamma > 0$

$$\text{mean density of eig}^{\text{s}} = f_n(z,\bar z) \int_{\mathcal{U}(n-1)} |\det(z I_{n-1} - \sqrt{\tilde G}\, u)|^2 \, du$$

$$= f_n(z,\bar z) \int_0^\infty \frac{\det\left(|z|^2 I_{n-1} + t\tilde G\right)}{(1+t)^{n+2}} \quad \text{by Thm 1}$$

with $\tilde G = \mathrm{diag}(1-\tilde\gamma, 1, \ldots, 1)$

Interesting regime : $n \to \infty$, $1 - |z|^2 \sim 1/n$

eigenv. of A 

$$E\left\{ \frac{\#\{\text{eigenv. } z_j \text{ of } A : \ \frac{2\alpha}{n} \le 1 - |z_j|^2 \le \frac{2\beta}{n}\}}{n} \right\} = \int_\alpha^\beta \rho(y)\,dy + o(1)_{n\to\infty}$$

where $$\rho(y) = -\frac{d}{dy}\left[\frac{e^{-y\left(\frac{2}{\gamma}-1\right)}}{y} \sinh y \right]$$

Feinberg - Zee ring

Consider random A with inv. matrix distr. $dP(A, A^*) \propto e^{-n \operatorname{tr} V(AA^*)} dA$
$V(t)$ is a polynomial. If $V(t) = t$ have Ginibre ensemble.

Note: A has complex eigvs, $W = AA^*$ has real eigvs.

$$\left\langle |\det(zI_n - A)|^2 \right\rangle_A = \left\langle |\det(zI_n - AU)|^2 \right\rangle_A = (n+1) \int_0^{+\infty} \frac{\langle \det(|z|^2 + t}{(1+t)^{n}}$$

$$= \exp[n\Phi(z) + o(n)]$$

where Φ is given in terms of limiting distribution, $d\sigma(\lambda)$, of eigvs of W

$$\Phi(z) = \begin{cases} \log |z|^2 & \text{if } |z| > m_1 = \int \lambda d\sigma(\lambda), \\ \int_0^\infty \log \lambda d\sigma(\lambda) & \text{if } 1/|z| > m_{-1} = \int \frac{d\sigma(\lambda)}{\lambda}, \\ |z|^2 + \int_0^\infty \log \frac{\lambda + t_0}{|z|^2 + t_0} d\sigma(\lambda) & \text{if } 1/m_{-1} < |z| < m_1 \end{cases}$$

where t_0 is the (unique) solution of $\int_0^\infty \frac{d\sigma(\lambda)}{\lambda + t} = \frac{1}{|z|^2 + t}$.

Have agreement with Feinberg & Zee ($\Delta\Phi = 0$ outside the ring).

Free probability link: if U and $A \geq 0$ are free and U is Haar unitary
Brown's measure of UA as computed by Haagerup-Larsen is exactly Δ

Conclusion and Outlook

- stochastic Horn problem
  singular values $\rightsquigarrow$ eigenvalues
- success for integer moments
- reproduce (but not prove) eigenv. density

  conjecture $E \log \det = \log E \det$ & strong non

- fractional moments or averages of ratios of dets WANTED

  $$\text{eigv. density} = \lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial \bar{z}} \lim_{z \to \zeta} \frac{\partial}{\partial \zeta} \frac{\det[(zI-A)(zI-A)^{*} + \varepsilon^2}{\det[(\zeta I-A)(\zeta I-A)^{*} + \varepsilon}$$

- other classical groups