Weak Stability for an Inverse Sturm-Liouville Problem with Finite Data and Complex Potential

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otation

$x=0$ $\qquad\qquad\qquad\qquad$ $x=1$

$(0) = 0$ $\qquad$ $-y'' + q_0(x)\,y = \lambda y$

$y(1) = 0$

$\Rightarrow$ eigenvalues $\lambda_j(q_0)$

$y'(1) = 0$

$\Rightarrow$ eigenvalues $\mu_j(q_0)$

Question

Suppose we know $\lambda_j(q_0)$ and $\mu_j(q_0)$ with accuracy $\epsilon > 0$ for $j = 1, 2, \ldots, N$. How accurately can we recover q_0 ?

- Hochstadt (1973): one spectrum fully known, one partially known.

- Ryabushko (1983): both spectra fully known with finite accuracy:

$$\|q - q_0\|_{L^2(0,1)} \le \text{const.} \Big(\|\lambda_j(q) - \lambda_j(q_0)\|_{\ell^2} + \|\mu_j(q) - \mu_j(q_0)\|_{\ell^2} \Big).$$

- M^{c}Laughlin (1987): also establishes a local diffeomorphism between potentials in $L^2(0,1)$ and spectral data in $\ell^2(\mathbb{N})$.

- Pöschel & Trubowitz (1987): perturbation of finitely many eigenvalues.

- Rundell & Sacks (1992): reconstruction algorithm which uses Levitan's ideas, works with finite data, no error analysis.

- Barnes (1997).

- Hitrik (1999).

- Suppose $q, q_0 \in L^2((0,1),\mathbb{C})$ have the same mean value.

- Let $a_j = |\lambda_j(q) - \lambda_j(q_0)|$
  $b_j = |\mu_j(q) - \mu_j(q_0)|$ $j = 1, 2, 3, \ldots$

- Let $\epsilon_0 > 0$ and $N_0 \in \mathbb{N}$ be fixed.

- Then there exists a constant C depending only on ϵ_0, N_0 and q_0 such that

  if $\max(a_1, \ldots, a_N, b_1, \ldots, b_N) < \epsilon$

  then

$$\sup_{x \in (0,1)} \left| \int_0^x (q - q_0) \right| \le C \exp(\|q\|_{L^2}) \left(\epsilon \log N + \frac{\|a\| + \|b\|}{\sqrt{N}} \right)$$

$\|q - q_0\|_{L^2}$

where $\|a\|$ and $\|b\|$ mean the $\ell^2(\mathbb{N})$-norms of these sequences.

$$\left\| \left(\frac{a_j}{j} \right) \right\|_1 + \left\| \left(\frac{b_j}{j} \right) \right\|_1$$

"Exact" Problem:

$$\begin{cases} -y'' + q_0(x)y = \lambda y \\ y(0)=0,\ y'(0)=1 \end{cases}$$

Solution:

$$s_0(x,\lambda);$$

$$s_0(1,\lambda_j(q_0)) = 0,$$

$$s_0'(1,\mu_j(q_0)) = 0.$$

"Comparison" Problem

$$\begin{cases} -y'' + q(x)y = \lambda y \\ y(0)=0,\ y'(0)=1 \end{cases}$$

Solution:

$$s(x,\lambda);$$

$$s(1,\lambda_j(q)) = 0,$$

$$s'(1,\mu_j(q)) = 0.$$

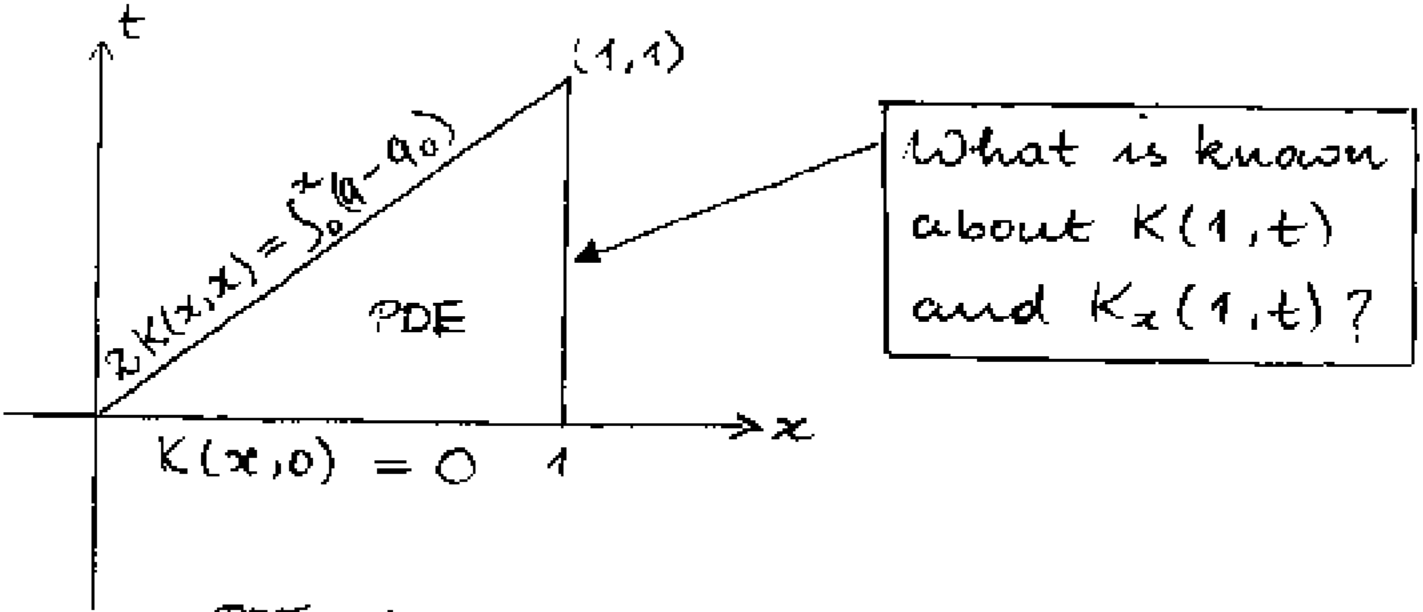
$$s(x,\lambda) = s_0(x,\lambda) + \int_0^x K(x,t)\,s_0(t,\lambda)\,dt$$

transformation operator

Relationship between $q - q_0$ and $K(\cdot,\cdot)$:

$$2K(x,x) = \int_0^x (q(t) - q_0(t))\,dt$$

Estimates on $q - q_0$ can therefore be found by obtaining estimates on K.



PDE solved by K:

$$(*) \quad K_{xx} - K_{tt} - (q(x) - q_0(t))K = 0.$$

Strategy of proof:

- obtain bounds for

$$f(t) := K(1,t)$$

$$g(t) := K_x(1,t);$$

•• Use bounds on f and g to get bounds on solution $K(\cdot,\cdot)$ of $(*)$, which is given by an appropriate series.

The solution is given by

$$K(x,t) = \sum_{n=0}^{\infty} K_n(x,t) \qquad (*)$$

where

$$\boxed{K_0(x,t) = \frac{1}{2}\int_{(x-t-1)}^{(x+t-1)} (f'(s)+g(s))\,ds}$$

and

$$K_n(x,t) = \frac{1}{2}\int_x^1 \int_{t+x-u}^{t-x+u} (q(u)-q_0(v)) K_{n-1}(u,v)\,dv\,du \qquad (**)$$

By induction one can prove that:

$$\boxed{|K_n(x,t)| \le \frac{\|K_0\|_\infty\, Q^n (1-x)^{3n/2}}{n!}}$$

where $Q^2 = \|q\|_2^2 + \|q_0\|_2^2$.

Therefore it is sufficient to obtain a bound on $\|K_0\|_\infty$.

Remark

(*), (**) do not define the usual Levitan series.

Recall that

$$s(x,\lambda) = s_0(x,\lambda) + \int_0^x K(x,t)\, s_0(t,\lambda)\, dt$$

Consequently

$$\boxed{s(1,\lambda) = s_0(1,\lambda) + \int_0^1 f(t)\, s_0(t,\lambda)\, dt} \quad (*)$$

and, by differentiation and $K(1,1) = 0$,

$$\boxed{s'(1,\lambda) = s_0'(1,\lambda) + \int_0^1 g(t)\, s_0(t,\lambda)\, dt} \quad (**)$$

We use these formulae to obtain bounds on the (generalized) Fourier coefficients of f and g.

Evaluate $(*)$ at $\lambda = \lambda_j(q)$ to obtain

$$0 = s_0(1,\lambda_j(q)) + \int_0^1 f(t)\, s_0(t,\lambda_j(q))\, dt$$

and at $\lambda = \lambda_j(q_0)$ to obtain

$$s(1,\lambda_j(q_0)) = \int_0^1 f(t)\, s_0(t,\lambda_j(q_0))\, dt =: \frac{\alpha_j}{j\pi}$$

ubtracting,

$$\frac{\alpha_j}{j\pi} = \overbrace{s_0(1, \lambda_j(q_0))}^{=0} - s_0(1, \lambda_j(q)) + \int_0^1 f(t)\big(s_0(t, \lambda_j(q_0)) - s_0(t, \lambda_j(q))\big)\,dt$$

hence

$$|\alpha_j| \leq j\pi\,(1 + \|f\|_1)\,\| s_0(\cdot, \lambda_j(q_0)) - s_0(\cdot, \lambda_j(q)) \|_\infty$$

$$\leq j\pi\,(1 + \|f\|_1)\,\underbrace{|\lambda_j(q_0) - \lambda_j(q)|}_{a_j} \cdot \underbrace{\sup_{\lambda \in [\lambda_j(q_0), \lambda_j(q)]} \left\| \frac{\partial s_0}{\partial \lambda}(\cdot, \lambda) \right\|_\infty}_{\leq \frac{\text{const}}{j^2}}$$

Thus

$$\boxed{|\alpha_j| = \left| j\pi \int_0^1 f(t)\, s_0(t, \lambda_j(q_0))\,dt \right| \leq C(1 + \|f\|_1)\,\frac{a_j}{j}}$$

are simple. Then

$$\{\psi_j\}_{j=1}^{\infty} := \{ j\pi \overline{S_0(\cdot, \lambda_j(q_0))} \}_{j=1}^{\infty}$$

is a <u>basis</u> of $L^2(0,1)$.

It can be shown to be quadratically close to orthonormal, and hence is a <u>Riesz</u> basis.

Hence

$$\|f\|_2^2 \leq C \sum_j |\overbrace{\langle f, \psi_j \rangle}^{\alpha_j}|^2 \leq \tilde{C} (1 + \|f\|_1)^2 \sum_j \frac{|a_j|^2}{j^2}$$

Also

$$\sum_j \frac{|a_j|^2}{j^2} \leq \frac{\pi^2}{6} \epsilon_0^2 + \frac{\|a\|^2}{N_0^2} \leq \frac{1}{4\tilde{C}}$$

$\forall$ suff. small ϵ_0 large N_0

These yield

$$\|f\|_1^2 \leq \|f\|_2^2 \leq 1/2 ,$$

and consequently

$$|\alpha_j| = |\langle f, \psi_j \rangle| \leq C (1 + \|f\|_1) \frac{|a_j|}{j}$$

$$\leq \frac{3C}{2} \frac{|a_j|}{j}$$

$$f(t) = \sum_{j=1}^{\infty} \alpha_j \varphi_j(t)$$

where

$$\varphi_j = \frac{\overline{\psi_j}}{\langle \psi_j, \overline{\psi_j} \rangle} \qquad \left(\psi_j(\cdot) = j\pi \overline{s_0(\cdot, \lambda_j(q_0))} \right)$$

- $\{\varphi_j\}$ is a Riesz basis [all $\lambda_j(q_0)$ simple];
- $\|\varphi_j\|_\infty \leq$ const [uniformly in j];

and so

$$\|f\|_\infty \leq \text{const.} \, \|(\alpha_j)\|_1$$

$$\leq \text{const.} \sum |a_j|/j$$

$$\leq \text{const.} \left[\epsilon \sum_{j=1}^{N} 1/j + \left(\sum |a_j|^2 \right)^{1/2} \left(\sum_{j=N+1}^{\infty} 1/j^2 \right)^{1/2} \right]$$

$$\Rightarrow \|f\|_\infty \leq \text{const.} \left[\epsilon \log N + \|a\|/\sqrt{N} \right]$$

[The bound on $\int g$ involves some additional technical complications.]

...

- The expression

$$s(t,\lambda) = s_0(t,\lambda) + \int_0^t K(t,x)\, s_0(x,\lambda)\, dx$$

must be differentiated with respect to λ to introduce the associated functions.

- Multiple eigenvalues split into clusters when $q_0 \rightsquigarrow q$ and it is necessary to use some result of Markusevich on polynomial interpolation in $\mathbb{C}$.

- The final result is the same.

These can be obtained immediately if one is prepared to make a-prior assumptions about smoothness of q_0 and q.

For example, if $q - q_0 \in H^n(0,1)$, $n > 0$, then for each $-1 \leq r \leq n$

$$\|q - q_0\|_{H^r} \leq C_r \left[\epsilon \log N + \frac{\|a\| + \|b\|}{\sqrt{N}} \right]^{(n-r)/(n+1)}.$$

This follows at once using standard interpolation inequalities, e.g.

$$\|f\|_{H^{(1-\theta)p_1 + \theta p_2}} \leq C \|f\|_{H^{p_1}}^{1-\theta} \|f\|_{H^{p_2}}^{\theta}$$

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