Some dynamical systems without ergodicity

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- When dynamics persistently fails to be ergodic
- Robust heteroclinic cycles and networks (with M Field)
- An example from convection (with O Podvigina)
- Product dynamics for heteroclinic attractors (with M Field)

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Convergence of averages

Flow $\phi_t(x)$ with initial condition x_0:

$$\mu_T(x_0) = \frac{1}{T} \int_{t=0}^{T} \delta_{\phi_t(x_0)}\, dt$$

determines long term statistical properties of ϕ_t.

Best possible case: for typical flows there is a finite set M of ergodic measures such that for almost all x_0

$$\mu_T(x_0) \to \tilde{\mu}$$

for some ergodic (natural) measure $\tilde{\mu} \in M$ that has nice properties.

However there are systems of physical interest where this is not the case; in particular

- Dynamics on phase spaces with invariant subspaces or other constraints
- Dynamics with symmetries

Nightmare: can get open sets of flows such that

$$\mu_T(x_0)$$

does not converge for an *open dense* set of x_0; those with robust heteroclinic-type attractors.

Robust heteroclinic attractors

heteroclinic connection $q(t)$ from equilbrium p_- to p_+ is a solution of the ODE with

$$q(t) \to p_\pm$$

as $t \to \pm\infty$.

heteroclinic cycle is a sequence of heteroclinic connections between equilibria such that one can return to any equilibrium via a sequence of connections.

Can find robust cycles where equilibria replaced by other transitive sets.

Robust heteroclinic cycles to equilibria

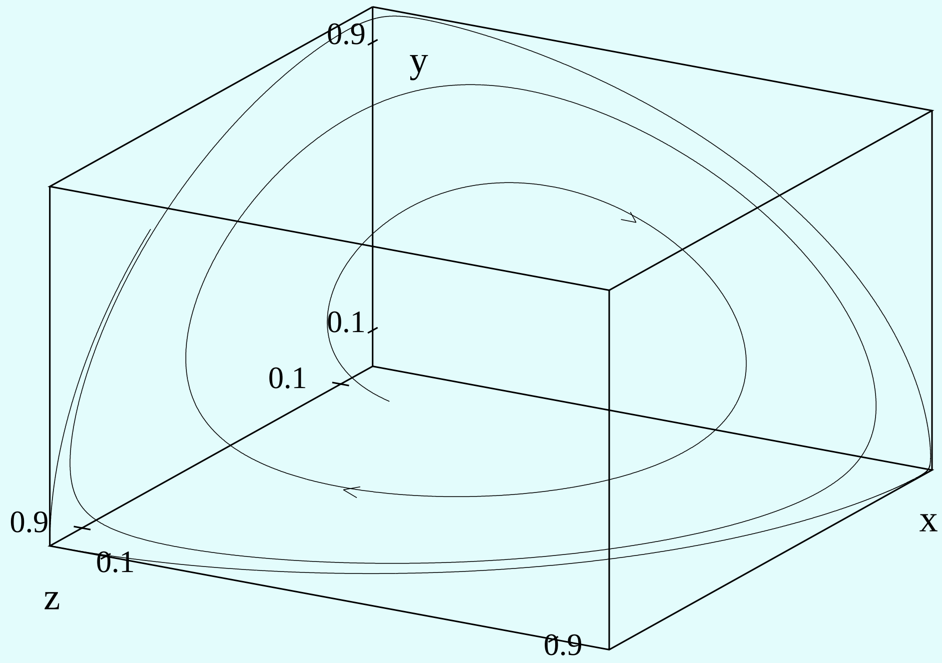
Simplest robust attracting cycle on $\mathbf{R}^3$ with symmetry Γ generated by reflections in coordinate planes and cycling the axes (x, y, z).

$$\begin{aligned} \dot{x} &= (\lambda + ax^2 + by^2 + cz^2)x \\ \dot{y} &= (\lambda + ay^2 + bz^2 + cx^2)y \\ \dot{z} &= (\lambda + az^2 + bx^2 + cy^2)z \end{aligned}$$

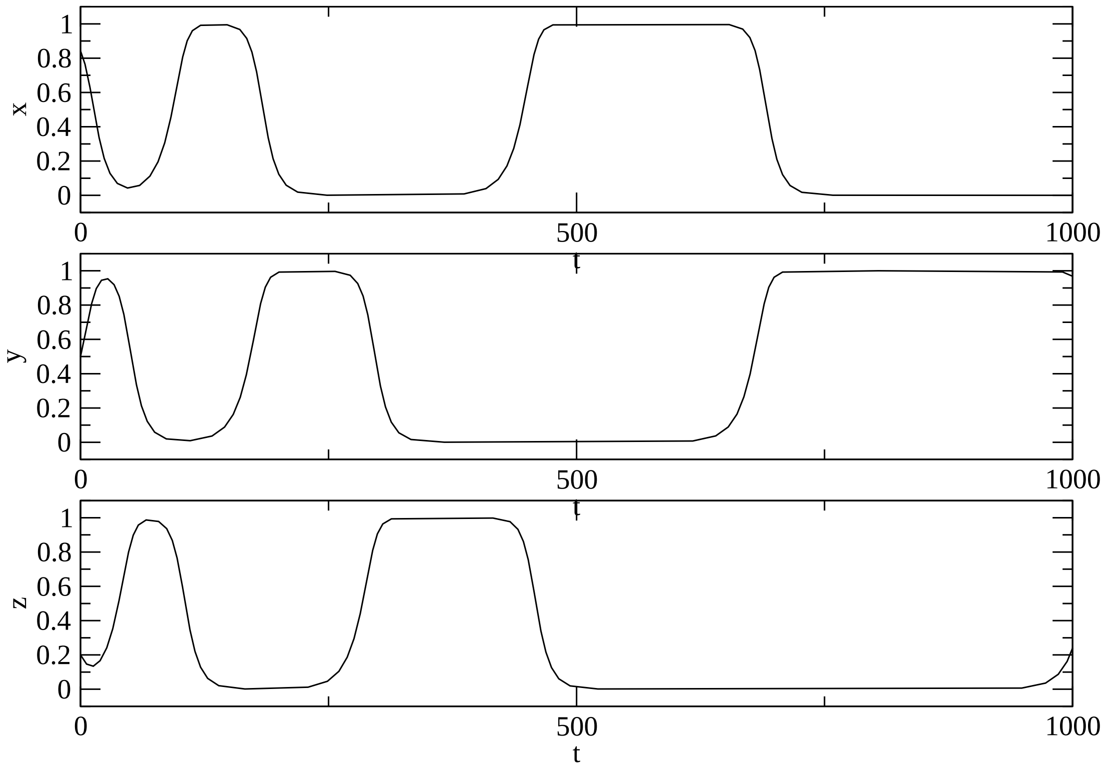
For open set of a, b, c, λ this has a heteroclinic cycle

- attracts an open dense set of initial conditions
- persists under any small enough perturbation in C_Γ.

In first octant $x \geq 0$, $y \geq 0$, $z \geq 0$ get equilibria that are attractors within the axes but have one-dimensional unstable manifolds.
$(a = -1, b = -0.98,\ c = -1.05)$



(Leonard, May, Busse, Guckenheimer, Holmes)



Timeseries of typical initial condition.

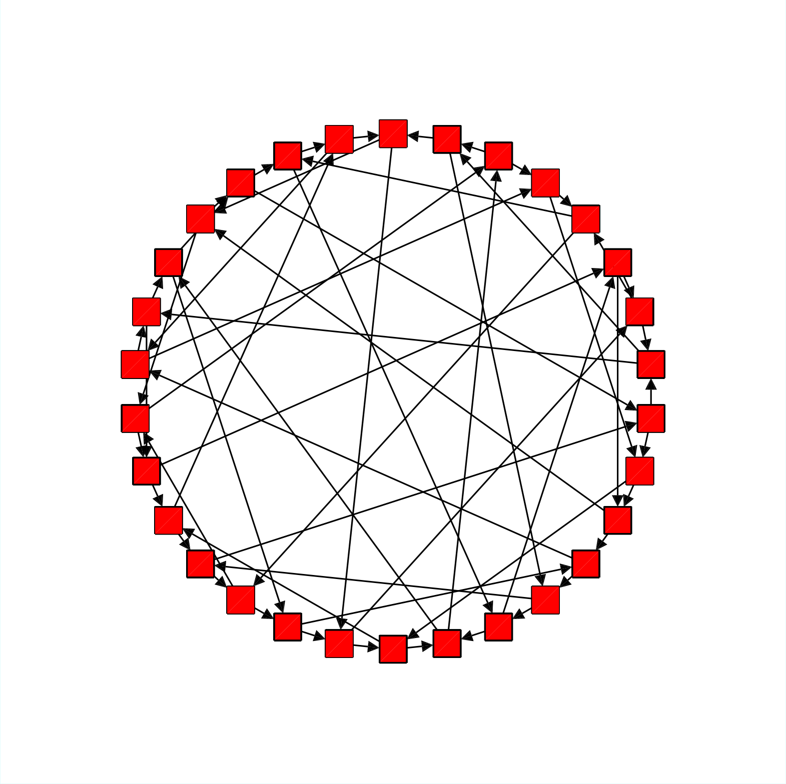
Example 1

System of globally coupled oscillators [Hansel et al, Kori & Kuramoto] with symmetry $\mathbf{S}_n$:

$$\dot{\theta}_i = \omega + \frac{1}{n}\sum_{j=1}^{n} g(\theta_i - \theta_j)$$

where $g(x) = -\sin(x+\alpha) + b\sin(2x+\beta)$.
Much richer dynamics than Kuramoto; ‘slow oscillations’ caused by noise-perturbed heteroclinic cycles.

Five oscillator case: for open set of parameters attractor as below where boxes represent synchronized clusters.



General structure of heteroclinic-like networks

For $x, y \in \Sigma$ and $\epsilon > 0$ there is an ϵ-pseudo orbit joining x to y if there is

$$\{x = x_0, y_0, x_1, \ldots, x_n, y_n = y\} \subset \Sigma$$

and $t_i \geq 1$, $0 \leq i < n$ s.t.

$$\begin{aligned} \rho(x_i, y_i) &< \epsilon, \\ x_{i+1} &= \phi_{t_i}(y_i). \end{aligned}$$

for $0 \leq i < n$.

Suppose given x, y and any ϵ there is an ϵ-p.o. from x to y then we say $x \to y$.

A set X is *chain recurrent* if $x \to y$ for any $x, y \in X$.

Structure of heteroclinic networks [A+Field]

Consider Σ a connected & compact invariant set for continuous flow

$$\phi_t : \Sigma \to \Sigma. \quad (1)$$

Interesting case is when Σ fails to have a dense orbit (not transitive) but is chain recurrent.

We say Σ is **indecomposable** if all points are nontrivially chain recurrent to themselves.

- If Σ is an ω-limit set then it is indecomposable.
- Chain recurrent + connected $\Rightarrow$ indecomposable.

For $x \in \Sigma$, let

$$\lambda(x) = \text{limits}_{t\to\infty}\{\phi_{\pm t}(x)\}$$

union of α and ω-limits. Say Σ is **recurrent** (or *transitive*) if there is an $x \in \Sigma$ such that $\lambda(x) = \Sigma$.

For any $S \subset \Sigma$ invariant, define

$$R(S) = \{x \in S \ : \ x \in \lambda(x)\}$$

the **set of recurrent points** (of S) and let

$$C(S) = S \setminus R(S)$$

the **set of connections** in S.
If $R(\Sigma)$ is a finite union of disjoint, compact, connected flow invariant subsets then say Σ has a **finite nodal set**.
For $S \subset \Sigma$ compact define

$$\lambda(S) \quad = \quad \overline{\cup_{x \in S} \lambda(x)},$$

Note that if X is invariant then

$$\lambda(X) \subset X$$

If X is recurrent then

$$\lambda(X) = X.$$

Suppose there is an N such that

$$\lambda^{N-1}(x) \neq \lambda^{N}(\Sigma) = R(\Sigma)$$

then say Σ has **depth** N.

Definition We say Σ is a **heteroclinic network** if

(a) Σ is indecomposable

(b) Σ has finite nodal set

(c) Σ has finite depth

Theorem [A, Field, 1999]
Let Σ be a heteroclinic network of depth N. For $N \geq n > 0$,

$$\Sigma_n := \lambda^n(\Sigma)$$

is a finite union of heteroclinic networks each with depth less than or equal to $N - n$.

Such networks may be **Asymptotically stable**.

There may be an open set of $x \in \mathbf{R}^n$ such that

$$\omega(x) = \Sigma.$$

However may have nontrivial ‘selection of connections’ resulting in

$$\omega(x) \subsetneq \Sigma$$

Consequences for ergodic averages Recall that

$$\mu_T(x_0) = \frac{1}{T}\int_{t=0}^{T} \delta_{\phi_t(x_0)}\, dt.$$

Easy: If x_0 has $\omega(x_0) \subset \Sigma$ where Σ is a heteroclinic network with recurrent set R then

$$L(x_0) = \text{limits}\{\mu_T(x_0)\ : T \to \infty\} \subset \text{Conv}(\mathcal{M}_{erg}(R)).$$

Hard: Which subset is this? (e.g. Sigmund, Hofbauer, Gaunersdorfer) What is the dynamics on Conv$(\mathcal{M}_{erg}(R))$?

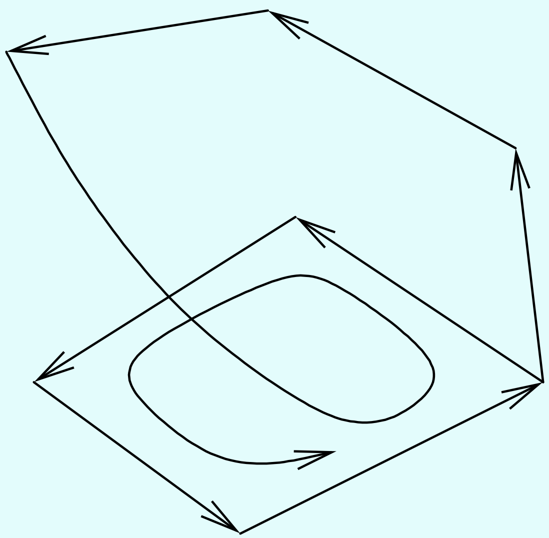
What I believe Generic smooth flows on finite dimensions have attractors composed of finite depth heteroclinic networks. On the networks there is a finite set M of ergodic measures such that

$$L(x_0) \subset \text{Conv}(M)$$

for almost every x_0.

A depth 2 example (Chawanya)

Flow given by replicator dynamics on 5-simplex. Has connection from equilibrium to 'child cycle'.



Not representable as transitive graph between equilibria! Infinite number of attracting p.o.s near cycle.

Example 2: a new robust depth 2 attractor from convection

(work with O. Podvigina)

Boussinesq convection problem in domain $(x, y, z) \in [0, L] \times [0, L] \times [0, 1]$

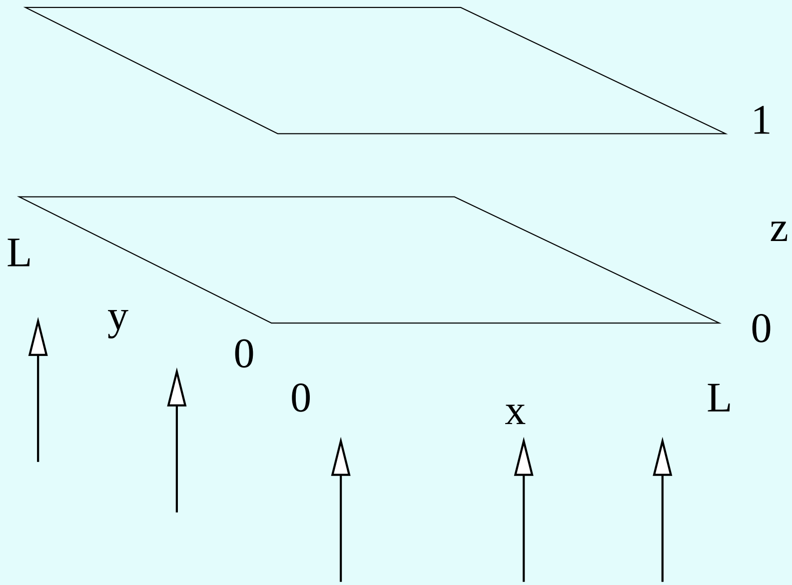
$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{v}) + P\Delta\mathbf{v} + PR\theta\mathbf{e}_z - \nabla p \tag{2}$$

with incompressibility

$$\nabla \cdot \mathbf{v} = 0 \tag{3}$$

and

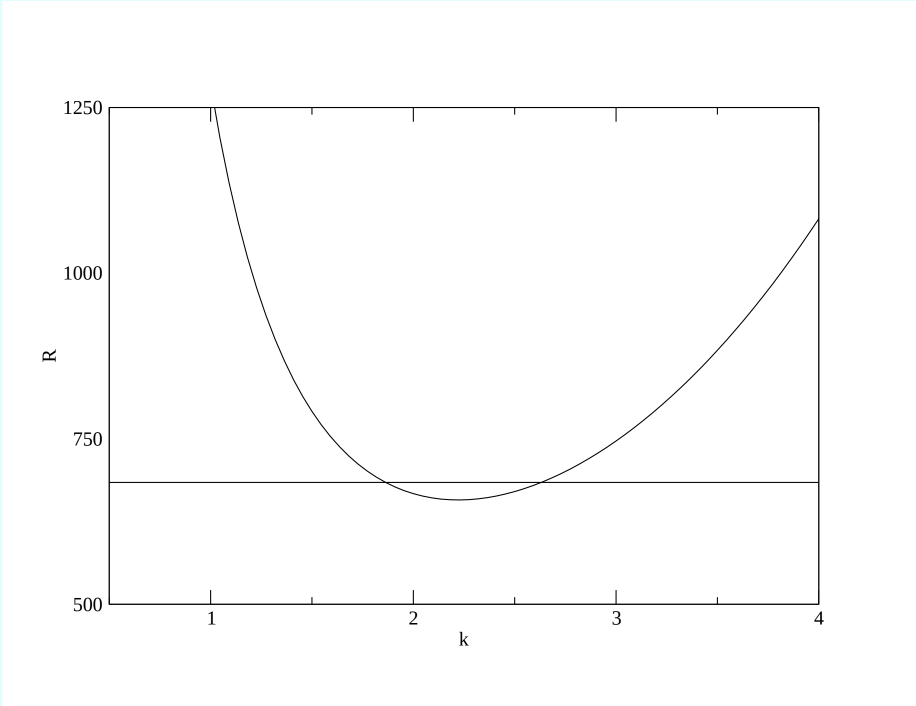
$$\frac{\partial \theta}{\partial t} = -(\mathbf{v} \cdot \nabla)\theta + \Delta\theta. \tag{4}$$



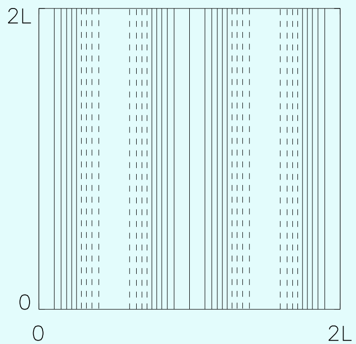
Boundary conditions: $v_{x;z} = v_{y;z} = v_z = 0$, $\theta = 0$ at $z = 0, 1$.
Periodic boundary conditions in x, y.

Stability

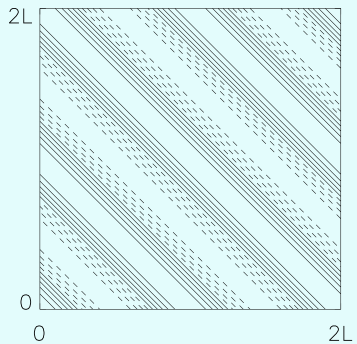
[Cox, Matthews, Proctor, Hirschberg, Knobloch] For fixed Prandtl number P there are two parameters to this problem: L and R. Trivial (conduction) state stable for $R < R_c = (k^2 + \pi^2)^3/k^2$ at which point it becomes unstable to convection rolls.



Examine stability with L that gives instability to rolls of wavelength L and to rolls of wavelength $L\sqrt{2}$; planforms of lines of equal v_z:



Large rolls (LR)



Small rolls (SR)

Unstable modes have form

$$z_1 e^{2\pi i x/L} + z_2 e^{2\pi i y/L} + z_3 e^{2\pi i (x+y)/L} + z_4 e^{2\pi i (x-y)/L} + \text{c.c.}$$

Normal form at bifurcation

(Truncated to cubic order)

$$\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + z_1(A_1|z_1|^2 + A_2|z_2|^2 + A_3(|z_3|^2 + |z_4|^2)) + A_4\bar{z}_1 z_3 z_4,\\
\dot{z}_2 &= \lambda_1 z_2 + z_2(A_1|z_2|^2 + A_2|z_1|^2 + A_3(|z_3|^2 + |z_4|^2)) + A_4\bar{z}_2 z_3 \bar{z}_4,\\
\dot{z}_3 &= \lambda_2 z_3 + z_3(A_5|z_3|^2 + A_6|z_4|^2 + A_7(|z_1|^2 + |z_2|^2)) + A_8(z_2^2 z_4 + z_1^2 \bar{z}_4),\\
\dot{z}_4 &= \lambda_2 z_4 + z_4(A_5|z_4|^2 + A_6|z_3|^2 + A_7(|z_1|^2 + |z_2|^2)) + A_8(\bar{z}_2{}^2 z_3 + z_1^2 \bar{z}_3),
\end{aligned} \tag{5}$$

z_i amplitudes of roll modes;
A_i parameters determined by centre manifold reduction;
λ_i correspond to perturbations to L and R.

Note presence of symmetries

$$\mathbb{T}^2 \times_s \mathbb{D}_4 \times \mathbb{Z}_2.$$

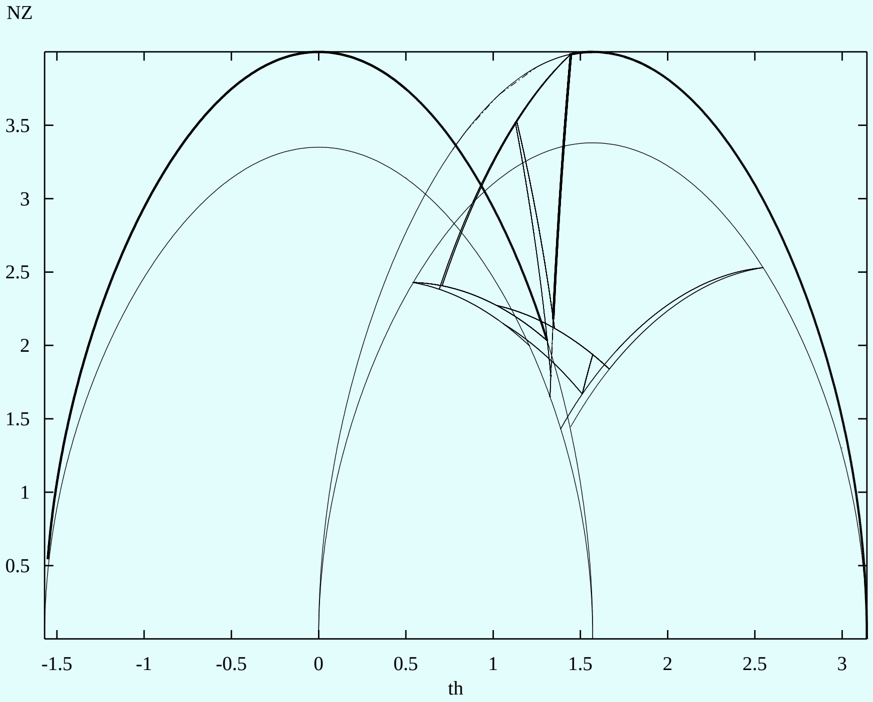
Physical symmetries of the domain

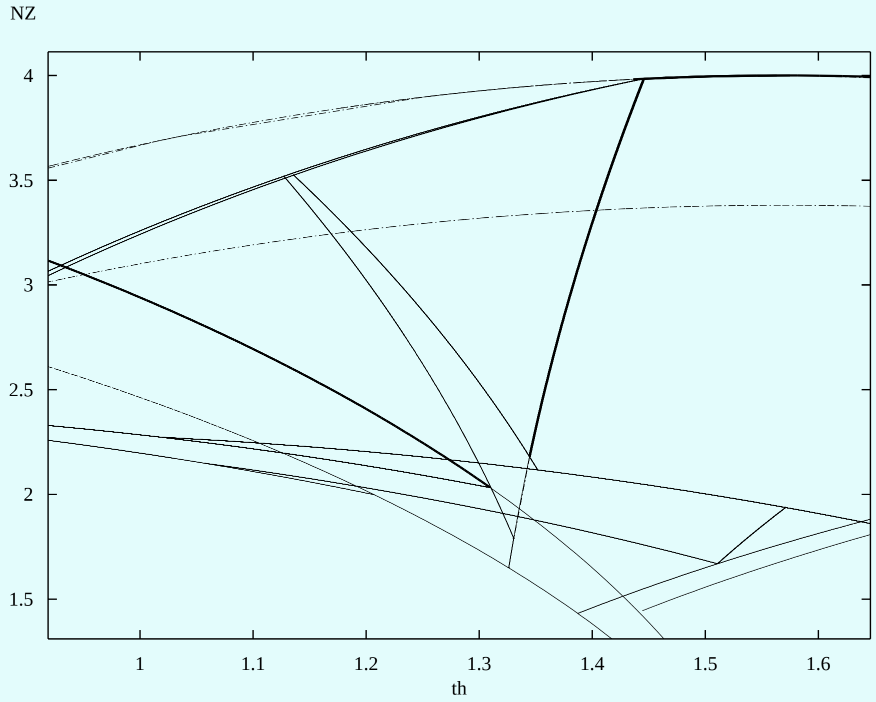
$$\mathbb{T}^2 \times_s \mathbb{D}_4$$

Boussinesq symmetry $\mathbb{Z}_2$ generated by $z \mapsto 1 - z$.
Forces the existence of more than 20 different types of symmetry-invariant subspace.

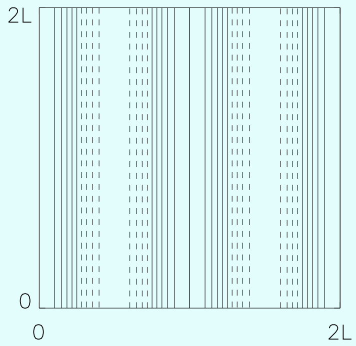
Bifurcation behaviour

Set $P = 1$ and take $\lambda_1 = \cos\theta$ and $\lambda_2 = \sin\theta$. ($NZ = \sqrt{\sum_k |z_k|^2}$)

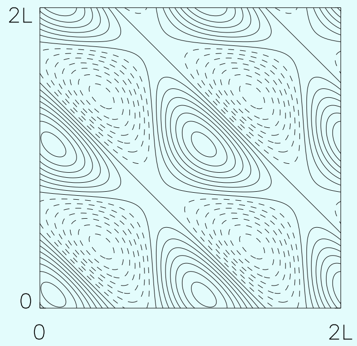




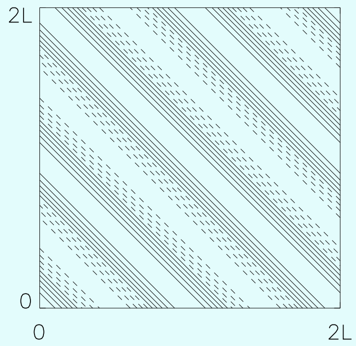
Bifurcation to stable solutions:



(LR) $(x, 0, 0, 0)$
for $\theta < 1.31$

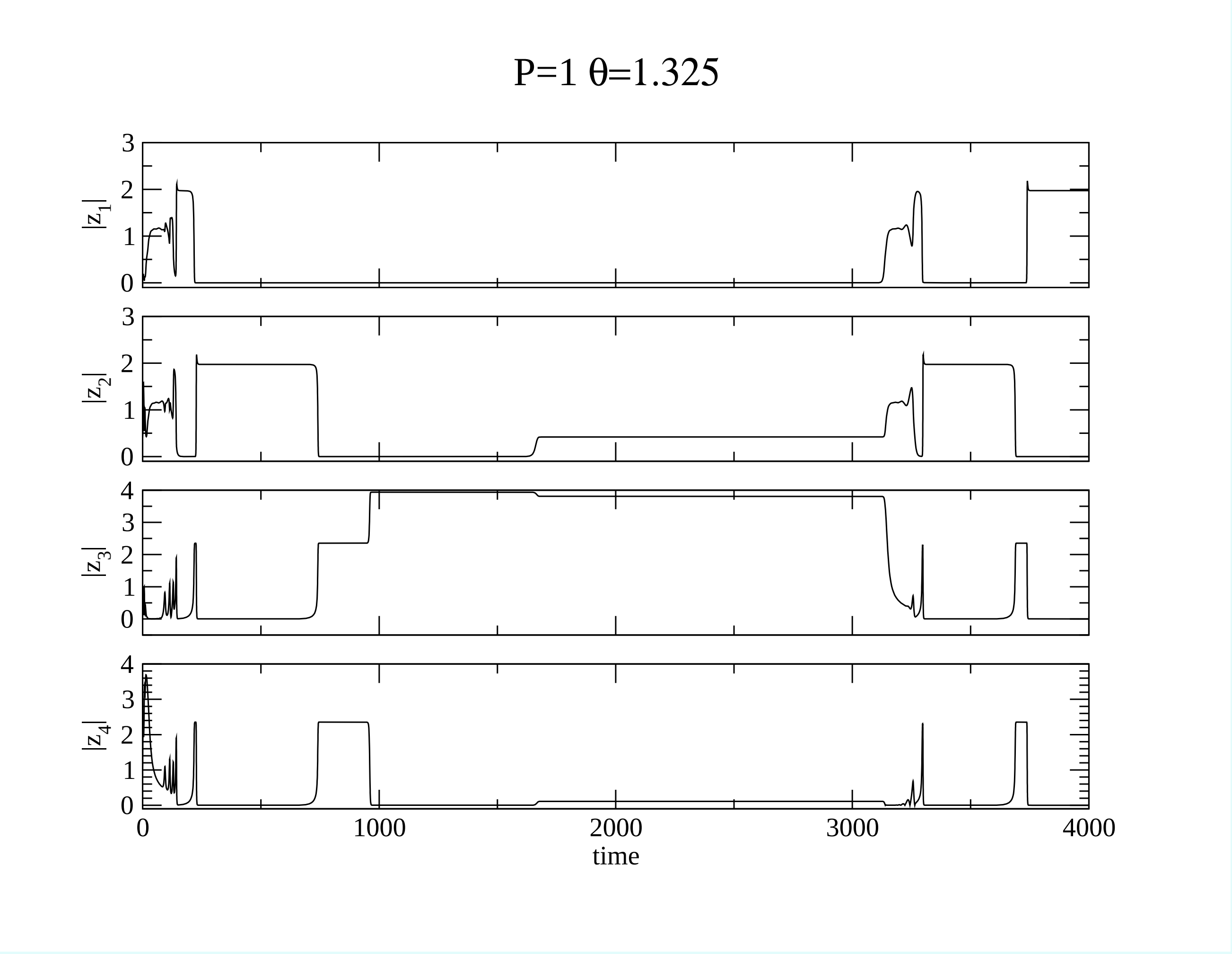


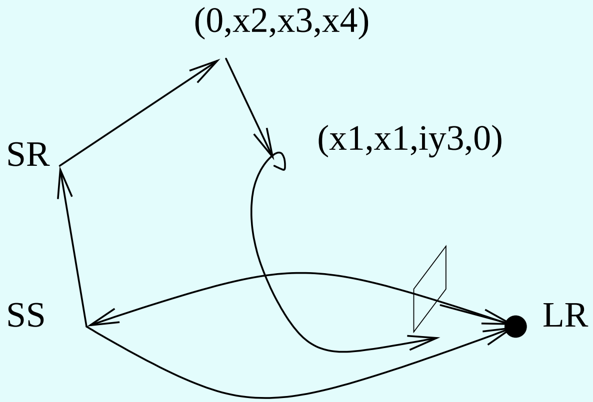
(WR3) $(x, x, iy, 0)$
for $1.344 < \theta < 1.446$



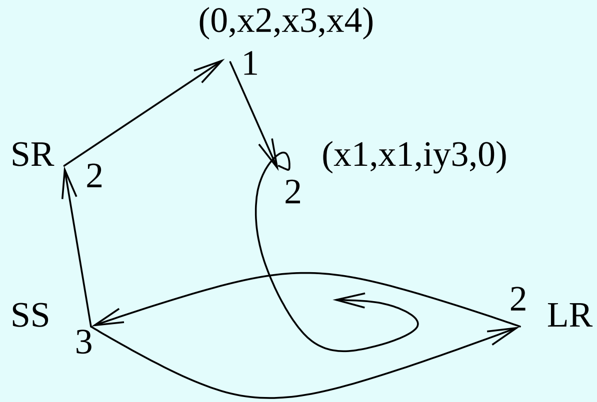
(SR) $(0, 0, x, 0)$
for $\theta > 1.446$

What about $\theta \in (1.31, 1.344)$?

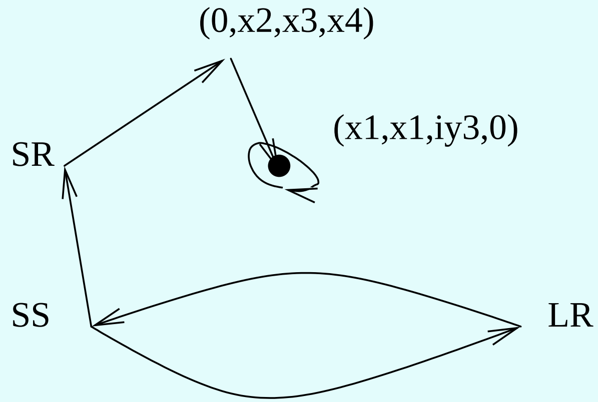




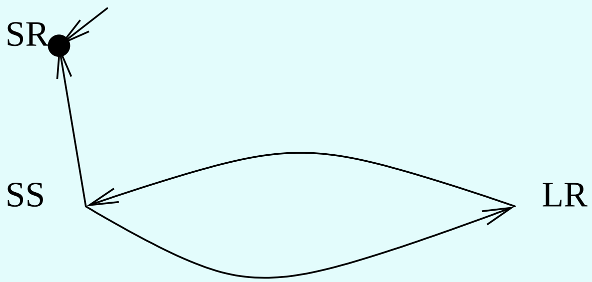
$\theta<1.31$



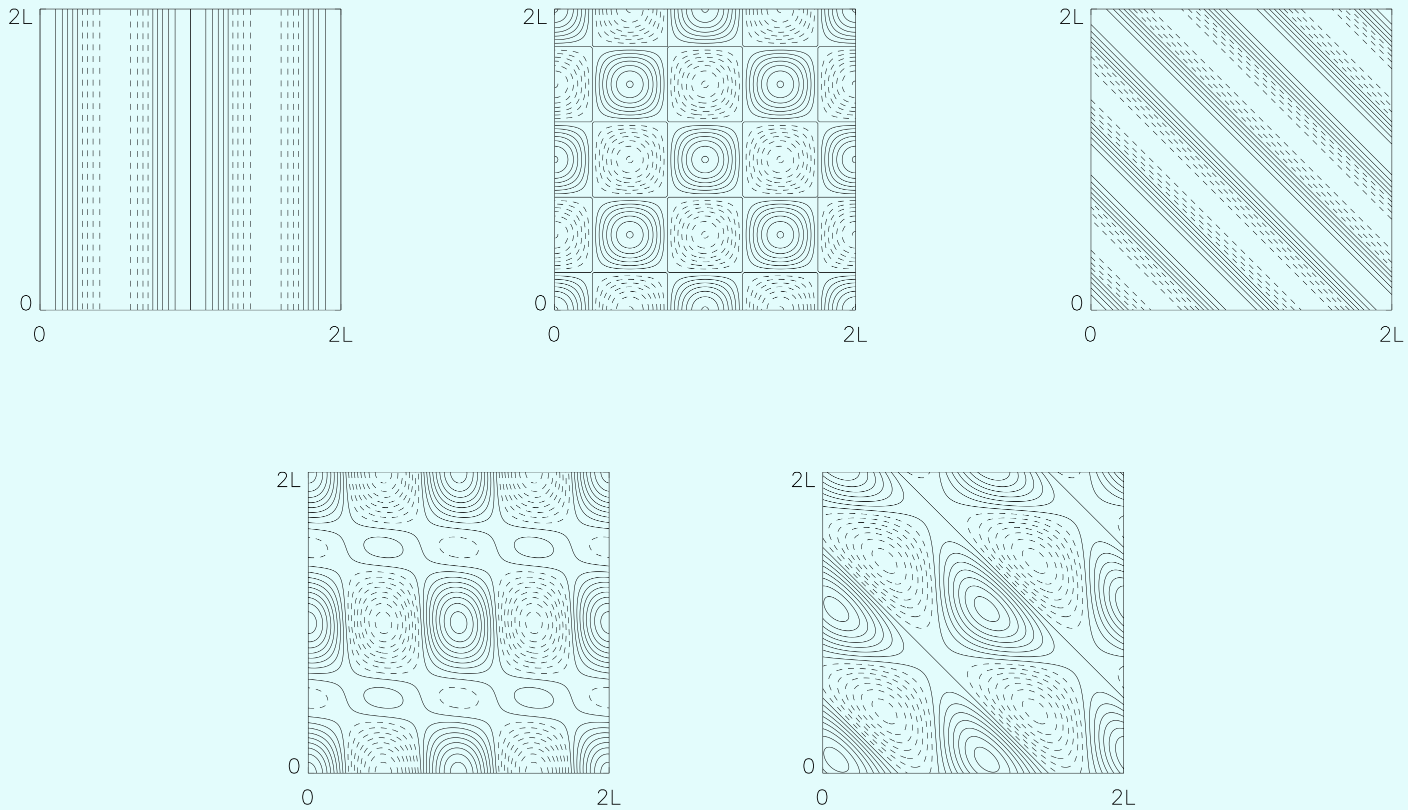
$1.31<\theta<1.344$



$1.344<\theta<1.446$



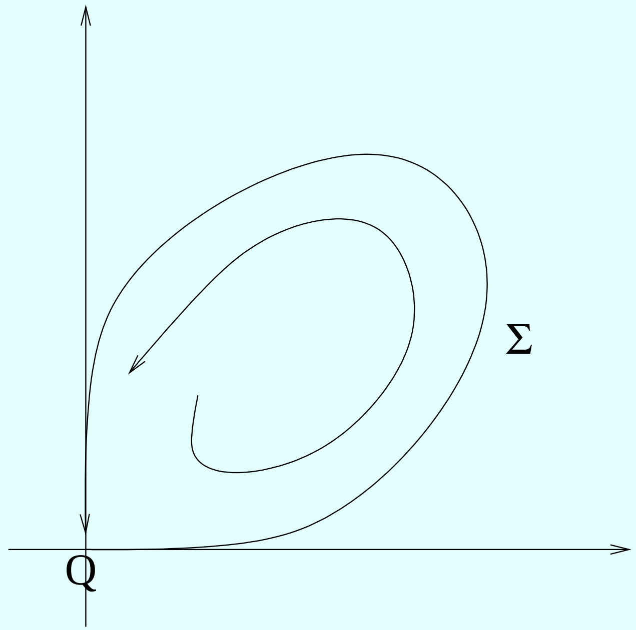
$1.446<\theta$


2L
0
0
2L

- Robust depth 2 heteroclinic network for this system.
- Several of the connections are multi-dimensional.
- Not completed: determination of largest chain recurrent set containing this network.
- Expect most trajectories to select the most unstable eigendirection.
- However multiple positive eigenvalues caused by symmetry!.

Product dynamics for heteroclinic attractors

Given two periodic attractors L_1 and L_2 of different flows, generically $L_1 \times L_2$ is (minimal) attractor for product system.
What about product of two heteroclinic attractors?



Can characterise attraction in terms of ‘geometric slowing down ratio’ $\lambda > 1$.

Consider a product of two planar ODEs with homoclinic attractors Σ_1, Σ_2 to equilibria Q_1, Q_2.

Theorem [A + Field 2005] The Milnor attractor for the product of two systems is typically NOT the product of the attractors, rather it is

$$(Q_1 \times \Sigma_2) \cup (\Sigma_1 \times Q_2).$$

This is because for typical $\lambda_1 > 1$, $\lambda_2 > 1$ and almost all a, b the sequence

$$\{a\lambda_1^n + b\lambda_2^m \;:\; (n,m) \in \mathbb{N}^2\}$$

typically has no accumulation points.

Some conclusions

Get robust non-ergodic behaviour in models for

- Fluid flows/magnetohydrodynamics
- Population/economic models
- Climate systems (D Cromellin)
- Coupled systems esp neuroscience (winnerless competition)

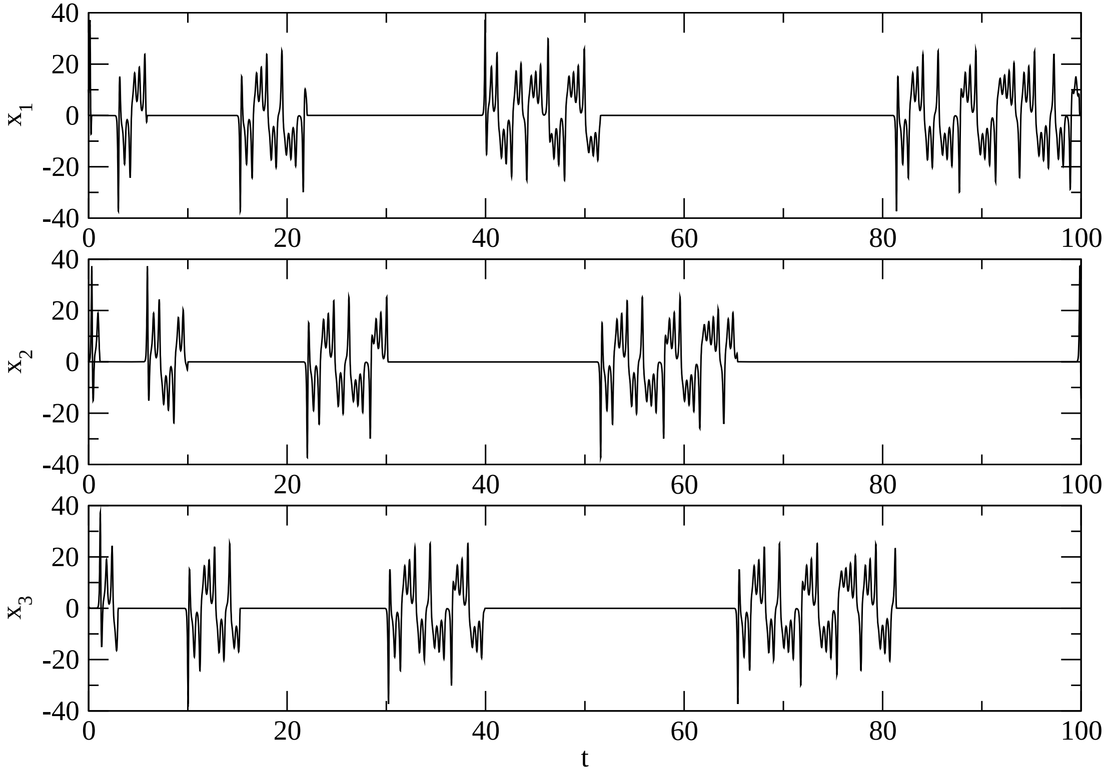
Singular behaviour on addition of noise; very poorly understood in general.

- Effect of noise on choice of trajectory?
- Effect of noise on choice of 'averaged' measure.

Heteroclinics between other invariant sets

Cycling chaos

System with same symmetry as Guckenheimer-Holmes example can get robust heteroclinic cycle between chaotic saddles. [Dellnitz & al]



Attracted trajectory switches between a number of chaotic sets for with ‘asymptotic slowing down’ of switching