Iterated Function Systems and Randomly Forced PDEs.

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A Simple Example

- First order system driven by a clocked random pulse sequence:

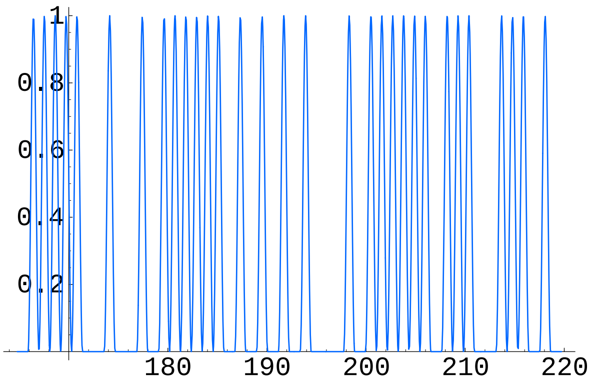
$$\frac{dv}{dt} = -v + \xi(t)$$

$$\xi(t) = \sum_p a_p g(t - p\tau)$$

  $\{a_p\} \in \{0, 1\}^{\mathbb{Z}^+}$ are the input symbols and g is supported on $(0, \tau)$.

- Symbols input at constant rate, τ^{-1}.

- In the examples, g is a raised cosine.

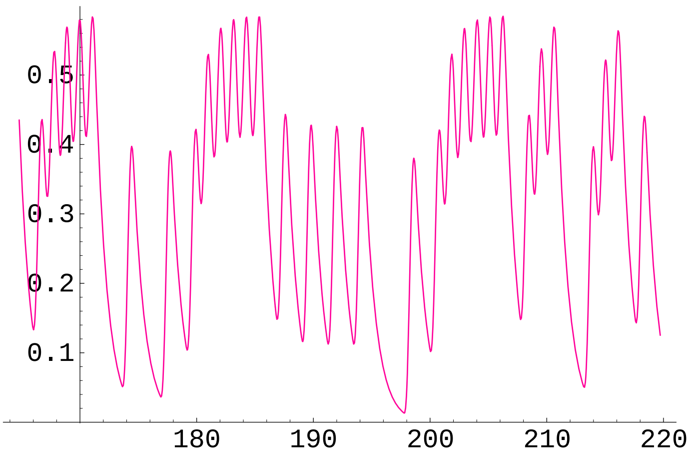


A typical segment of $\xi(t)$

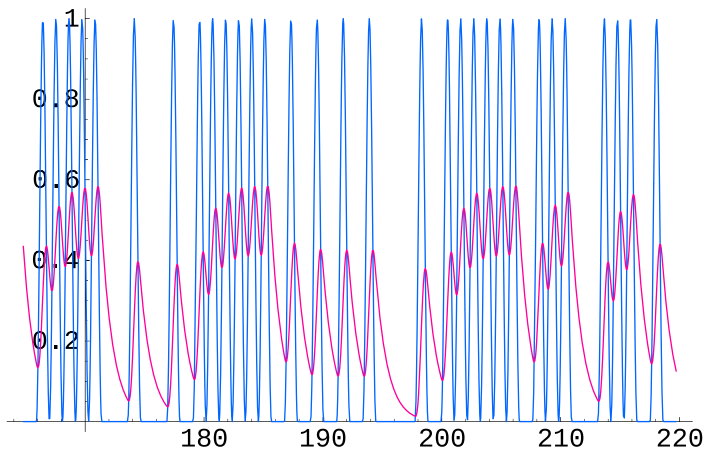
Randomly Forced First Order ODE (cont.)

- Can be solved using elementary undergraduates.

- The important parameter is τ .



A typical response $v(t)$, when $\tau = \log 3$.

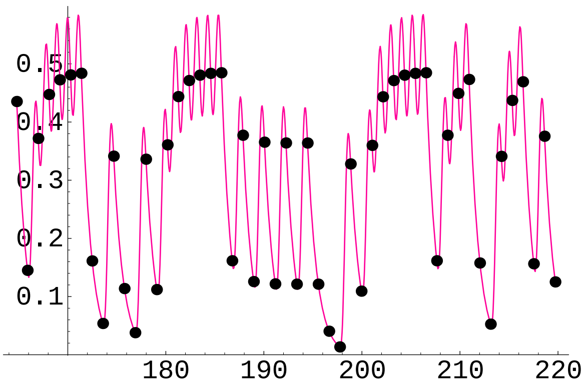


Input and output: comparing $\xi(t)$ and $v(t)$.

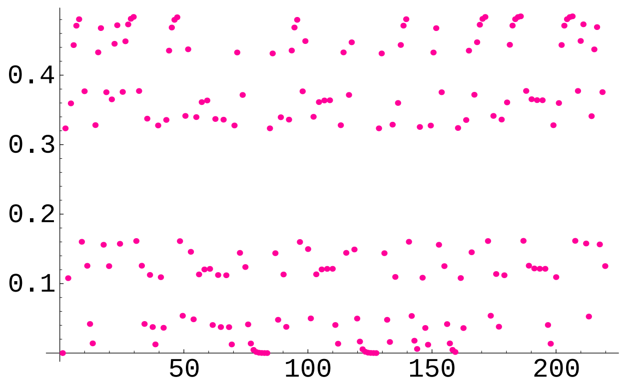
Sampling the Output

- When processing signals, it is usual to sample the output.

- A simple picture emerges if we sample at the symbol input rate.



The response $v(t)$ (curve) and samples $v(p\tau)$ (dots).

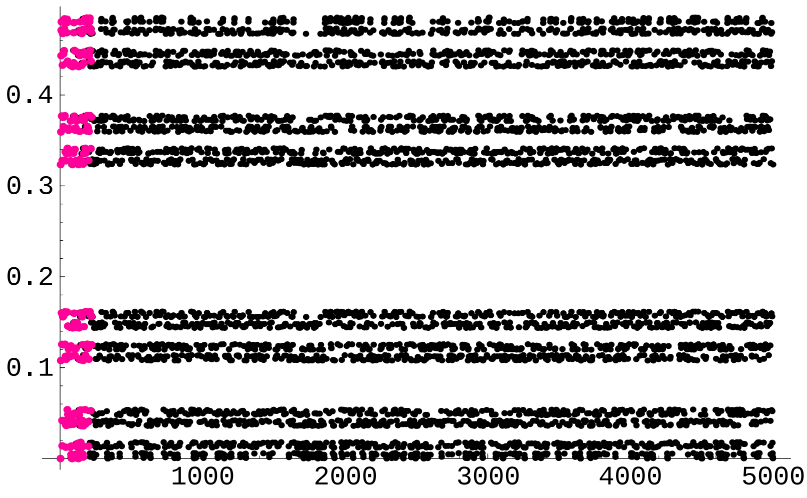


More samples—can you see a pattern?

Sampling the Output

- A pattern becomes clear with a longer time series.

- The samples $\{v(p\tau)\}$ seem to be distributed as a middle thirds Cantor set.



A longer time series—red shows data of previous figure.

From Another Point of View

- Integrate the ODE for one sample period, τ.

- Depending on the input symbol the output changes according to:

$$\begin{aligned} v &\mapsto \lambda v \\ v &\mapsto \lambda v + b \end{aligned}$$

  where $\lambda = e^{-\tau}$ and $b = e^{-\tau} \int_0^\tau e^t g(t) dt$

- The sampled output is given by random composition of these maps.

- A skew product system: base dynamics a full shift on $\{0, 1\}^{\mathbb{Z}^+}$; fibre dynamics given by ODE.

- An Iterated Function System (IFS)

Iterated Function Systems

- Let $\mathcal{C}$ be the set of nonempty compact subsets of a complete metric space, $(\mathbf{X}, \mathbf{d})$

- Equipped with the Hausdorff metric, $\mathcal{C}$ is a complete metric space

- Define the map $F : \mathcal{C} \to \mathcal{C}$

$$FU = \bigcup_{a=1}^{|\mathcal{A}|} f_a U$$

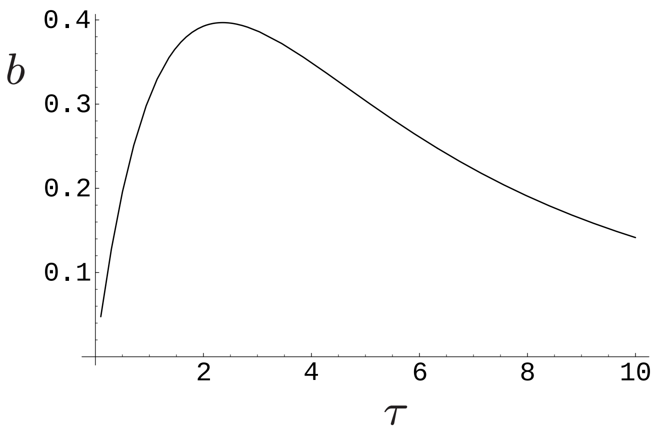
were $\mathcal{A}$ is a finite alphabet and the maps $\{f_a : a \in \mathcal{A}\}$ act on $\mathbf{X}$.

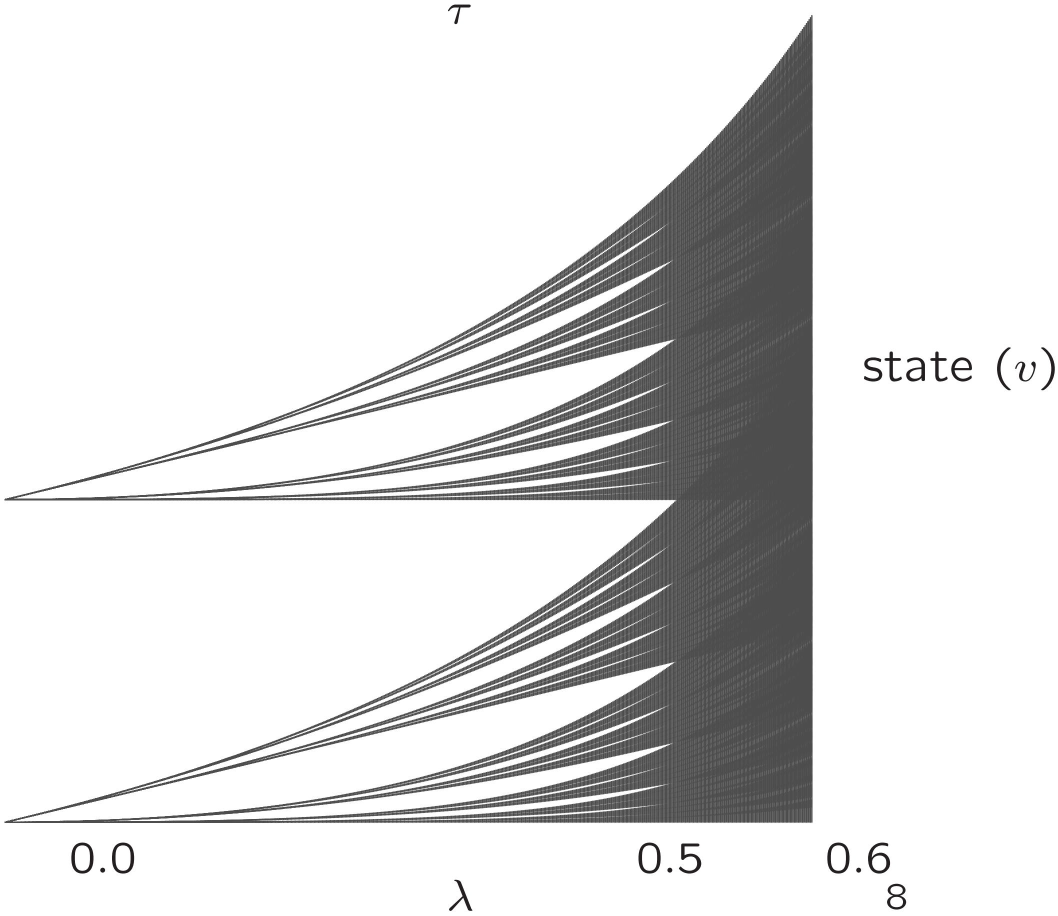
Theorem 1 *Given that the $\{f_a\}$ are contraction maps, $F : \mathcal{C} \to \mathcal{C}$ has a unique fixed point K, and for any $A \in \mathcal{C}$, the sequence $F^n A \to K$ in the Hausdorff metric.*

Parameter Dependence of Attractor

- Rescale $v \mapsto v/b$:

$$\{v \mapsto \lambda v, v \mapsto \lambda v + 1 : \lambda = e^{-\tau} \in [0, 1)\}$$





The Basic Model

- The cable equation:

$$RC\partial_t v = \partial_x^2 v - RAv + RI_e$$

  with R the resistance per unit length of the conductor, C and A the capacitance and conductance per unit length of the insulation and I_e is the input current.

- IFS model of digital channels.

- The cable equation is used to model (passive) sections of axon or dendrite: where the term Av is a linear approximation of the membrane current.

- Within each τ second interval, input one of a finite number of possible finite-duration pulse—assume no overlap of inputs.

Specifying the Model

- Finite cable $x \in \Omega = [0, l]$ with zero current boundary conditions:

$$\partial_x v(0, t) = 0 \text{ and } \partial_x v(l, t) = 0 \quad \forall t$$

- Rescale time $t \mapsto t/RC$ and introduce the dimensionless parameter $\rho = RA$.

- The cable equation (with $J_e(x, t) = RI_e(x, t)$):

$$\partial_t v = \partial_x^2 v - \rho v + J_e \qquad (1)$$

- The input $J_e(x, t)$ is supplied as a spatially-coded, pulse sequence

The IFS Consists of Contractions

- IFS consists of a finite set of maps $\{f_a : L^2(\Omega) \to L^2(\Omega), a \in \mathcal{A}\}$

- Given $v(x,0) \in L^2(\Omega)$ integrate for time τ with input $J_e^a(x,t),\ t \in (0,\tau)$ corresponding to a symbol $a \in \mathcal{A}$ to obtain $v(x,\tau) \in L^2(\Omega)$ and write

$$v(x,\tau) \stackrel{\text{def}}{=} f_a v$$

Lemma 2 *If $\rho > 0$, the $\{f_a : a \in \mathcal{A}\}$ are contractions in the $L^2(\Omega)$ norm.*

- For two arbitrary states v_1, v_2; by linearity of PDE:

$$\partial_t \|v_1 - v_2\|^2 = -2\rho\|v_1 - v_2\|^2 - 2\|\partial_x(v_1 - v_2)\|^2$$

  from which it follows that:

$$\|v_1(\tau) - v_2(\tau)\| = \|(f_a v_1 - f_a v_2)\| \le e^{-\rho\tau}\|(v_1 - v_2)\|$$

Existence of a Unique Attractor

- Let $\mathcal{B}$ be the set of nonempty, closed bounded subsets of $(\mathbf{X}, \mathbf{d})$ and extend F to act on $\mathcal{B}$

- Equipped with the Hausdorff metric, $\mathcal{B}$ is a complete metric space and $\mathcal{C} \subseteq \mathcal{B}$

Theorem 3 *Given hypotheses in Theorem 1, for any $A \in \mathcal{B}$, $F^n A \to K$ in the Hausdorff metric.*

- Proof by approximating $A \in \mathcal{B}$ by an r-net $P_r(A)$ and showing that

$$FP_r(A) = P_{r\lambda_0}(FA)$$

where $\lambda_0 = e^{-\rho\tau} < 1$ is the contractivity of F. Note that by Theorem 1,

$$F^n P_r(A) \to K$$

The Cable Equation Attractor

- So we have a compact attractor, K, for the forced cable equation.

- A simple corollary to Theorem 3 shows the attractor is finite-dimensional

Corollary 4 *The box-counting dimension—and hence the Hausdorff dimension—of K is bounded*

$$\dim_H(K) \leq \dim_B(K) \leq \frac{\log |\mathcal{A}|}{|\log \lambda_0|}$$

- This bound is linear in the symbol input rate:

$$\dim_H(K) \leq \dim_B(K) \leq \frac{\log |\mathcal{A}|}{\rho\tau}$$

Being More Explicit

- Write the input as:

$$J_e(x,t) = \frac{1}{2}J_0(t) + \sum_{k=1}^{\infty} J_k(t)\cos(\frac{\pi k x}{l})$$

- with spatio-temporal symbols:

$$J_k(t) = \sum_p J_k^{(a_p)} g(t - p\tau)$$

- Introduce a Fourier series solution:

$$v(x,t) = \frac{1}{2}v_0(t) + \sum_{k=1}^{\infty} v_k(t)\cos(\frac{\pi k x}{l})$$

- Evolution of Fourier coefficients:

$$\dot{v}_k(t) = -(\rho + (\pi k/l)^2)v_k(t) + J_k(t)$$

An Infinite-dimensional Affine IFS

- Gives an infinite-dimensional IFS:

$$\{v_k \mapsto \lambda_k v_k + b_k J_k^{(a)} : k \in \mathbb{Z}^+, a \in \mathcal{A}\}$$

- Where the contractions are given by:

$$\lambda_k = \exp[-\tau(\rho + (\pi k/l)^2)]$$

- and the offsets $b_k J_k^{(a)}$ contain the term:

$$b_k = e^{[-\tau(\rho+(\pi k/l)^2)]} \int_0^\tau e^{[t(\rho+(\pi k/l)^2)]} g(t)dt$$

- Later we truncate to get IFSs of affine maps $\tilde{f}_a : \mathbb{R}^n \to \mathbb{R}^n$

$$\tilde{f}_a x = Tx + \beta_a$$

- T is diagonal with elements $\{\lambda_k\}_{k=0}^{n-1}$
- Note that T is independent of the symbol being input.

Dimension of Self-affine Attractors

Theorem 5 (Falconer,Solomyak) *Let* $\{T_a : a \in \mathcal{A}\}$ *be linear contractions such that* $\max\{\|T_a\| : a \in \mathcal{A}\} < 1/2$ *and let* $\{\beta_a \in \mathbb{R}^n : a \in \mathcal{A}\}$ *be vectors. If* K *is an affine invariant set satisfying:*

$$K = \bigcup_{a=1}^{|\mathcal{A}|} (T_a(K) + \beta_a)$$

Then $\dim_H K = \dim_B K = d(T_1, T_2, \ldots T_{|\mathcal{A}|})$ *for almost all* $(\beta_1, \beta_2, \ldots \beta_{|\mathcal{A}|}) \in \mathbb{R}^{|\mathcal{A}|n}$ *in the sense of* $|\mathcal{A}|n$*-dimensional Lebesgue measure.*

- $d(T_1, T_2, \ldots T_{|\mathcal{A}|})$ is the singularity dimension (next slide).

Theorem 6 (Solomyak) *Let* $T_a = T \; \forall \, a \in \mathcal{A}$*. If the eigenvalues of* T *are such that all the images* $TK + \beta_a$ *of the attractor* K *are pairwise disjoint, then Falconer's formula holds for almost all* $(\beta_1, \beta_2, \ldots \beta_{|\mathcal{A}|}) \in \mathbb{R}^{|\mathcal{A}|n}$.

The Singularity Dimension

- Think of the following as a multiplicative interpolation:

$$\phi^s(T) = \sigma_1\sigma_2\ldots\sigma_{r-1}\sigma_r^{s-r+1}$$

  where the σ_i are the ordered singular values of T and $r \in \mathbb{Z}^+$ is such that $r-1 < s \le r$.

- ϕ^s is strictly decreasing and continuous.

- Now we sum over words of length q:

$$\Sigma_q^s = \sum_{a_1,\ldots a_q} \phi^s(T_{a_1} \circ T_{a_2} \circ \ldots T_{a_q})$$

- Submultiplicative: $\Sigma_{q_1+q_2}^s \le \Sigma_{q_1}^s \Sigma_{q_2}^s$.

- Therefore, $\Sigma_\infty^s = \lim_{q\to\infty}(\Sigma_q^s)^{\frac{1}{q}}$ exists.

- If $\Sigma_\infty^n \le 1$, there is a unique value of s—called d—such that $\Sigma_\infty^d = 1$.

The Cable Equation Attractor Dimension

- Use an n-dimensional truncation (n arbitrarily large).

- Then: $\Sigma_q^s = |\mathcal{A}|^q \phi^s(T^q)$

- Where $\phi^s(T^q) = (\phi^s(T))^q$ can be written explicitly.

- It follows taking the limiting geometric mean gives: $\Sigma_\infty^s = |\mathcal{A}| \phi^s(T)$

- The required value of d is the solution of

$$|\mathcal{A}| \phi^d(T) = 1$$

- Linear interpolation of the cumulative sum of the ordered list of Lyapunov exponents and $\log |\mathcal{A}|$ (the maximal entropy of the shift invariant measures on $\mathcal{A}^{\mathbb{Z}^+}$).

Testing the Hypotheses

Proposition 7 *Consider the IFS consisting of* $|\mathcal{A}| = 2$ *maps* $\tilde{f}_a : \mathbb{R}^n \to \mathbb{R}^n$ *where* $\tilde{f}_a x = Tx + \beta_a$ *and* T *is diagonal with non-vanishing elements* $T_{kk} = \lambda_{k-1}$. *Then for sufficiently large* n, *all the images* $TK + \beta_a$ *of the attractor* K *are pairwise disjoint for almost all* $(\beta_1, \beta_2) \in \mathbb{R}^{2n}$ *in the sense of* $2n$*-dimensional Lebesgue measure.*

- For a proof, take the nth component of the IFS.

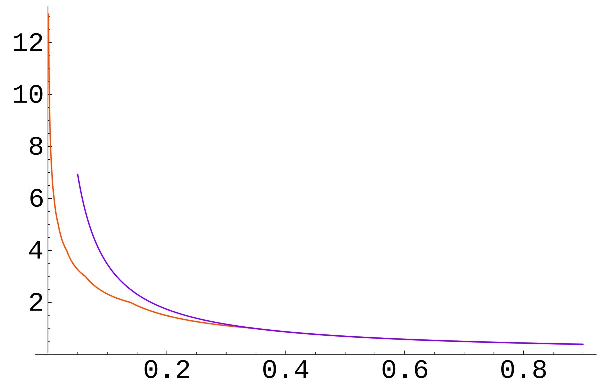
- Problem reduces to case at very beginning of talk

- So no overlap iff $\lambda_n < 1/2$.

- This can always be arranged for large enough n.

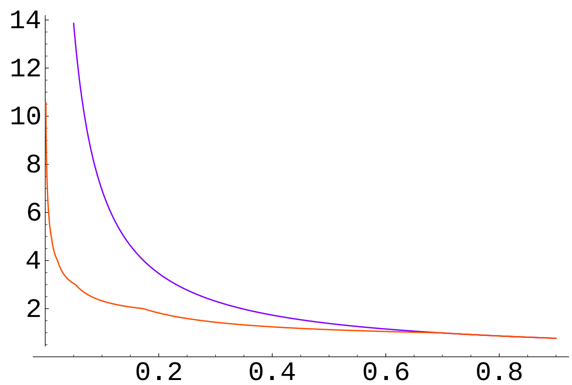
Numerical Values of the Attractor Dimension

- For almost all forms of input current distribution, we can, therefore, find $\dim_H K$ using a simple numerical root finder.



Red: $\dim_H K$ vs τ in the case $\rho = 2.0$

Blue: The upper bound from Corollary 4



Red: $\dim_H K$ vs τ in the case $\rho = 0.5$

Blue:: The upper bound from Corollary 4

The Effect of Noise

- Things are easier with noise.
- Add i.i.d. random shifts $y_i \in D \subset \mathbb{R}^n$ at each application of a map from the IFS.
- Where the y_i have a.c. distribution with bdd density, supported on an arbitrarily small disc D at the origin.
- For sample path y the attractor is K^y

Theorem 8 (Jordan,Pollicott,Simon) *Given a contracting self-affine IFS of the form assumed in Theorem 5. For $\mathbb{P}$-almost all $y \in \mathcal{D}^\infty$ then:*

1. *if $d(T_1, \ldots, T_{|\mathcal{A}|}) \leq n$ then*
$$\dim_H(K^y) = d(T_1, \ldots, T_{|\mathcal{A}|})$$
2. *if $d(T_1, \ldots, T_{|\mathcal{A}|}) > n$ then $m(K^y) > 0$.*

Concluding Remarks

- Finite-dimensional attractors for randomly forced, noisy, extended systems.

- Generalise the noise model?

- Generalise to neural systems where the timing is random?

- Also can introduce nonlinearities—maybe need to extend Theorems 1 and 3 to allow for non-contracting flows.

- Information theory—channel capacity?

- Delay embedding for IFSs—results of Robinson.