Occurrence of normal and anomalous diffusion in polygonal billiard channels

David P. Sanders & Hernán Larralde

Centro de Ciencias Físicas, UNAM
Cuernavaca, Mexico

www.fis.unam.mx/~dsanders
dsanders@fis.unam.mx

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Introduction

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  - electron gas in metal (Lorentz 1905)
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  - one of simplest physical systems with macroscopic transport

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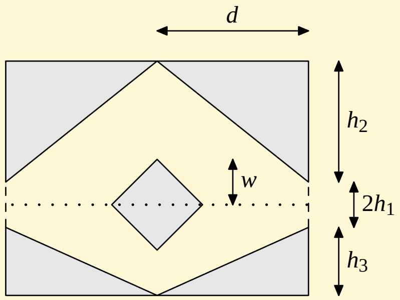
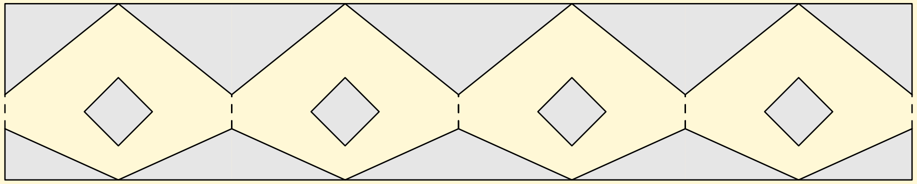
- Necessary microscopic conditions for macroscopic transport?
- Corners separate nearby trajectories: “randomising” effect

- Models:





Polygonal billiard channels

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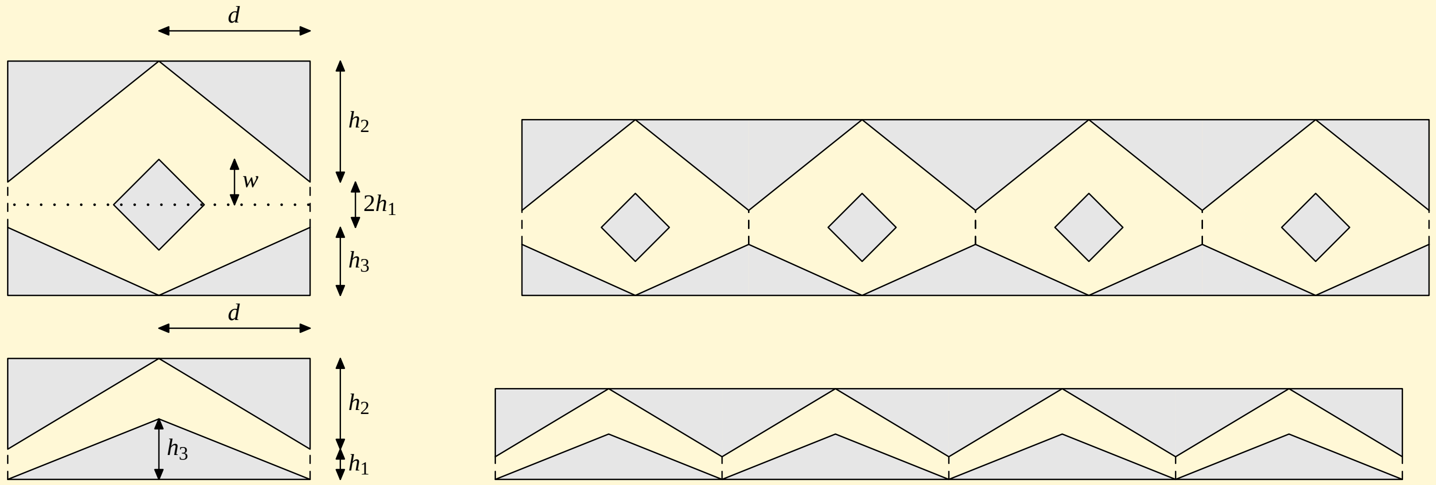

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Introduction
- Motivation
- Properties of billiards
- Channels
- Results

Normal diffusion

Anomalous diffusion

Conclusions

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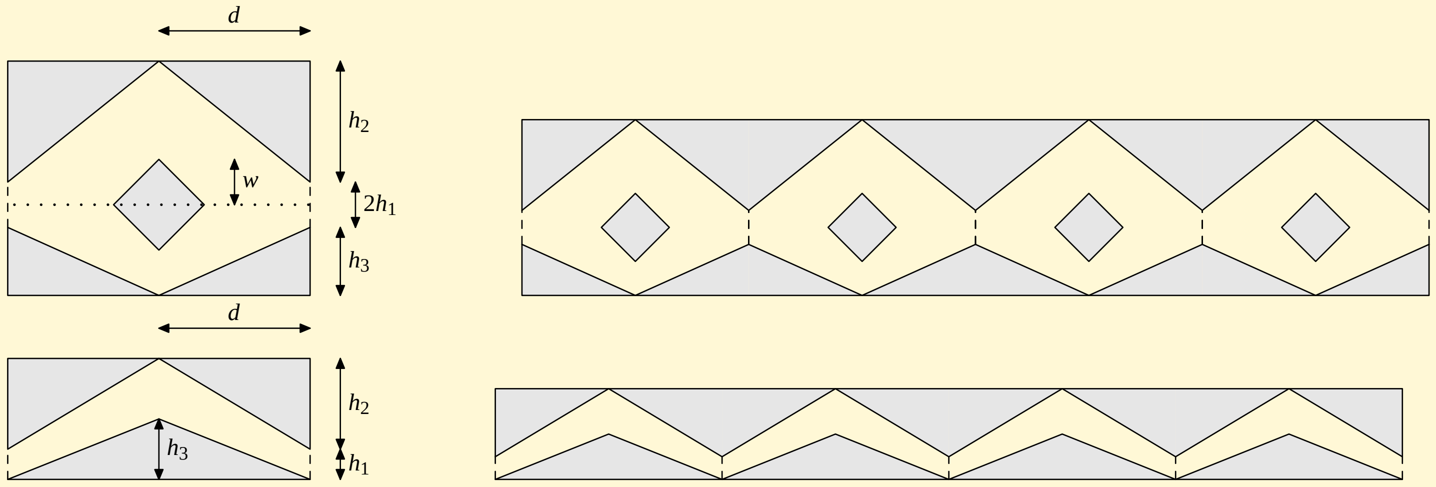

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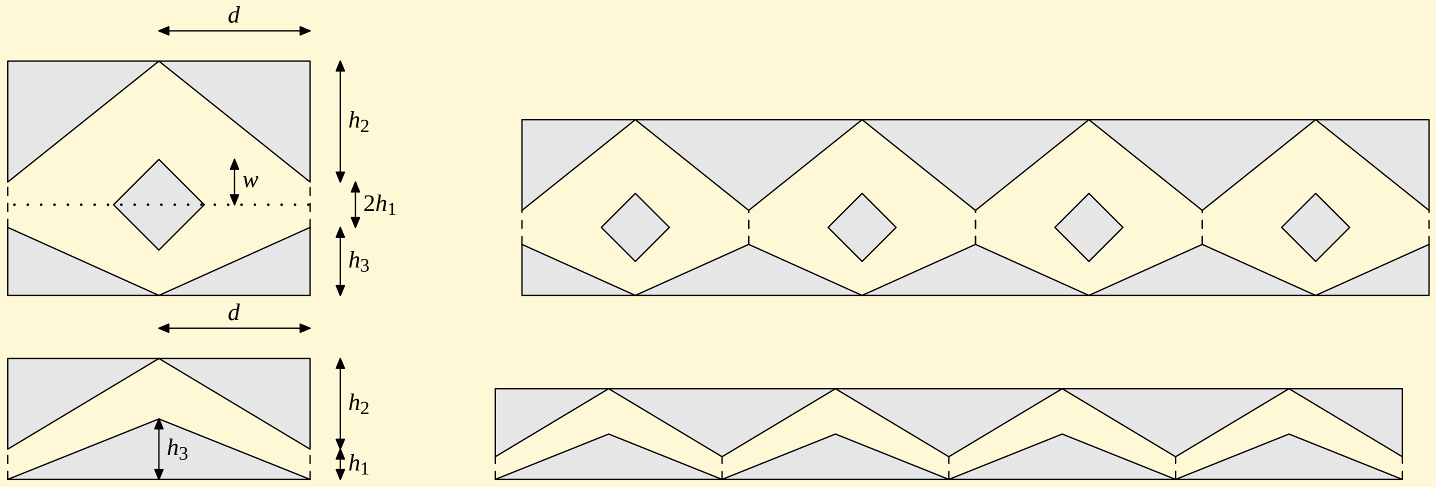

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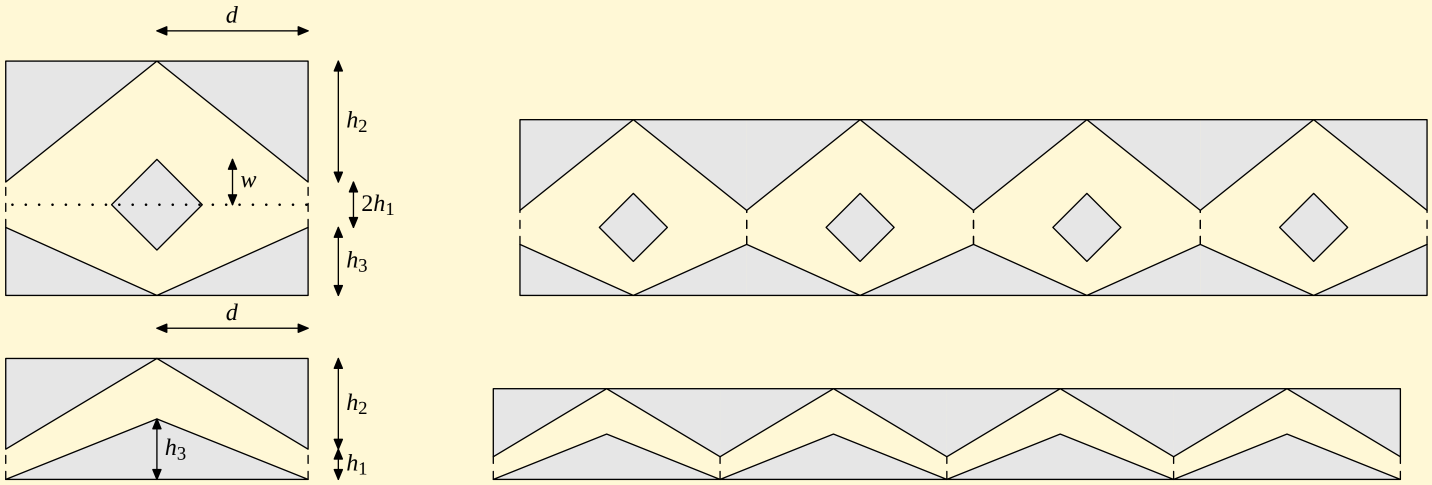

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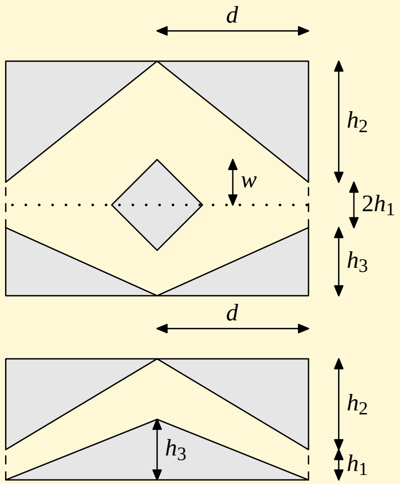

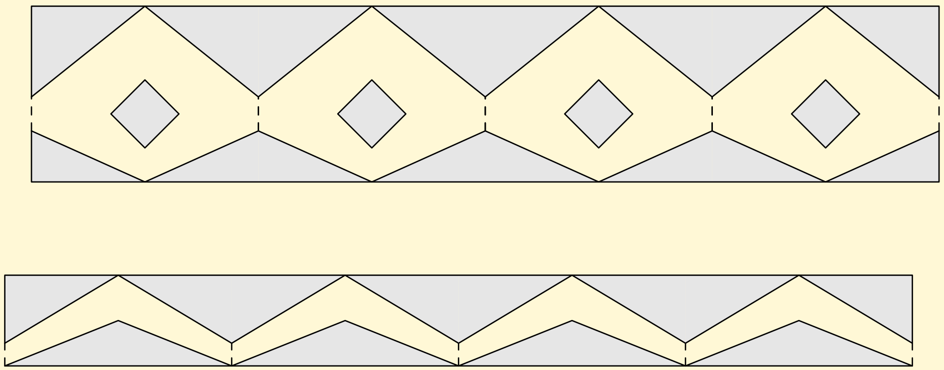

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- Jepps & Rondoni 2006

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condition	$\sigma^2(t)$ asymptotic	diffusion
generic	t	normal
infinite horizon	$t \log t$	marginal anomalous
parallel scatterers	$t^\alpha, \quad \alpha > 1$	anomalous superdiffusion

Normal diffusion

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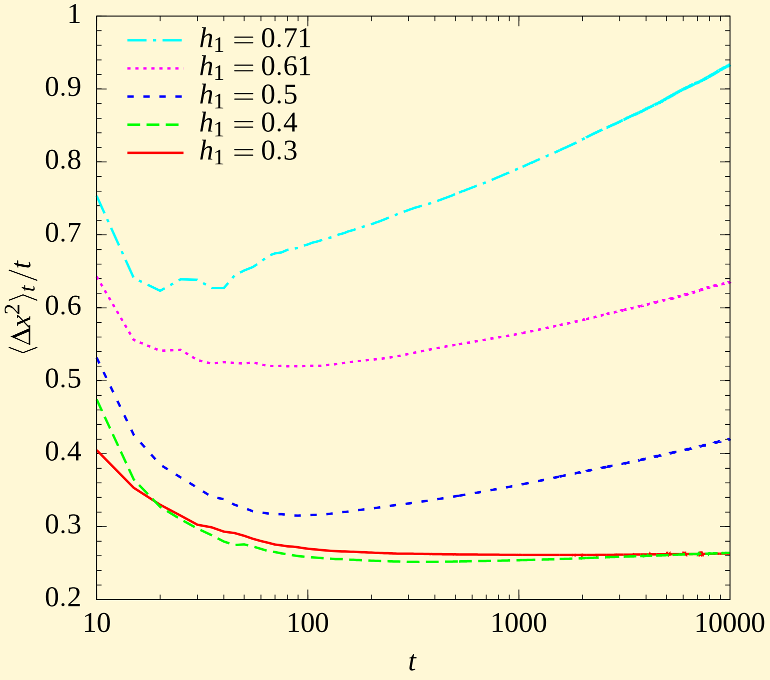
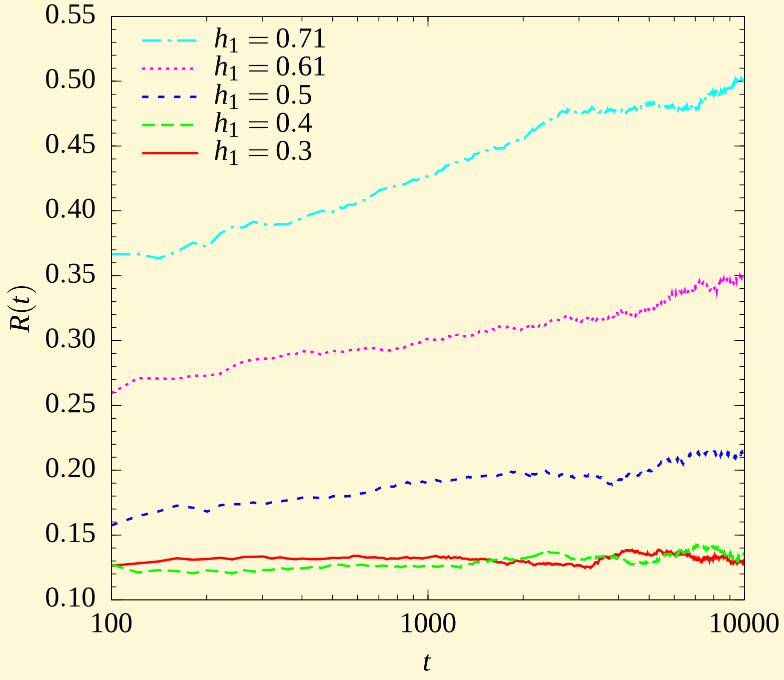


$\sigma^2(t)/t$



$R(t)$

Fine structure of distributions

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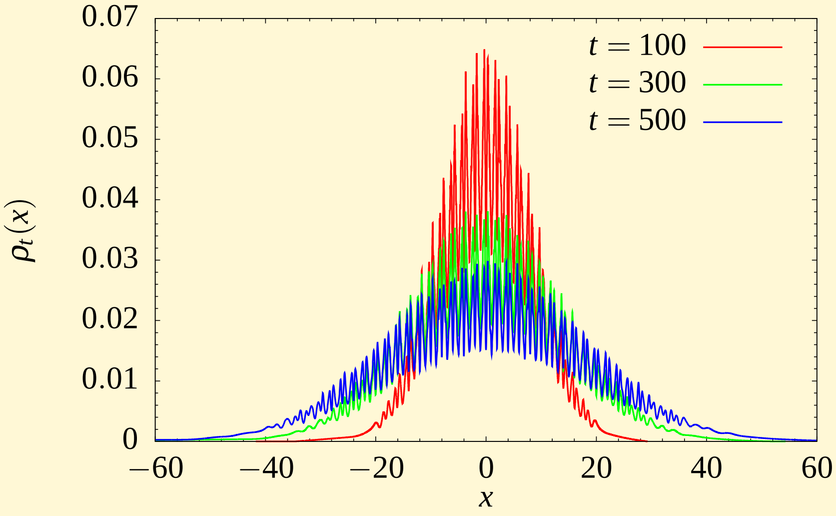

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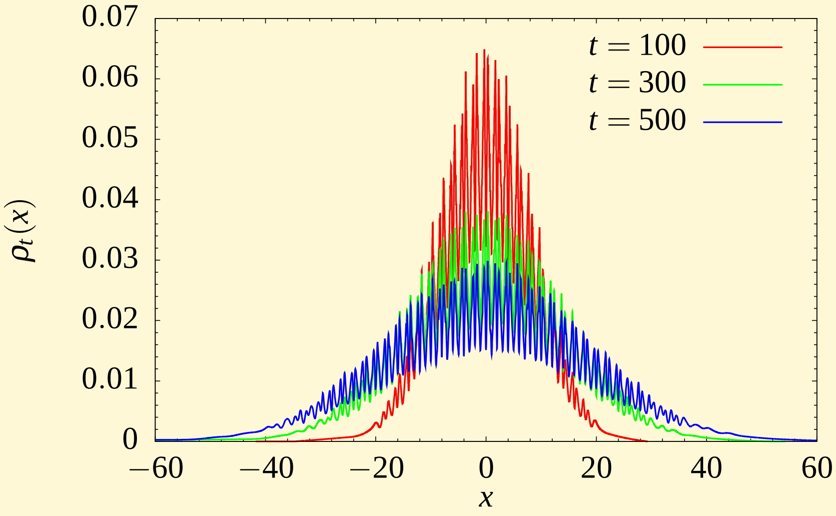

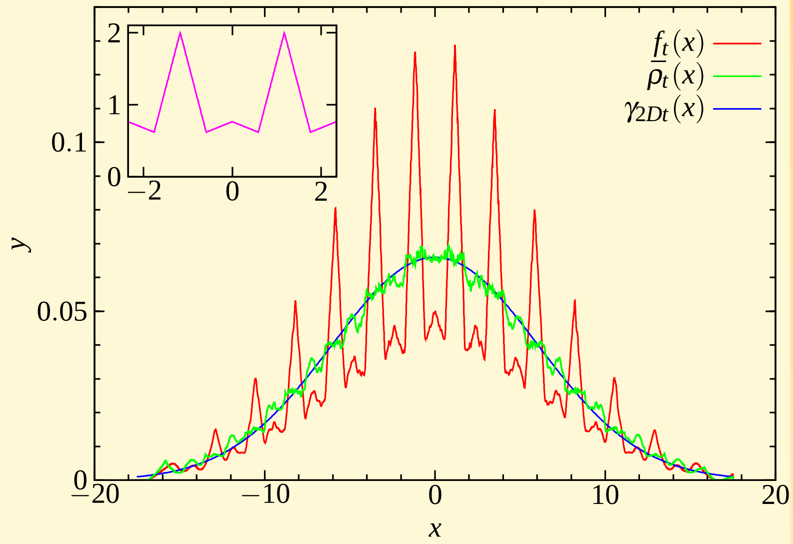

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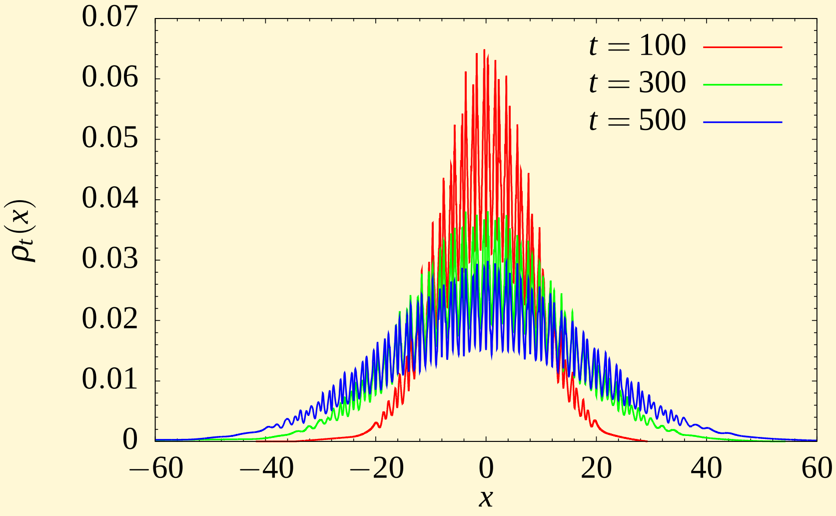
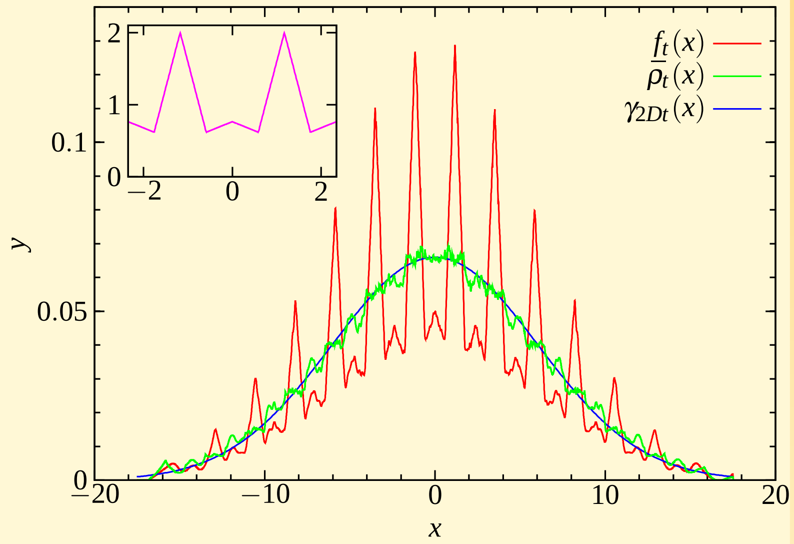
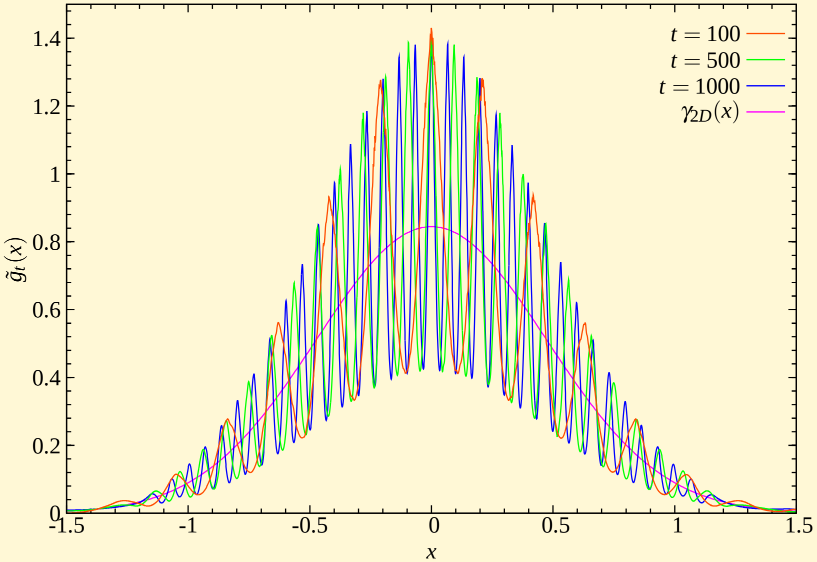




- Demodulate by available height $h(x)$:

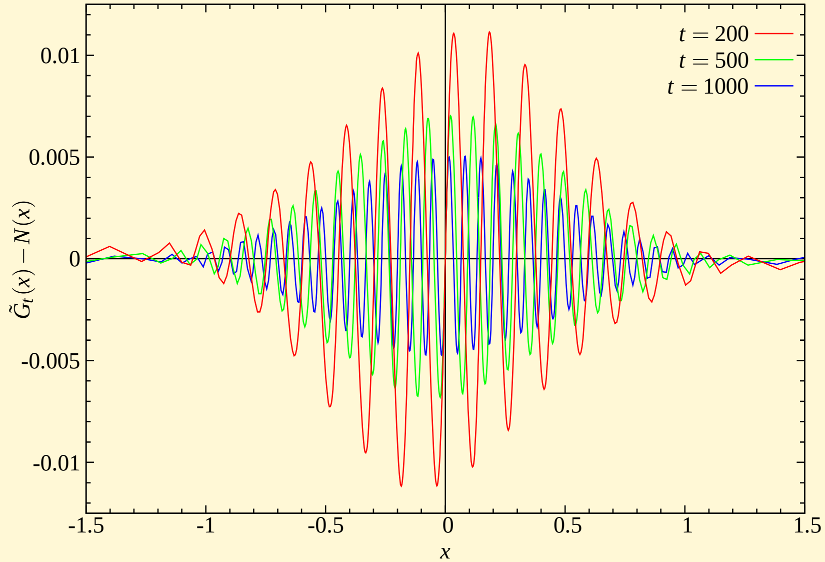
$$f_t(x) := \frac{\rho_t(x)}{h(x)}$$

Weak convergence (Lorentz gas channel)



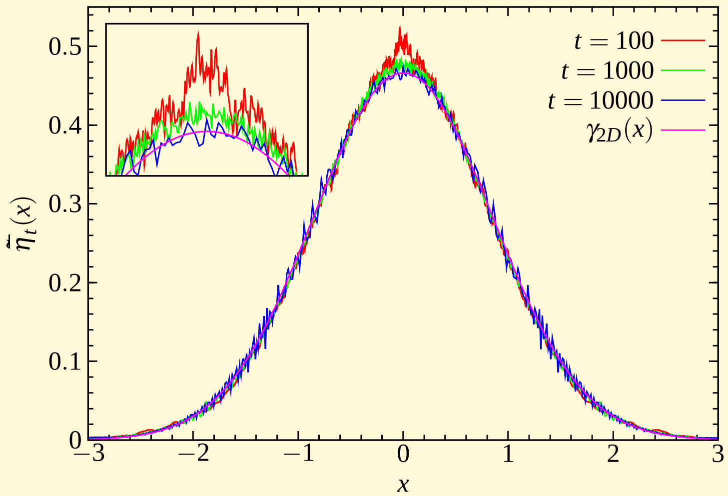
rescaled densities

$\tilde{f}_t(x) := \sqrt{t}\, f_t(x\sqrt{t})$

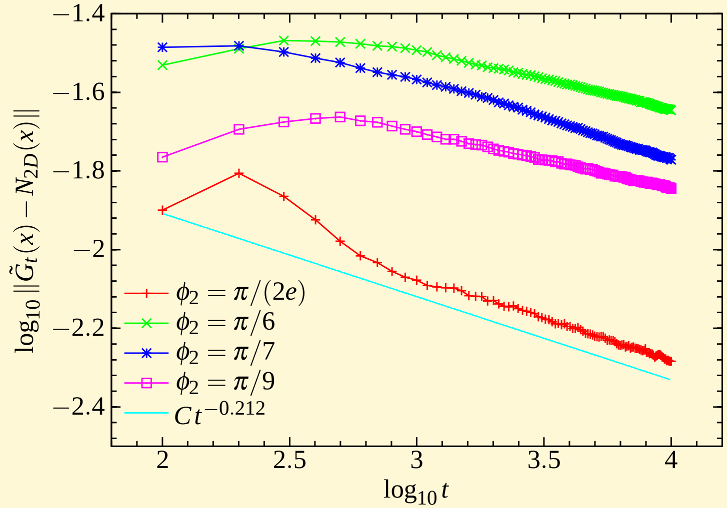


distance of
cumulative distribution
from normal

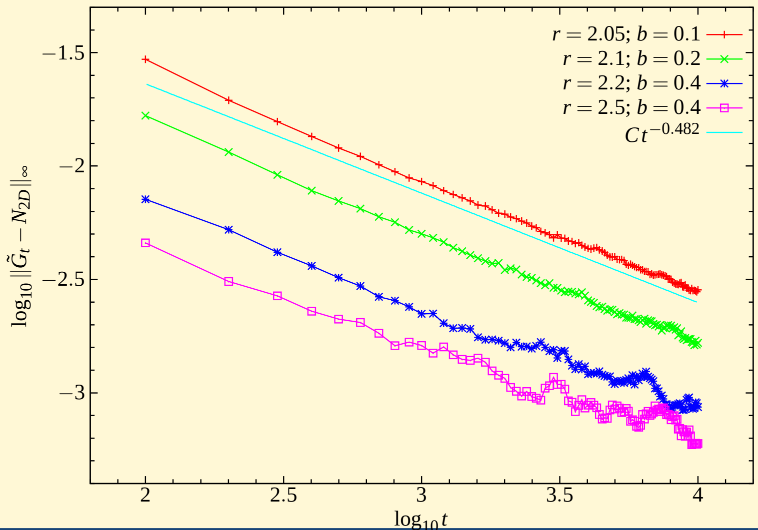
Central limit theorem



demodulated, $\sqrt{t}$-rescaled



max. distance from normal



polygonal: distance $\sim t^{-0.21}$
Lorentz: distance $\sim t^{-0.48}$

Pène (2002): faster than $t^{-1/6}$
Heuristic: slower than $t^{-1/2}$

Anomalous super-diffusion

Parallel scatterers

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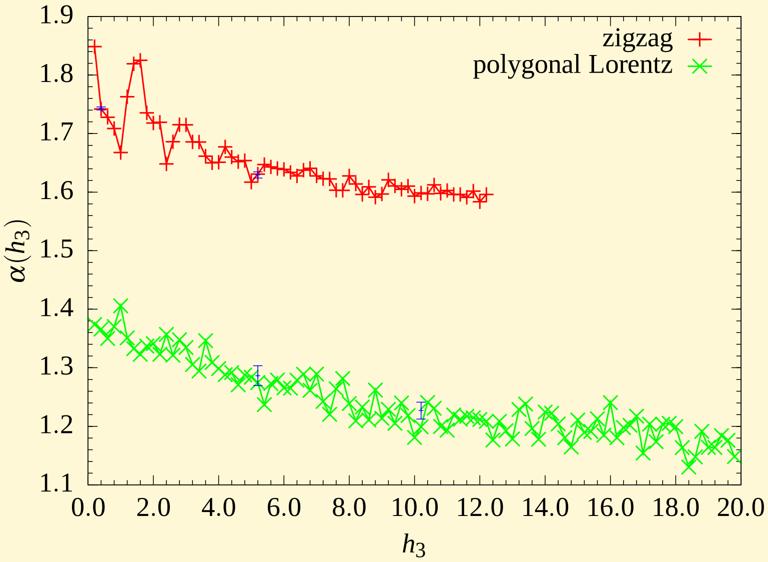

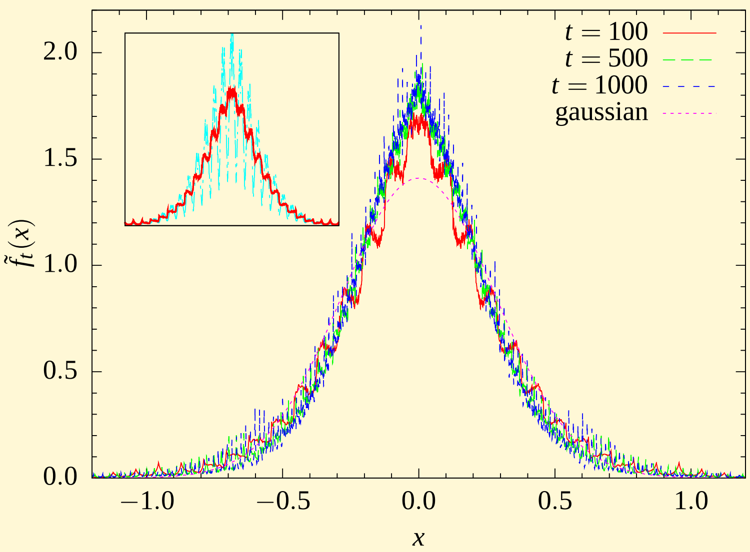

exponent α

rescaled densities $\tilde{f}_t(x) := t^{\alpha/2} f_t(x t^{\alpha/2})$

Qualitative results

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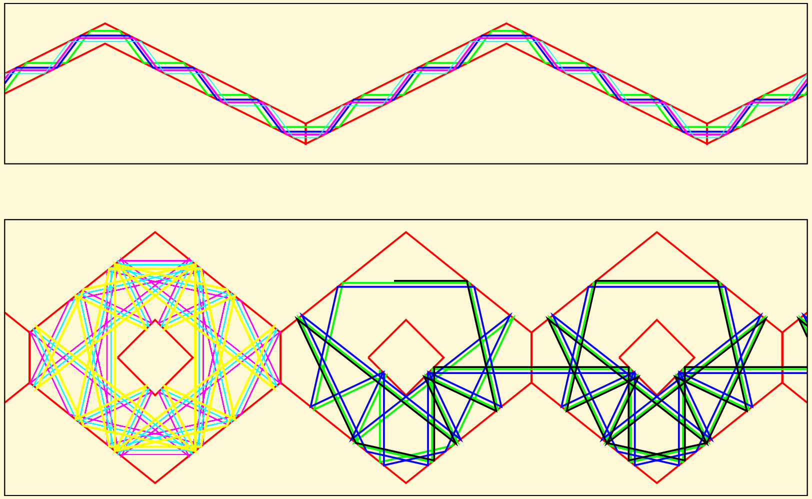

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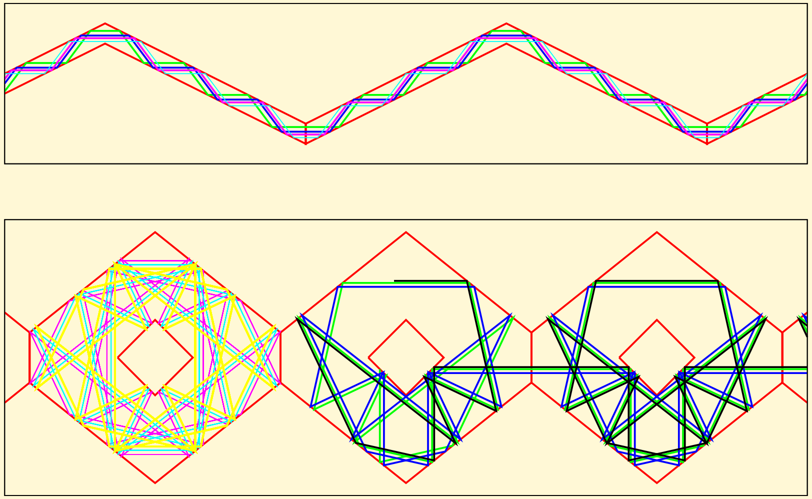
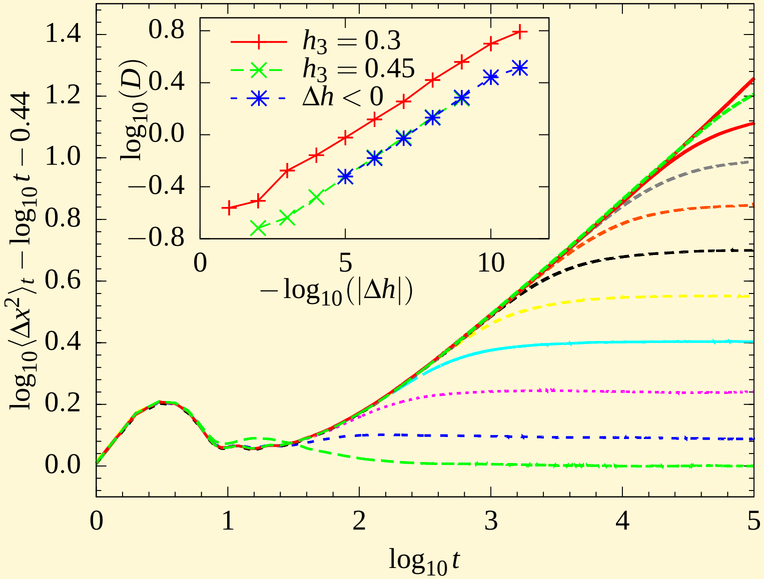
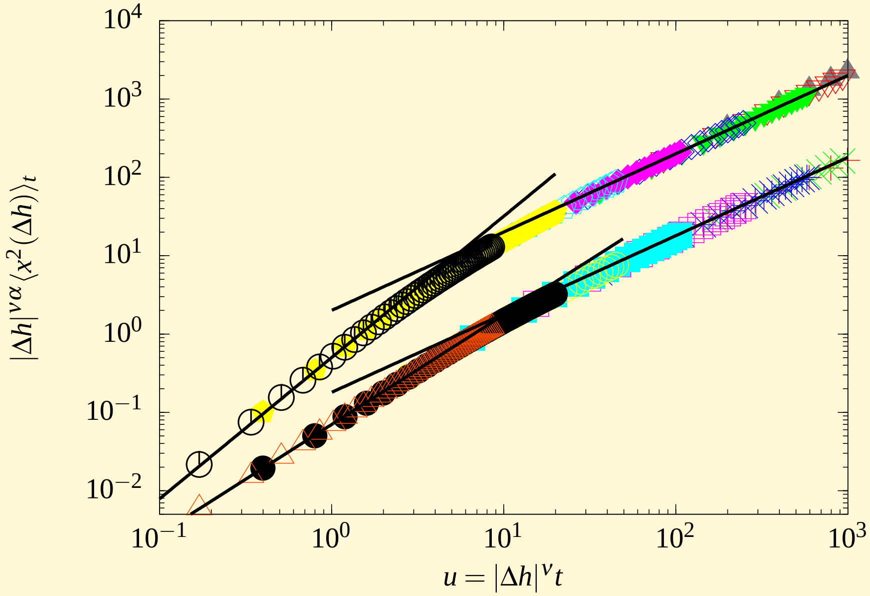


- Much more likely when parallel scatterers
- Model with continuous-time random walks (DPS+HL 2006, Schmiedeberg & Stark 2006)

Crossover from normal to anomalous





- $\langle x^2(\Delta h)\rangle_t \sim \begin{cases} D(\Delta h)t & \text{for } t > T_c \\ t^\alpha & \text{for } t < T_c \end{cases}; \qquad T_c \sim |\Delta h|^{-\nu}$
- Data collapse of $|\Delta h|^{\nu\alpha}\langle x^2(\Delta h)\rangle_t$ as function of $u := |\Delta h|^\nu t$:

$$\phi(u) \sim \begin{cases} u & \text{for } u \gg 1 \\ u^\alpha & \text{for } u \ll 1 \end{cases}$$

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References

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