Dynamical Systems and Statistical Mechanics

3-13 July 2006, LMS Durham Symposium

Statistical Limit Theorems for non-invertible transformations

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Limit theorems

$(Y, \mathcal{B}, \nu)$ - measure space with $\nu(Y) = 1$

$T : Y \to Y$ - transformation preserving the measure ν

$h : Y \to \mathbb{R}$ - measurable function

Question:

Are there sequences $b_n > 0$, c_n, and a non-degenerate random variable η such that

$$\frac{1}{b_n}\sum_{j=0}^{n-1} h \circ T^j - c_n \to^d \eta?$$

$$\eta_n \to^d \eta :\Longleftrightarrow \lim_{n\to\infty}\int_Z f(z)\mu_n(dz) = \int_Z f(z)\mu(dz), \quad f \in C(Z)$$

η_n, η - random variables with values in a metric space Z and distributions μ_n, μ

Functional limit theorems

Define the process $\{w_n(t) : t \in [0, 1]\}$ by

$$w_n(t) = \frac{1}{B_n} \sum_{j=0}^{[nt]-1} h \circ T^j - C_{[nt]} \text{ for } t \in [0, 1],\ n \geq 1.$$

Is there a process $\mathcal{W}$ such that the sequence w_n converges weakly to $\mathcal{W}$

$$w_n(\cdot) \to^d \mathcal{W}(\cdot)?$$

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Perron-Frobenius operator

$\mathcal{P}_T : L^1(Y, \mathcal{B}, \nu) \to L^1(Y, \mathcal{B}, \nu)$:

for all $f \in L^1(Y, \mathcal{B}, \nu), g \in L^\infty(Y, \mathcal{B}, \nu)$

$$\int \mathcal{P}_T f(y) g(y) \nu(dy) = \int f(y) g(T(y)) \nu(dy).$$

'Standard' central limit theorems

Theorem 1. *Let* $h \in L^2(Y, \mathcal{B}, \nu)$ *be such that* $\mathcal{P}_T h = 0$.

Then

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{[n\cdot]-1} h \circ T^j \to^d \sqrt{E(h^2|\mathcal{I})}\, w(\cdot),$$

where $\mathcal{I}$ *is the* σ*-algebra of* T*-invariant sets and* w *is a standard Brownian motion independent of* $E(h^2|\mathcal{I})$.

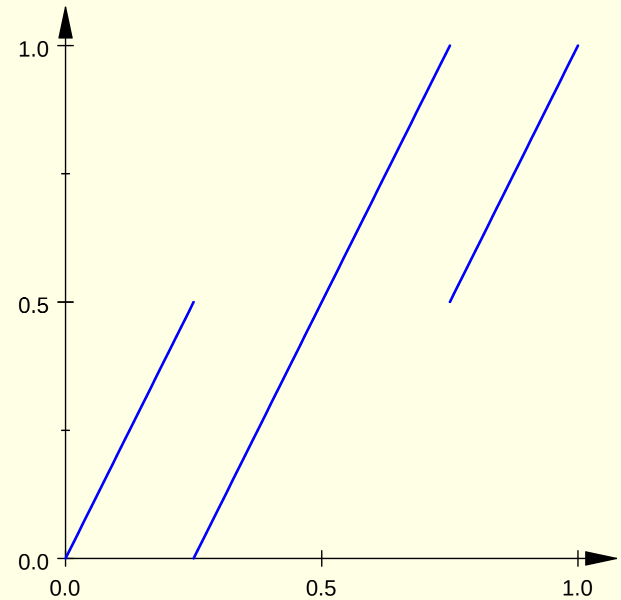
Note:

$\mathcal{P}_T h = 0 \Longrightarrow \int h(y)\nu(dy) = 0$;

T-ergodic $\Longrightarrow \sqrt{E(h^2|\mathcal{I})} = \|h\|_2$.

Example. Consider $T : [0,1] \to [0,1]$

$$T(y) = \begin{cases} 2y, & y \in [0, \frac{1}{4}) \\ 2y - \frac{1}{2}, & y \in [\frac{1}{4}, \frac{3}{4}), \\ 2y - 1, & y \in [\frac{3}{4}, 1]. \end{cases}$$



ν - Lebesgue measure

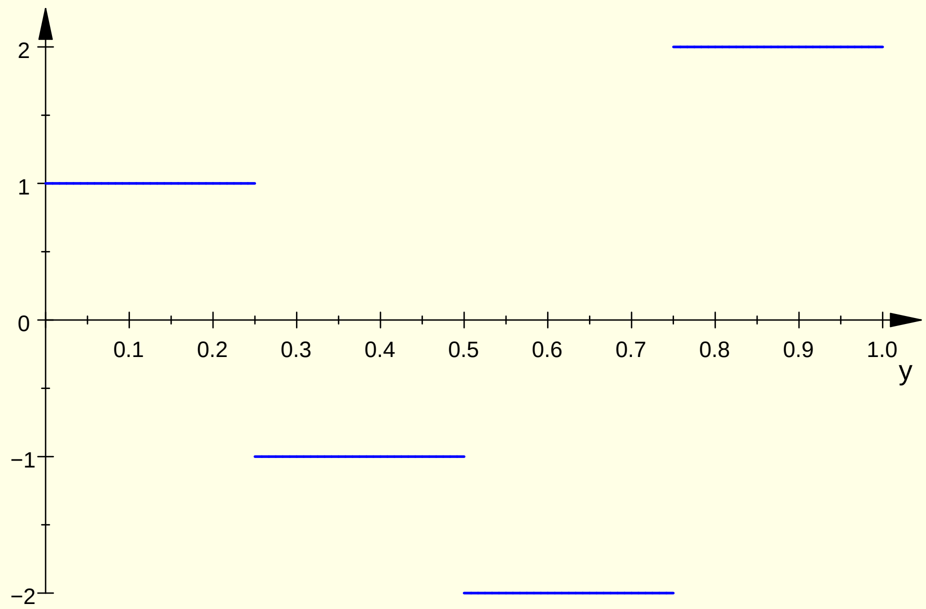
T- not ergodic: $T^{-1}([0, \frac{1}{2}]) = [0, \frac{1}{2}]$ and $T^{-1}([\frac{1}{2}, 1]) = [\frac{1}{2}, 1]$

The Perron-Frobenius operator

$$\mathcal{P}_T f(y) = \frac{1}{2} f\left(\frac{1}{2}y\right) 1_{[0,\frac{1}{2})}(y) + \frac{1}{2} f\left(\frac{1}{2}y + \frac{1}{4}\right) + \frac{1}{2} f\left(\frac{1}{2}y + \frac{1}{2}\right) 1_{[\frac{1}{2},1]}(y).$$

Let

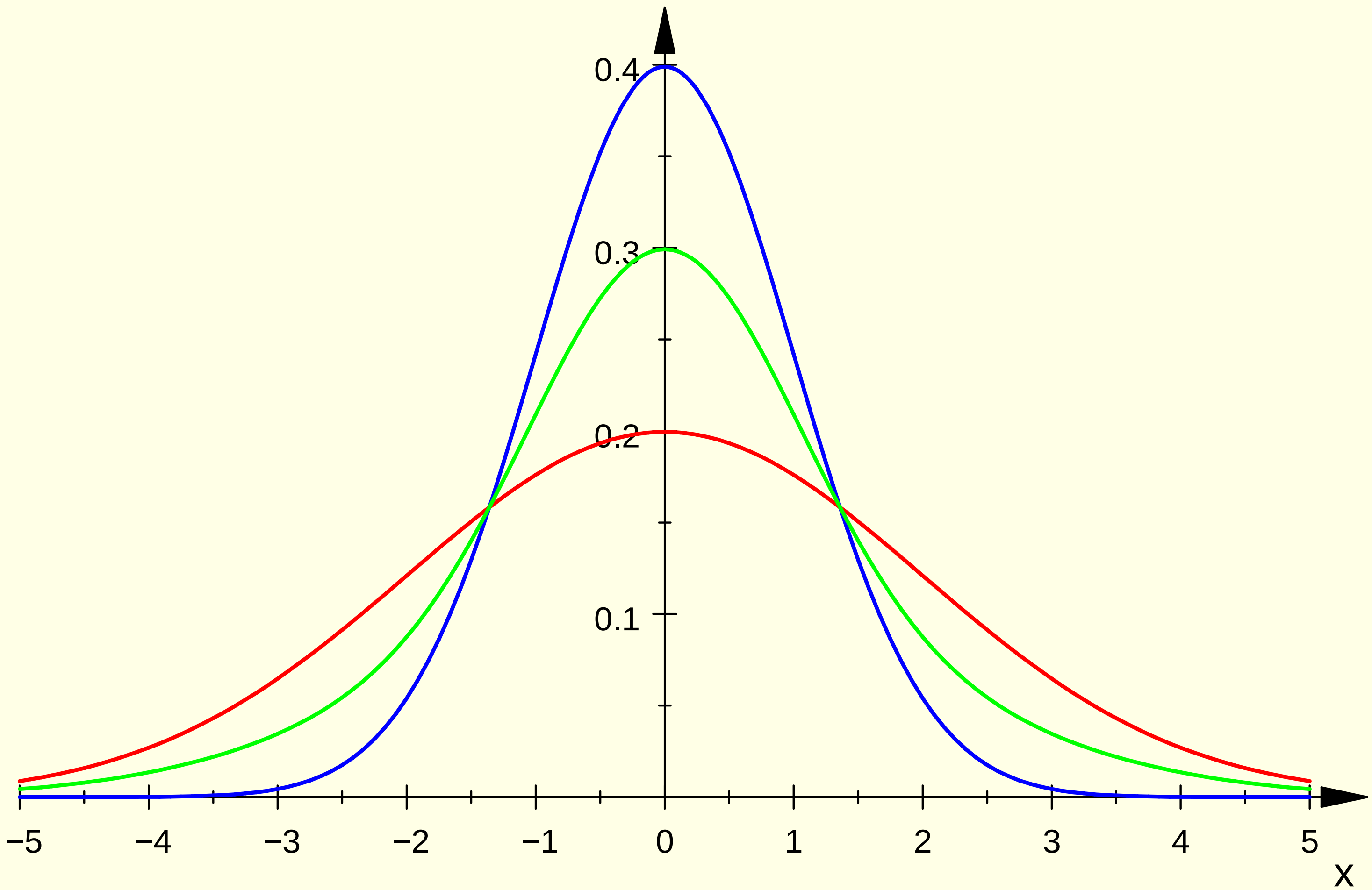
$$h(y) = \begin{cases} 1, & y \in [0, \frac{1}{4}) \\ -1, & y \in [\frac{1}{4}, \frac{1}{2}), \\ -2, & y \in [\frac{1}{2}, \frac{3}{4}), \\ 2, & y \in [\frac{3}{4}, 1]. \end{cases}$$



Then $\mathcal{P}_T h = 0$ and $E(h^2|\mathcal{I}) = 1_{[0,\frac{1}{2}]} + 4\, 1_{[\frac{1}{2},1]}$.

The density of $\sqrt{E(h^2|\mathcal{I})}w(t)$ equals to

$$\frac{1}{2}\frac{1}{\sqrt{2t\pi}}\exp\left(-\frac{x^2}{2t}\right)+\frac{1}{2}\frac{1}{\sqrt{8t\pi}}\exp\left(-\frac{x^2}{8t}\right), \quad x\in\mathbb{R}.$$



When the problem can be reduced to Theorem 1?

Theorem 2. *Let $h \in L^2(Y, \mathcal{B}, \nu)$ be such that $\int h(y)\nu(dy) = 0$. Suppose that*

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \Big\| \sum_{k=1}^{n} \mathcal{P}_T^k h \Big\|_2 < \infty. \tag{1}$$

Then

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{[n\cdot]-1} h \circ T^j \to^d \sqrt{E(\tilde{h}^2|\mathcal{I})}\, w(\cdot),$$

where $\tilde{h} \in L^2(Y, \mathcal{B}, \nu)$ is such that

$$\mathcal{P}_T \tilde{h} = 0 \qquad \textit{and} \qquad \frac{1}{\sqrt{n}} \Big\| \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \Big\|_2 \to 0. \tag{2}$$

Note:

If $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \|\mathcal{P}_T^n h\|_2 < \infty$ then (1) holds.

Sufficient decay of correlations $\Longrightarrow$ condition (1)

$$\Big|\int f(y)g(T^n(y))\nu(dy) - \int f d\nu \int g d\nu\Big| \le l(n)\|f\|_{\mathcal{L}_1}\|g\|_{\mathcal{L}_2}$$

- coupling method: [Young, 1998, Young, 1999]

  $\mathcal{L}_1$ - Hölder continuous functions, $\mathcal{L}_2 = L^\infty$

  $||\mathcal{P}_T^n h||_1 \le l(n)\|h\|_{\mathcal{L}_1}$

  Note:

  $$||h||_1 \le ||h||_2 \le ||h||_\infty$$

  $$||\mathcal{P}_T^n h||_2 \le ||h||_\infty^{1/2}||\mathcal{P}_T^n h||_1^{1/2}$$

- Birkhoff metrics: [Liverani, 1995]
- estimates in BV norms

Example. T is ergodic and cyclic: $Y = Y_1 \cup \ldots \cup Y_r$, $T(Y_i) = Y_{i+1}$, $i = 1, \ldots, r-1$, $T(Y_r) = Y_1$.

Let $h \in L^2(Y, \mathcal{B}, \nu)$ be such that $\int h(y)\nu(dy) = 0$ and

$$\left\| \sum_{k=0}^{n-1} \mathcal{P}_T^{rk} h_r \right\|_2 = O(1), \quad \text{where} \quad h_r = \sum_{k=0}^{r-1} h \circ T^k.$$

Then

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{[n\cdot]-1} h \circ T^j \to^d \sigma w(\cdot), \quad \text{where}$$

$$\sigma^2 = \int_{Y_1} h_r^2(y)\nu(dy) + 2\sum_{j=1}^{\infty} \int_{Y_1} h_r(y) h_r(T^{rj})\nu(dy);$$

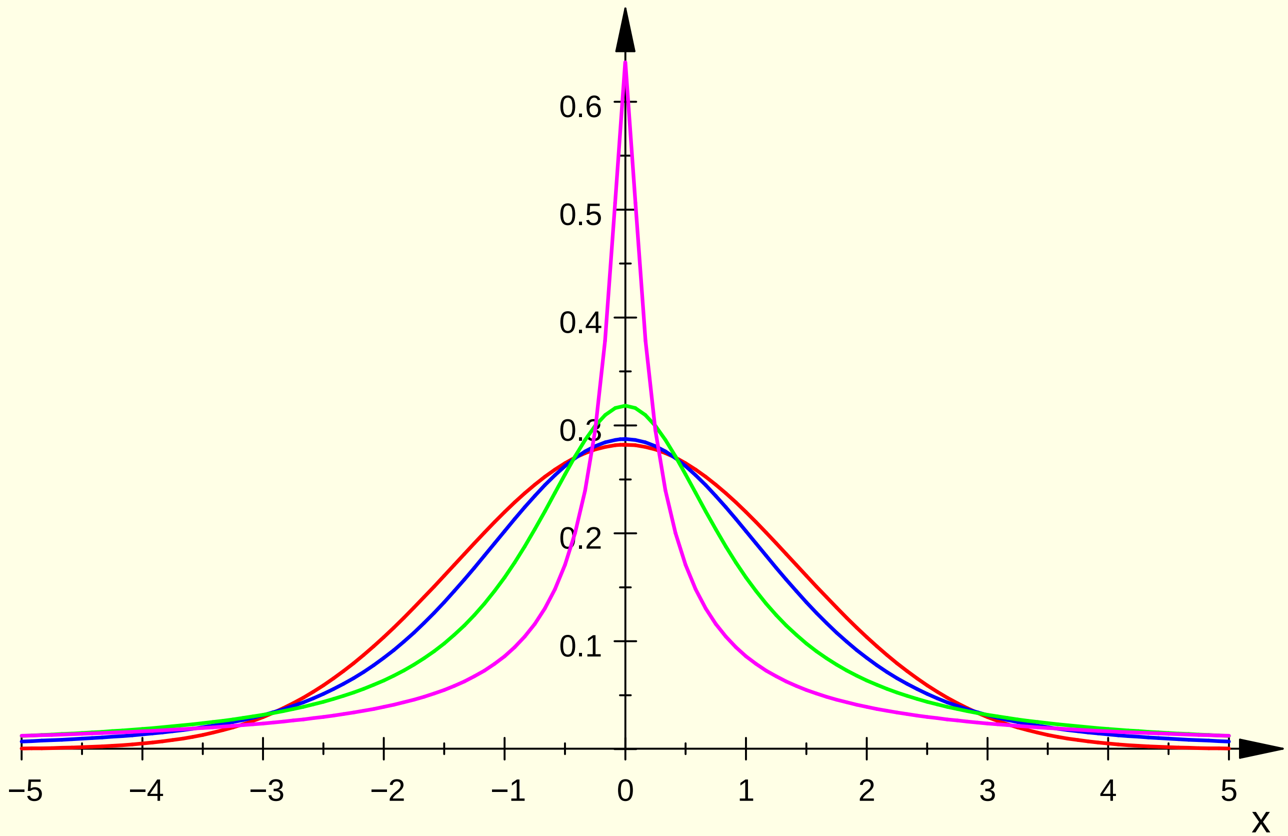
$$\sigma = 0 \Longleftrightarrow h_r = f \circ T^r - f \quad \text{for some } f.$$

[Melbourne and Nicol, 2004] $\mathcal{P}_T$ quasi-compact on $\mathcal{L}$

α-stable random variable

η_α is said to be α-stable for some $\alpha \in (0,2)$ if it has characteristic function

$$Ee^{i\theta\eta_\alpha} = \begin{cases} \exp(i\theta a - \sigma_\alpha|\theta|^\alpha(1 - i\beta \mathrm{sign}(\theta)\tan(\pi\alpha/2))), & \alpha \neq 1, \\ \exp(i\theta a - \sigma_\alpha|\theta|(1 + i\beta(2/\pi)\mathrm{sign}(\theta)\ln(\theta)), & \alpha = 1. \end{cases}$$



$\beta = 0,\ a = 0$

Gaussian law

$\alpha = 1.5$

Cauchy law

$\alpha = 0.5$

h is in a domain of attraction of an α-stable law

$$\nu(|h| > x) \sim x^{-\alpha} L(x),\ x \to \infty,$$

where $\alpha \in (0, 2)$, L is a slowly varying function at ∞ and

$$\lim_{x\to\infty} \frac{\nu(h > x)}{\nu(|h| > x)} = p \quad \text{and} \quad \lim_{x\to\infty} \frac{\nu(h < -x)}{\nu(|h| > x)} = q$$

with $0 \le p \le 1$ and $p + q = 1$.

Let b_n be such that

$$\lim_{n\to\infty} n\nu(|h| > b_n) = 1. \tag{3}$$

Then

$$n\nu(b_n^{-1} h \in \cdot) \to^v \lambda_\alpha(\cdot), \quad \text{where}$$

$$\lambda_\alpha(dx) = \left(p\alpha x^{-\alpha-1} 1_{(0,\infty)}(x) + q\alpha(-x)^{-\alpha-1} 1_{(-\infty,0)}(x)\right) dx.$$

Assumptions:

(C1) T is ergodic.

(C2) h is in a domain of attraction of an α-stable law with $\alpha \in (0, 2)$

(C3) for every set V which is a finite sum of disjoint and separated from 0 intervals of the form $(c, d]$

$$\lim_{n\to\infty} \nu(\tau_{h^{-1}(b_n V)} \geq n) = \exp(-\lambda_\alpha(V)), \quad (4)$$

where $\tau_A(y) = \inf\{k \geq 1 : T^k(y) \in A\}$.

Note:

Let $U_n = h^{-1}(b_n V)$. Then $\nu(U_n) \xrightarrow[n\to\infty]{} 0$ and

$$(4) \iff \lim_{n\to\infty} \nu\left(\tau_{U_n} \geq \frac{t}{\nu(U_n)}\right) = \exp(-t)$$

Convergence to α-stable random variable η_α

Theorem 3. *Assume (C1)-(C3).*

(i) If $0 < \alpha < 1$*, then* $\frac{1}{b_n}\sum_{j=0}^{n-1} h \circ T^j \to^d \eta_\alpha.$

(ii) If $1 \le \alpha < 2$ *and for all* $\varepsilon > 0$

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \nu(|S_n(0,\delta] - ES_n(0,\delta]| > \varepsilon) = 0, \tag{5}$$

where $S_n(0,\delta] = b_n^{-1}\sum_{j=0}^{n-1}(1_{|h|^{-1}((0,\delta b_n])} h) \circ T^j$*, then there is* c_n

$$\frac{1}{b_n}\sum_{j=0}^{n-1} h \circ T^j - c_n \to^d \eta_\alpha.$$

Convergence to α-stable random variable η_α

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where $S_n(0,\delta] = b_n^{-1}\sum_{j=0}^{n-1}(1_{|h|^{-1}((0,\delta b_n])}h) \circ T^j$, *then there is* c_n

$$\frac{1}{b_n}\sum_{j=0}^{n-1} h \circ T^j - c_n \to^d \eta_\alpha.$$

Remark. Extensions of (4) and (5) lead to functional limit theorem with α-stable process in the limit.

Example. Tent map

$T(y) = 1 - 2|y|$, $y \in [-1, 1]$, ν- normalized Lebesgue measure on $[-1, 1]$,

$$\mathcal{P}_T h(y) = \frac{1}{2} h\left(\frac{y-1}{2}\right) + \frac{1}{2} h\left(\frac{1-y}{2}\right).$$

$h(y) = -h(-y)$, $y \in [-1, 1] \Longrightarrow \mathcal{P}_T h = 0$.

Let $h(y) = y^\gamma$, $y > 0$ and $h(-y) = -h(y)$:

$$2\gamma + 1 > 0 \Longrightarrow \frac{1}{\sqrt{n}} \sum_{j=0}^{[n\cdot]-1} h \circ T^j \to^d \frac{1}{\sqrt{2\gamma + 1}} w(\cdot),$$

$$2\gamma + 1 < 0 \Longrightarrow \frac{1}{n^{-\gamma}} \sum_{j=0}^{[n\cdot]-1} h \circ T^j \to^d X_{-\frac{1}{\gamma}}(\cdot),$$

X_α - symmetric α-stable process

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