The Painlevé Equations — Nonlinear Special Fuctions

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Reference:

P A Clarkson, Painlevé equations — nonlinear special functions, in "*Orthogonal Polynomials and Special Functions: Computation and Application*" [Editors F Marcellàn and W van Assche], *Lect. Notes Math.*, **1883**, Springer, Berlin (2006) pp 331–411

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Painlevé Equations

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 6w^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \qquad \mathrm{P_V}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\} \qquad \mathrm{P_{VI}}$$

where α, β, γ and δ are arbitrary constants.

History of the Painlevé Equations

- Derived by Painlevé, Gambier and colleagues in the late 19th/early 20th centuries
- Studied in Minsk, Belarus by Erugin, Lukashevich, Gromak *et al.* since 1950's; much of their work is published in the journal *Diff. Eqns.*, the translation of *Diff. Urav.*
- Barouch, McCoy, Tracy & Wu [1973, 1976] showed that the correlation function of the two-dimensional Ising model is expressible in terms of solutions of P_{III}.
- Ablowitz & Segur [1977] demonstrated a close connection between completely integrable PDEs solvable by inverse scattering, the **soliton equations**, such as the Korteweg-de Vries and nonlinear Schrödinger equations, and the Painlevé equations.
- Flaschka & Newell [1980] introduced the **isomonodromy deformation method** (inverse scattering for ODEs), which expresses the Painlevé equation as the compatibility condition of two **linear** systems of equations and are studied using **Riemann-Hilbert** methods. Subsequent developments by Deift, Fokas, Its, Zhou, ...
- Algebraic and geometric studies of the Painlevé equations by Okamoto in 1980's. Subsequent developments by Noumi, Umemura, Yamada, ...
- The Painlevé equations are a chapter in the "*Digital Library of Mathematical Functions*", which is a rewrite/update of Abramowitz & Stegun's "*Handbook of Mathematical Functions*" due to appear — see `http://dlmf.nist.gov`.

Properties of the Painlevé Equations

- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **Bäcklund transformations** which map solutions of a given Painlevé equation to solutions of the same Painlevé equation, but with different values of the parameters.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **rational** and **algebraic solutions** for certain values of the parameters.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **special function solutions** expressed in terms of the classical special functions [Airy $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$, Bessel $J_\nu(z)$, $Y_\nu(z)$, parabolic cylinder $D_\nu(z)$, Whittaker $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ and hypergeometric ${}_2F_1(a,b;c;z)$], for certain values of the parameters.
- These rational, algebraic and special function solutions of $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ can often be written in **determinantal form**.
- $\mathrm{P_{I}}$–$\mathrm{P_{VI}}$ can be written as a (non-autonomous) **Hamiltonian system**.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have associated **Affine Weyl groups** which act on the parameter space.
- $\mathrm{P_{I}}$–$\mathrm{P_{VI}}$ possess **Lax pairs** (**isomonodromy problems**).
- $\mathrm{P_{I}}$–$\mathrm{P_{VI}}$ can be written in **bilinear form**.
- $\mathrm{P_{I}}$–$\mathrm{P_{VI}}$ form a **coalescence cascade**.

$$\begin{array}{ccccccc}
\mathrm{P_{VI}} & \longrightarrow & \mathrm{P_{V}} & \longrightarrow & \mathrm{P_{IV}} & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{P_{III}} & \longrightarrow & \mathrm{P_{II}} & \longrightarrow & \mathrm{P_{I}}
\end{array}$$

Why study Painlevé-type equations?

- They are very attractive! The solutions of the Painlevé equations, called the **Painlevé transcendents**, are interesting mathematical functions.
- The **general solutions** of the Painlevé equations are **transcendental**, i.e. irreducible in the sense that they cannot be expressed in terms of previously known functions, such as rational functions or the classical special functions.
- The Painlevé equations may be viewed as **nonlinear special functions**.
- The Painlevé property is deeply connected to the notion of "**integrability**" of differential equations, which is not well understood.
- Progress on problems involving Painlevé-type equations frequently has many benefits to the theory of differential equations.
- They have numerous significant mathematical and physical applications.

Combinatorics and $\mathrm{P_{II}}$

Let S_N be the group of permutations $\boldsymbol{\pi}$ of the numbers $1, 2, \ldots, N$. For $1 \le j_1 < j_2 < \ldots < j_k \le N$, we say that $\boldsymbol{\pi}(j_1), \boldsymbol{\pi}(j_2), \ldots, \boldsymbol{\pi}(j_k)$ is an **increasing subsequence** of $\boldsymbol{\pi}$ of length k if

$$\boldsymbol{\pi}(j_1) < \boldsymbol{\pi}(j_2) < \cdots < \boldsymbol{\pi}(j_k)$$

Let $\ell_N(\boldsymbol{\pi})$ be the length of the longest subsequence of $\boldsymbol{\pi}$.

For example, if $N = 5$ and $\boldsymbol{\pi}$ is the permutation

$$\boldsymbol{\pi}(1, 2, 3, 4, 5) = (5, 1, 3, 2, 4)$$

then 134 and 124 are both longest increasing subsequences of $\boldsymbol{\pi}$ and so $\ell_5(\boldsymbol{\pi}) = 3$.

Define

$$q_N(m) \equiv \mathrm{Prob}(\ell_N(\boldsymbol{\pi}) \le m)$$

then determine

$$\lim_{N\to\infty} q_N(m)$$

which can be expressed in terms of solutions of the special case of $\mathrm{P_{II}}$ with $\alpha = 0$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw \qquad \mathrm{P_{II}}$$

Define

$$\chi_N(\boldsymbol{\pi}) = \frac{\ell_N(\boldsymbol{\pi}) - 2\sqrt{N}}{N^{1/6}}$$

then Baik, Deift & Johansson [1999] showed that

$$\lim_{N\to\infty} \textbf{Prob}(\chi_N(\boldsymbol{\pi}) \le t) = \exp\left\{-\int_t^\infty (z-t)w_{\rm HM}^2(z)\,\mathrm{d}z\right\}$$

where $w(z)$ satisfies the special case of $\mathrm{P_{II}}$ with $\alpha = 0$

$$w_{\rm HM}'' = 2w_{\rm HM}^3 + zw_{\rm HM} \tag{1}$$

together with the boundary conditions

$$\begin{aligned} w_{\rm HM}(z) &\sim \mathrm{Ai}(z), && \text{as} \quad z \to \infty \\ w_{\rm HM}(z) &\sim \left(-\tfrac{1}{2}z\right)^{1/2}, && \text{as} \quad z \to -\infty \end{aligned} \tag{2}$$

with $\mathrm{Ai}(z)$ the Airy function satisfying

$$\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0$$

and boundary conditions

$$\begin{aligned} \mathrm{Ai}(z) &\sim \tfrac{1}{2}\pi^{-1/2}z^{-1/4}\exp\left(-\tfrac{2}{3}z^{3/2}\right), && \text{as} \quad z \to \infty \\ \mathrm{Ai}(z) &= \pi^{-1/2}|z|^{-1/4}\sin\left(\tfrac{2}{3}|z|^{3/2} + \tfrac{1}{4}\pi\right) + o(|z|^{-1/4}), && \text{as} \quad z \to -\infty \end{aligned}$$

Hastings & McLeod [1980] proved that there is a unique solution of (1) satisfying the boundary conditions (2).

Theorem (Tracy & Widom [1994])

In Random Matrix Theory, the limiting distribution for the normalized largest eigenvalue in the Gaussian Unitary Ensemble of $N \times N$ Hermitian matrices in the edge scaling limit, is

$$\lim_{N\to\infty} \textbf{Prob}\left(\left(\lambda_{\max} - 2\sqrt{N}\right)\sqrt{2}\,N^{1/6} \le t\right) = F_2(t)$$

where

$$F_2(t) = \exp\left\{-\int_t^{\infty}(z-t)w_{\mathit{HM}}^2(z)\,\mathrm{d}z\right\}$$

Theorem (Baik, Deift & Johansson [1999])

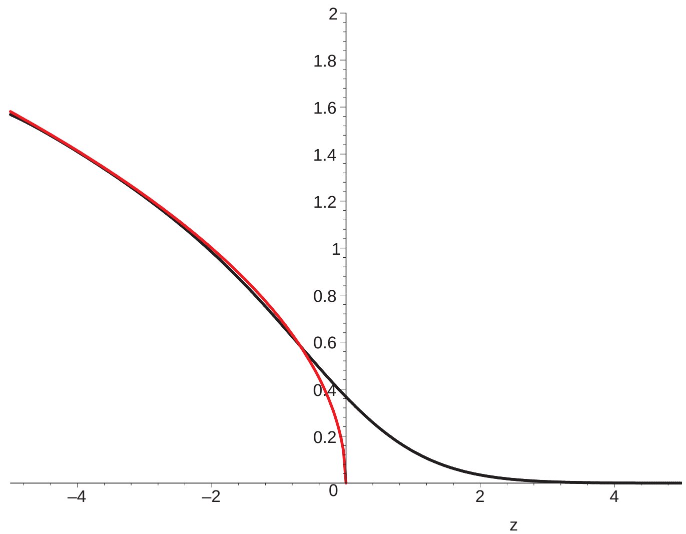
Let χ be a random variable whose distribution function is the distribution function $F_2(t)$. Then, as $N \to \infty$,

$$\chi_N := \frac{\ell_N(\boldsymbol{\pi}) - 2\sqrt{N}}{N^{1/6}} \to \chi$$

in distribution, i.e.

$$\lim_{N\to\infty} \textbf{Prob}\left(\chi_N \le t\right) = F_2(t)$$

- The function $F_2(t)$ is known as the **Tracy-Widom distribution**.
- The solution $w_{\mathrm{HM}}(z)$ is known as the **Hastings-McLeod solution**.

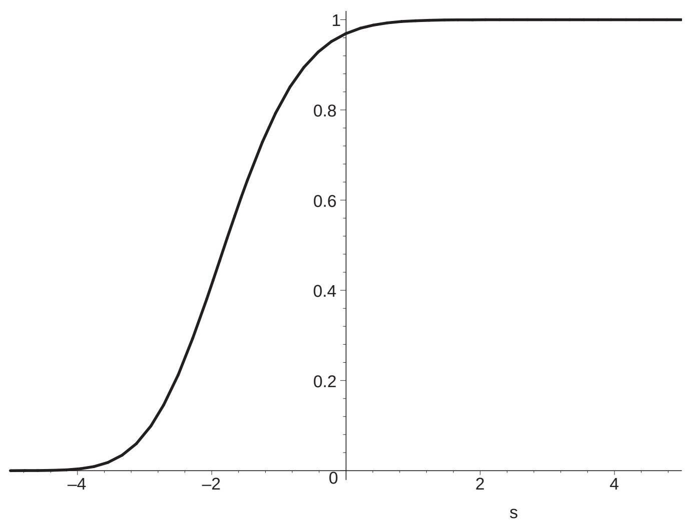


$$w_{\mathrm{HM}}'' = 2w_{\mathrm{HM}}^3 + z w_{\mathrm{HM}}$$

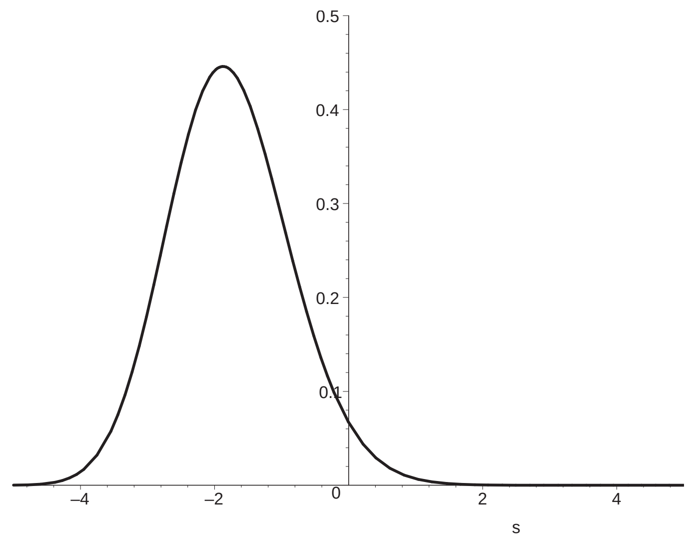
$$w_{\mathrm{HM}}(z) \sim \begin{cases} \mathrm{Ai}(z), & \text{as } z \to +\infty \\ \left(-\frac{1}{2}z\right)^{1/2}, & \text{as } z \to -\infty \end{cases}$$

$$F_2(t) = \exp\left\{ -\int_t^\infty (z-t) w_{\mathrm{HM}}^2(z)\,\mathrm{d}z \right\}$$

$w_{\mathrm{HM}}(z)$



$F_2(t)$



$$\frac{\mathrm{d}F_2}{\mathrm{d}t}(t)$$

The "classic" boundary value problem for $\mathrm{P_{II}}$ with $\alpha = 0$

$$w''_{\mathrm{HM}} = 2w^3_{\mathrm{HM}} + z w_{\mathrm{HM}} \tag{1}$$

with

$$w_{\mathrm{HM}}(z) \sim \begin{cases} \mathrm{Ai}(z), & \text{as} \quad z \to \infty \\ \left(-\tfrac{1}{2}z\right)^{1/2}, & \text{as} \quad z \to -\infty \end{cases}$$

the "Hastings-McLeod solution", arises in many applications:

- Spherical electric probe in a continuum plasma
- Görtler vortices in boundary layers
- Nonlinear optics
- Random matrix theory: Orthogonal, Unitary and Sympletic Emsembles
- Length of longest increasing subsequences, patience sorting and random walks
- Buses in Cuernavaca (Mexico), Aztec diamond tiling and airline boarding
- Universality of the edge scaling for nongaussian Wigner matrices
- Shape fluctuations in polynuclear growth models
- Distribution of eigenvalues for covariance matrices and Wishart distributions
- Distribution of zeros of the Riemann zeta function
- Bose-Einstein condensation, Superheating fields of superconductors

Elliptic Asymptotics of $\mathrm{P_{II}}$

(Boutroux [1913])

Making the transformation

$$w(z) = z^{1/2}u(\zeta), \qquad \zeta = \tfrac{2}{3}z^{3/2}$$

in $\mathrm{P_{II}}$

$$w'' = 2w^3 + zw + \alpha$$

gives

$$\frac{\mathrm{d}^2u}{\mathrm{d}\zeta^2} = 2u^3 + u - \frac{1}{\zeta}\frac{\mathrm{d}u}{\mathrm{d}\zeta} + \frac{u}{9\zeta^2} + \frac{2\alpha}{3\zeta}$$

Thus, in three sectors of angle $\frac{2}{3}\pi$, the generic $\mathrm{P_{II}}$ function has the asymptotics

$$w(z) \sim z^{1/2}u(\zeta), \qquad \zeta = \tfrac{2}{3}z^{3/2}$$

where $u(\zeta)$ satisfies the Jacobian elliptic equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\zeta}\right)^2 = u^4 + u^2 + K$$

with K an arbitrary constant. The parameters in the elliptic function $u(\zeta)$ change across the Stokes lines at 0 and $\pm\frac{2}{3}\pi$ from the positive real axis of the complex z-plane.

Asymptotics of General Painlevé II

There is a family of solutions of $\mathrm{P_{II}}$

$$w'' = 2w^3 + zw + \alpha$$

with the asymptotic behaviour

$$w(z) \sim -\frac{\alpha}{z} \sum_{n=0}^{\infty} \frac{b_n}{z^{3n}}, \qquad \text{as} \quad |z| \to \infty \tag{1}$$

where $b_0 = 1$ and

$$b_{n+1} = (3n+2)(3n+1)b_n - 2\alpha^2 \sum_{k=0}^{n} \sum_{m=0}^{k} b_m b_{k-m} b_{n-k}$$

The first few coefficients are

$$\begin{aligned} b_1 &= -2(\alpha^2 - 1), & b_3 &= -8(\alpha^2 - 1)(12\alpha^4 - 117\alpha^2 + 280) \\ b_2 &= 4(\alpha^2 - 1)(3\alpha^2 - 10), & b_4 &= 16(\alpha^2 - 1)(55\alpha^6 - 1091\alpha^4 + 7336\alpha^2 - 15400) \end{aligned}$$

The series (1) is divergent and the arbitrary constants arise from exponentially small terms which are "beyond all orders". As $|z| \to \infty$, in sectors

$$w(z) = w_1(z) + kz^{-1/4} \exp\left(-\tfrac{2}{3} z^{3/2}\right) \left\{1 + \mathcal{O}\left(|z|^{-3/4}\right)\right\} + \mathcal{O}\left(z^{-7/4} \exp\left(-\tfrac{4}{3} z^{3/2}\right)\right)$$

where $w_1(z) \sim -\alpha/z$ and k is an arbitrary constant (Its & Kapaev [2003]).

There is a second family of solutions with the asymptotic behaviour

$$w(z) \sim \pm\frac{\mathrm{i}\,z^{1/2}}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{c_n}{z^{3n/2}}, \qquad \text{as} \quad |z|\to\infty \tag{2}$$

with $c_0 = 1$, $c_1 = \mp\frac{1}{2}\sqrt{2}\,\mathrm{i}\,\alpha$ and

$$c_{n+2} = \frac{1-9n^2}{8}c_n - \frac{1}{2}\left\{\sum_{k=1}^{n+1}c_kc_{n+2-k} + \sum_{k=1}^{n+1}\sum_{m=0}^{k}c_mc_{k-m}c_{n+2-k}\right\}$$

The first few coefficients are

$$c_2 = \frac{6\alpha^2+1}{8}, \qquad c_3 = \pm\sqrt{2}\,\mathrm{i}\,\frac{\alpha(16\alpha^2+11)}{16}$$

$$c_4 = -\frac{420\alpha^4+708\alpha^2+73}{128}, \qquad c_5 = \mp\sqrt{2}\,\mathrm{i}\,\frac{\alpha(768\alpha^4+2504\alpha^2+1021)}{128}$$

The series (2) is also divergent and the arbitrary constants arise from exponentially small terms which are "beyond all orders". As $|z|\to\infty$, in sectors

$$w(z) = w_2(z) + k|z|^{-(6\alpha+1)/4}\exp\left(-\tfrac{2}{3}\sqrt{2}\,|z|^{3/2}\right)\left\{1+\mathcal{O}(|z|^{-1/4})\right\}$$

where $w_2(z) \sim \pm\dfrac{\mathrm{i}\,z^{1/2}}{\sqrt{2}}$ and k is an arbitrary constant (Kapaev [2004]).

Asymptotics of $\mathbf{P}_{\mathbf{II}}$

$$w_k'' = 2w_k^3 + zw_k$$

with boundary condition

$$w_k(z) \sim k\,\mathrm{Ai}(z) \qquad \text{as} \quad z \to \infty$$

- If $|k| < 1$, then as $z \to -\infty$

$$w_k(z) = d|z|^{-1/4} \sin\left(\tfrac{2}{3}|z|^{3/2} - \tfrac{3}{4}d^2 \ln|z| - \theta_0\right) + o(|z|^{-1/4})$$

- If $|k| = 1$, then as $z \to -\infty$

$$w_k(z) \sim \mathrm{sgn}(k)\left(-\tfrac{1}{2}z\right)^{1/2}$$

- If $|k| > 1$, then $w_k(z)$ blows up at a finite z_0

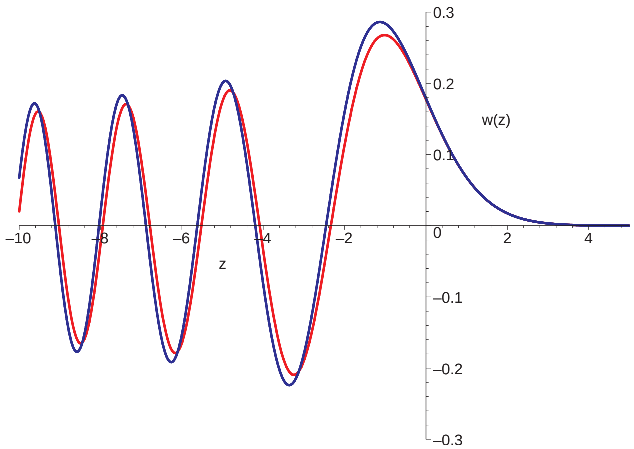
$$w_k(z) \sim \mathrm{sgn}(k)(z - z_0)^{-1} \qquad \text{as} \quad z \downarrow z_0$$

- Connection formulae

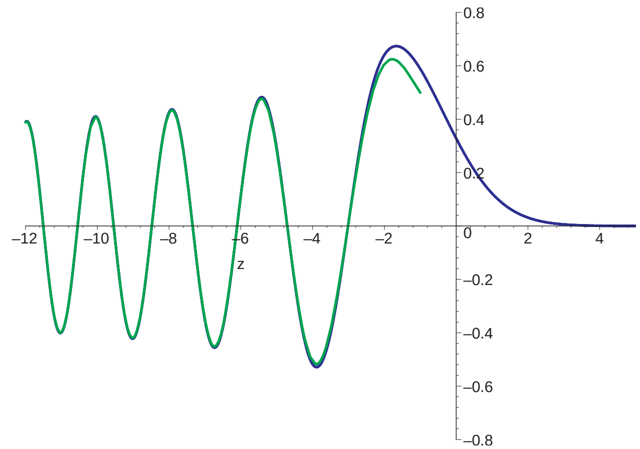
$$\begin{aligned} d^2(k) &= -\pi^{-1}\ln(1-k^2) \\ \theta_0(k) &= \tfrac{3}{2}d^2\ln 2 + \arg\left\{\Gamma\left(1 - \tfrac{1}{2}\mathrm{i}d^2\right)\right\} - \tfrac{1}{4}\pi \end{aligned}$$

(Ablowitz & Segur [1977, 1981], Hastings & McLeod [1980], Suleimanov [1987], Bassom, PAC, Law & McLeod [1998], PAC & McLeod [1988], Deift & Zhou [1993, 1995])

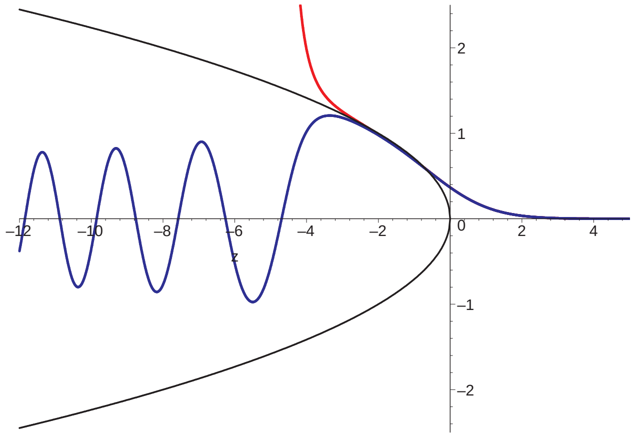
$$w_k'' = 2w_k^3 + zw_k, \qquad w_k(z) \sim k\,\mathrm{Ai}(z) \text{ as } z \longrightarrow +\infty$$



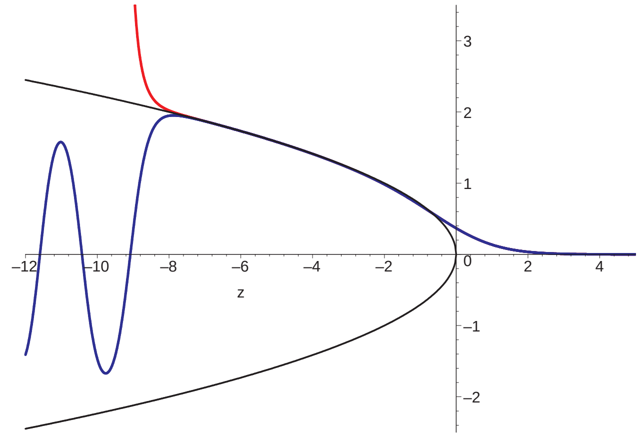
$w_k(z)$ with $k = 0.5$ and $0.5\,\mathrm{Ai}(z)$



$w_k(z)$ and asymptotic expansion for $k = 0.9$



$w_k(z)$ with $k = 0.999, 1.001$



$w_k(z)$ with $k = 1 \pm 10^{-4}$

Second Painlevé Equation

$$w'' = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$\mathrm{P_{II}}$ arises as a reduction of the mKdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0 \tag{1}$$

through the scaling reduction

$$u(x,t) = (3t)^{-1/3} w(z), \qquad z = x/(3t)^{1/3}$$

The mKdV equation (1) is solvable by inverse scattering through the integral equation

$$\mathcal{K}(x,y;t) = \mathcal{F}\,(x+y;t) + \tfrac{1}{4}\int_x^\infty\!\!\int_x^\infty \mathcal{K}(x,z;t)\,\mathcal{F}\,(z+s;t)\,\mathcal{F}\,(z+y;t)\;\mathrm{d}z\,\mathrm{d}s$$

where $\mathcal{F}(x,t)$ is expressed in terms of the initial data and satisfies the linear equation

$$\mathcal{F}_t + \mathcal{F}_{xxx} = 0$$

and $u(x,t)$ is obtained through

$$u(x,t) = \mathcal{K}(x,x;t)$$

Theorem (Ablowitz & Segur [1977])

Consider the integral equation

$$K(z,\xi) = k\,\mathrm{Ai}\left(\frac{z+\xi}{2}\right) + \frac{k^2}{4}\int_z^\infty\!\!\int_z^\infty K(z,s)\,\mathrm{Ai}\left(\frac{s+t}{2}\right)\mathrm{Ai}\left(\frac{t+\xi}{2}\right)\mathrm{d}s\,\mathrm{d}t \qquad (1)$$

Then $w(z) = K(z,z)$ *satisfies*

$$w'' = 2w^3 + zw$$

which is the special case of $\mathrm{P_{II}}$ *with* $\alpha = 0$*, and the boundary condition*

$$w(z) \sim k\,\mathrm{Ai}(z), \qquad \text{as} \quad z \to \infty$$

The integral equation (1) is derived by making the scaling reduction

$$\mathcal{K}(x,y;t) = (3t)^{1/3}K(z,\xi), \qquad z = x/(3t)^{1/3}$$
$$\mathcal{F}(x+y;t) = (3t)^{1/3}F(z+\xi), \qquad \xi = y/(3t)^{1/3}$$

in the integral equation

$$\mathcal{K}(x,y;t) = \mathcal{F}\left(x+y;t\right) + \tfrac{1}{4}\int_x^\infty\!\!\int_x^\infty \mathcal{K}(x,z;t)\,\mathcal{F}\left(z+s;t\right)\mathcal{F}\left(z+y;t\right)\,\mathrm{d}z\,\mathrm{d}s$$

for solving the mKdV equation by inverse scattering.

Isomonodromy Deformation Method for P_{II}

(Flaschka & Newell [1980])

The second Painlevé equation

$$w'' = 2w^3 + zw + \alpha \qquad \mathrm{P}_{\mathrm{II}}$$

is the compatibility condition of the linear system

$$\frac{\partial \boldsymbol{\Psi}}{\partial z} = \begin{pmatrix} -\mathrm{i}\lambda & w \\ w & \mathrm{i}\lambda \end{pmatrix} \boldsymbol{\Psi}, \qquad \frac{\partial \boldsymbol{\Psi}}{\partial \lambda} = \begin{pmatrix} -\mathrm{i}(4\lambda^2 + 2w^2 + z) & 4\lambda w + 2\mathrm{i}w' + \alpha/\lambda \\ 4\lambda w - 2\mathrm{i}w' + \alpha/\lambda & \mathrm{i}(4\lambda^2 + 2w^2 + z) \end{pmatrix} \boldsymbol{\Psi}$$

- The connection formulae

$$d^2(k) = -\pi^{-1} \ln(1 - k^2), \qquad \theta_0(k) = \tfrac{3}{2} d^2 \ln 2 + \arg \left\{ \Gamma\left(1 - \tfrac{1}{2}\mathrm{i}d^2\right) \right\} - \tfrac{1}{4}\pi$$

  for solutions of P_{II} with $\alpha = 0$ were derived heuristically using the isomonodromy deformation method by Suleimanov [1987] and Its & Kapaev [1988]. Subsequently proved rigorously by Deift & Zhou [1993, 1995] using a nonlinear version of the classical steepest descent method for oscillatory Riemann-Hilbert problems, which is rather complex.

- Bassom, PAC, Law & McLeod [1998] developed a uniform approximation method. This procedure, which is rigorous, removes the need to match solutions and can, in principle, lead to simpler solutions of connection problems.

The Lax pair for the modified KdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0$$

is

$$\boldsymbol{\psi}_x = \begin{pmatrix} -\mathrm{i}k & u \\ u & \mathrm{i}k \end{pmatrix} \boldsymbol{\psi}, \qquad \boldsymbol{\psi}_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \boldsymbol{\psi}$$

with

$$\begin{aligned} A &= -4\mathrm{i}k^3 - 2\mathrm{i}ku^2 \\ B &= 4k^2 u + 2\mathrm{i}ku_x - u_{xx} + 2u^3 \\ C &= 4k^2 u - 2\mathrm{i}ku_x - u_{xx} + 2u^3 \end{aligned}$$

Making the reduction

$$\begin{aligned} &u(x,t) = (3t)^{-1/3} w(z), && z = x/(3t)^{1/3} \\ &\boldsymbol{\psi}(x,t;k) = \boldsymbol{\Psi}(z;\lambda), && \lambda = k(3t)^{1/3} \end{aligned}$$

yields the monodromy pair for $\mathrm{P_{II}}$

$$\frac{\partial \boldsymbol{\Psi}}{\partial z} = \begin{pmatrix} -\mathrm{i}\lambda & w \\ w & \mathrm{i}\lambda \end{pmatrix} \boldsymbol{\Psi}, \qquad \frac{\partial \boldsymbol{\Psi}}{\partial \lambda} = \begin{pmatrix} -\mathrm{i}(4\lambda^2 + 2w^2 + z) & 4\lambda w + 2\mathrm{i}w' + \alpha/\lambda \\ 4\lambda w - 2\mathrm{i}w' + \alpha/\lambda & \mathrm{i}(4\lambda^2 + 2w^2 + z) \end{pmatrix} \boldsymbol{\Psi}$$

$$\frac{\partial \boldsymbol{\Psi}}{\partial z} = \begin{pmatrix} -\mathrm{i}\lambda & w \\ w & \mathrm{i}\lambda \end{pmatrix} \boldsymbol{\Psi} \equiv \mathbf{A}(z,\lambda)\boldsymbol{\Psi}$$

$$\frac{\partial \boldsymbol{\Psi}}{\partial \lambda} = \begin{pmatrix} -\mathrm{i}(4\lambda^2 + 2w^2 + z) & 4\lambda w + 2\mathrm{i}w' + \alpha/\lambda \\ 4\lambda w - 2\mathrm{i}w' + \alpha/\lambda & \mathrm{i}(4\lambda^2 + 2w^2 + z) \end{pmatrix} \boldsymbol{\Psi} \equiv \mathbf{B}(z,\lambda)\boldsymbol{\Psi}$$

Note that

$$\frac{\partial^2 \boldsymbol{\Psi}}{\partial z \partial \lambda} = \frac{\partial^2 \boldsymbol{\Psi}}{\partial \lambda \partial z} \qquad \Longleftrightarrow \qquad \frac{\partial \mathbf{A}}{\partial \lambda} - \frac{\partial \mathbf{B}}{\partial z} + \mathbf{AB} - \mathbf{BA} = 0$$

if and only if $w(z)$ satisfies $\mathrm{P_{II}}$

$$w'' = 2w^3 + zw + \alpha$$

$$\begin{aligned} \lambda = 0 \quad &\text{— regular singular point if } \alpha \neq 0 \\ \lambda = \infty \quad &\text{— irregular singular point} \end{aligned}$$

Direct Problem

- Obtain the monodromy data given $w(z_0)$ and $w'(z_0)$.

Inverse Problem

- Reconstruct $w(z_0)$ from the monodromy data.

Fokas & Ablowitz [1983], Fokas & Zhou [1992] show that the Riemann-Hilbert problem associated with $\mathrm{P_{II}}$ consists of finding the piecewise holomorphic 2×2 matrix valued function $\boldsymbol{\Psi}(\lambda)$, the fundamental solution, such that

- $\boldsymbol{\Psi}(\lambda)$ is holomorphic for $\lambda \in \mathbb{C} \setminus \bigcup_{k=1}^{6} \Gamma_k$, where Γ_k are the rays

$$\Gamma_k = \{\lambda \in \mathbb{C} \colon \ \arg\lambda = \tfrac{1}{6}(2k-1)\pi\}, \qquad k = 1, \ldots, 6$$

  oriented from zero to infinity, and as $\lambda \to \infty$

$$\boldsymbol{\Psi}(\lambda) = [\mathbf{I} + \mathcal{O}(\lambda^{-1})] \exp\left\{-\mathrm{i}(\tfrac{4}{3}\lambda^3 + z\lambda)\boldsymbol{\sigma}_3\right\}, \qquad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- On the rays Γ_k the jump conditions hold

$$\boldsymbol{\Psi}_{k+1}(\lambda) = \boldsymbol{\Psi}_k(\lambda)\mathbf{S}_k, \quad \lambda \in \Gamma_k$$

  where the **Stokes multipliers $\mathbf{S}_k$** are

$$\mathbf{S}_{2k-1} = \begin{pmatrix} 1 & 0 \\ s_{2k-1} & 1 \end{pmatrix}, \quad \mathbf{S}_{2k} = \begin{pmatrix} 1 & s_{2k} \\ 0 & 1 \end{pmatrix}$$

  and the **the monodromy data** s_k do not depend either on z or on λ and satisfy the constraints

$$\begin{aligned} & s_{k+3} = s_k, && k = 1, 2, 3 \\ & s_1 + s_2 + s_3 + s_1 s_2 s_3 = 2\mathrm{i}\sin(\pi\alpha) \end{aligned}$$

Theorem

(Flaschka & Newell [1980])

The monodromy data for the system

$$\frac{\partial \boldsymbol{\Psi}}{\partial \lambda} = \begin{pmatrix} -\mathrm{i}(4\lambda^2 + 2w^2 + z) & 4\lambda w + 2\mathrm{i}w' + \alpha/\lambda \\ 4\lambda w - 2\mathrm{i}w' + \alpha/\lambda & \mathrm{i}(4\lambda^2 + 2w^2 + z) \end{pmatrix} \boldsymbol{\Psi}$$

do not depend upon z *if and only if* $w(z)$ *satisfies* $\mathrm{P_{II}}$

$$w'' = 2w^3 + zw + \alpha$$

For the special case of $\mathrm{P_{II}}$ with $\alpha = 0$

$$w'' = 2w^3 + zw$$

then

$$s_2 = \frac{s_1^* - s_1}{1 - s_1 s_1^*}, \qquad s_3 = -s_1^*$$

and so the monodromy data is characterized by the complex parameter s_1.

Isomonodromy

- Each Painlevé equation has associated with it a linear equation — involving as parameters $w(z)$, $w'(z)$ and z — whose monodromy data is independent of z if $w(z)$ satisfies the Painlevé equation.

$$w(z) \sim k\,\mathrm{Ai}(z), \qquad z \to \infty$$

$$w(z) = d|z|^{-1/4} \sin\left(\tfrac{2}{3}|z|^{3/2} - \tfrac{3}{4}d^2 \ln|z| - \theta_0\right) + o(|z|^{-1/4}), \qquad z \to -\infty$$

Objective

- Express the parameters k, d and θ_0 in terms of the monodromy data s_1. Since s_1 is independent of z then we obtain the requisite connection formulae

Asymptotics as $z \to \infty$

- Since w and w' decay exponentially to zero then the computation of the monodromy data is reduced to the evaluation of an integral using the WKB method.

Asymptotics as $z \to -\infty$

- Replace w and w' in the monodromy problem by the leading terms in their asymptotic expansions and this obtain a singular perturbation systems in a small parameter. Applying a nonlinear version of the classical steepest descent method for oscillatory Riemann-Hilbert problems, yields the connection formulae

$$d^2(k) = -\pi^{-1} \ln(1 - k^2)$$

$$\theta_0(k) = \tfrac{3}{2}d^2 \ln 2 + \arg\left\{\Gamma\left(1 - \tfrac{1}{2}\mathrm{i}d^2\right)\right\} - \tfrac{1}{4}\pi$$

Classical Solutions of $\mathrm{P_{II}}$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

Theorem

1. $\mathrm{P_{II}}$ *has rational solutions if and only if*

$$\alpha = n$$

with $n \in \mathbb{Z}$*.*

2. $\mathrm{P_{II}}$ *has solutions expressible in terms of the Riccati equation*

$$\varepsilon w' = w^2 + \tfrac{1}{2}z$$

if and only if

$$\alpha = n + \tfrac{1}{2}$$

with $n \in \mathbb{Z}$*. The Riccati equation has solution*

$$w(z) = -\varepsilon\varphi'(z)/\varphi(z)$$

where

$$\varphi(z) = C_1\,\mathrm{Ai}(\zeta) + C_2\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/2}z$$

with $\mathrm{Ai}(\zeta)$ *and* $\mathrm{Bi}(\zeta)$ *Airy functions.*

Bäcklund Transformations

Definition

- A **Bäcklund transformation** maps solutions of a given Painlevé equation to solutions of the same Painlevé equation, though with different values of the parameters.

Example

(Gambier [1910])

Suppose that $w(z;\alpha)$ is a solution of $\mathrm{P_{II}}$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

then

$$\boldsymbol{\mathcal{S}} \qquad w(z;-\alpha) = -w(z;\alpha)$$

$$\boldsymbol{\mathcal{T}}_{\pm} \qquad w(z;\alpha\pm 1) = -w(z;\alpha) - \frac{2\alpha \pm 1}{2w^2(z;\alpha) \pm 2w'(z;\alpha) + z}$$

are also solutions of $\mathrm{P_{II}}$, provided that

$$2w^2(z;\alpha) \pm 2w'(z;\alpha) + z \neq 0$$

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\boldsymbol{\mathcal{T}}_+} & w(z;\alpha-1) & \xrightarrow{\boldsymbol{\mathcal{T}}_+} & w(z;\alpha) & \xrightarrow{\boldsymbol{\mathcal{T}}_+} & w(z;\alpha+1) & \xrightarrow{\boldsymbol{\mathcal{T}}_+} & \ldots \\ \ldots & \xleftarrow{\boldsymbol{\mathcal{T}}_-} & w(z;\alpha-1) & \xleftarrow{\boldsymbol{\mathcal{T}}_-} & w(z;\alpha) & \xleftarrow{\boldsymbol{\mathcal{T}}_-} & w(z;\alpha+1) & \xleftarrow{\boldsymbol{\mathcal{T}}_-} & \ldots \end{array}$$

Associated Difference Equations

Example

(Fokas, Grammaticos & Ramani [1993])

Suppose that $w(z;\alpha)$ is a solution of $\mathrm{P_{II}}$

$$w'' = 2w^3 + zw + \alpha$$

Then

$$w(z;\alpha+1) = -w(z;\alpha) - \frac{2\alpha+1}{2w^2(z;\alpha) + 2w'(z;\alpha) + z}$$
$$w(z;\alpha-1) = -w(z;\alpha) - \frac{2\alpha-1}{2w^2(z;\alpha) - 2w'(z;\alpha) + z}$$

are also solutions of $\mathrm{P_{II}}$. Eliminating $w'(z;\alpha)$ yields

$$\frac{2\alpha+1}{w(z;\alpha+1) + w(z;\alpha)} + \frac{2\alpha-1}{w(z;\alpha) + w(z;\alpha-1)} + 4w^2(z;\alpha) + 2z = 0$$

Hence setting

$$w_{\alpha\pm1} = w(z;\alpha\pm1), \qquad w_\alpha = w(z;\alpha)$$

gives

$$\boxed{\frac{2\alpha+1}{w_{\alpha+1} + w_\alpha} + \frac{2\alpha-1}{w_\alpha + w_{\alpha-1}} + 4w_\alpha^2 + 2z = 0}$$

which is an alternative form of $\mathrm{dP_I}$

Therefore hierarchies of solutions of $\mathrm{P_{II}}$ satisfy both:

- a differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

- a difference equation

$$\frac{2\alpha+1}{w_{\alpha+1}+w_\alpha} + \frac{2\alpha-1}{w_\alpha+w_{\alpha-1}} + 4w_\alpha^2 + 2z = 0 \qquad \text{a-dP}_\mathrm{I}$$

Remarks

- This is analogous to the situation for classical special functions such as Bessel functions and Hermite functions which satisfy both a differential equation and a difference equation.
- For $\mathrm{P_{II}}$, the independent variable z varies and the parameter α is fixed, whilst for a-dP$_\mathrm{I}$, z is a fixed parameter and α varies.
- The asymptotics of w_α as $\alpha \to \infty$ can be studied through the asymptotics of $w(z;\alpha)$ as $\alpha \to \infty$.

Rational Solutions of $\mathrm{P_{II}}$ — Vorob'ev–Yablonskii Polynomials

Theorem (Yablonskii & Vorob'ev [1965])

Suppose that $Q_n(z)$ satisfies the recursion relation

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4\left[Q_nQ_n'' - (Q_n')^2\right]$$

with $Q_0(z) = 1$ and $Q_1(z) = z$. Then the rational function

$$w(z;n) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{\frac{Q_{n-1}(z)}{Q_n(z)}\right\} = \frac{Q_{n-1}'(z)}{Q_{n-1}(z)} - \frac{Q_n'(z)}{Q_n(z)}$$

satisfies $\mathrm{P_{II}}$

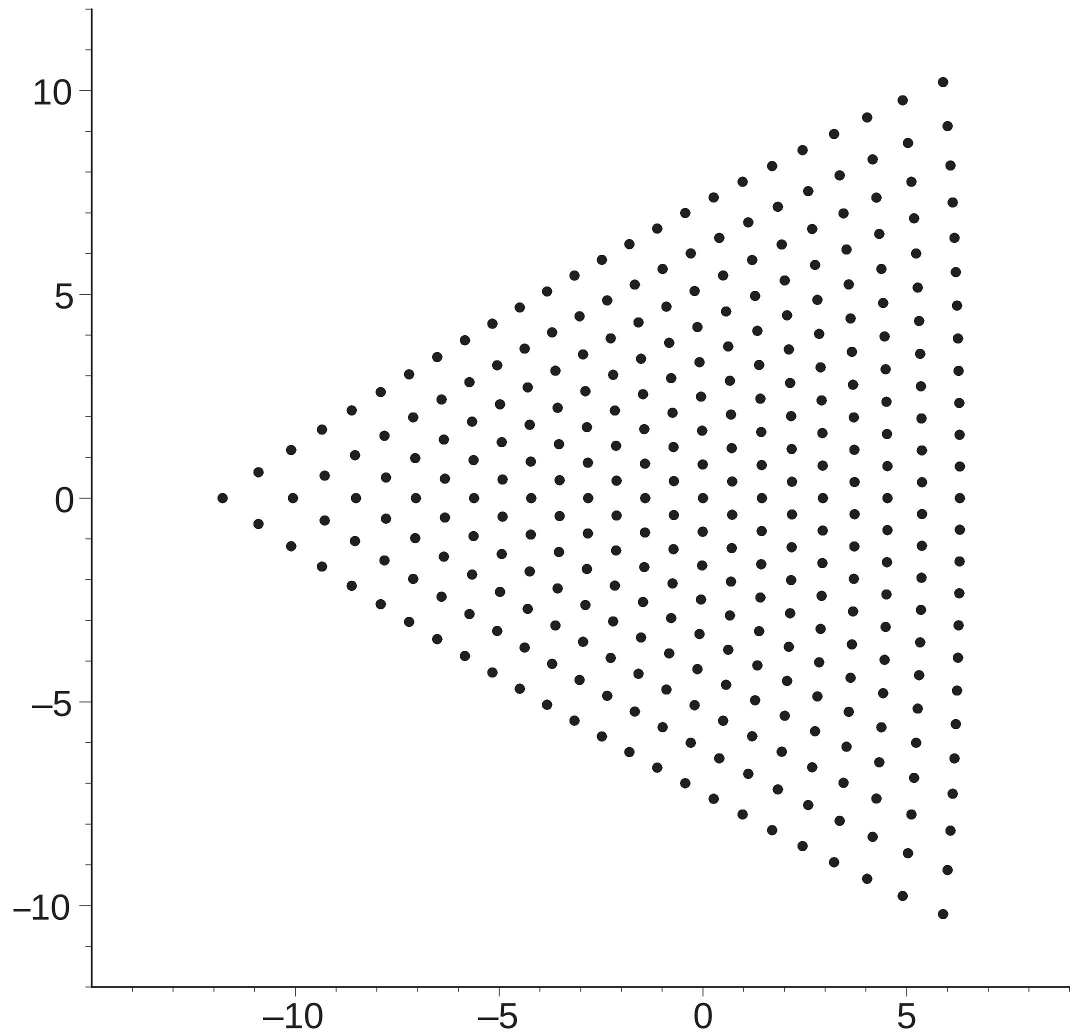
$$w'' = 2w^3 + zw + \alpha$$

with $\alpha = n \in \mathbb{Z}^+$. Further $w(z;0) = 0$ and $w(z;-n) = -w(z;n)$.

The Yablonskii–Vorob'ev polynomials are monic polynomials of degree $\frac{1}{2}n(n+1)$

$$\begin{aligned}
Q_2(z) &= z^3 + 4\\
Q_3(z) &= z^6 + 20z^3 - 80\\
Q_4(z) &= z^{10} + 60z^7 + 11200z\\
Q_5(z) &= z^{15} + 140z^{12} + 2800z^9 + 78400z^6 - 313600z^3 - 6272000\\
Q_6(z) &= z^{21} + 280z^{18} + 18480z^{15} + 627200z^{12} - 17248000z^9 + 1448832000z^6\\
&\qquad + 19317760000z^3 - 38635520000
\end{aligned}$$

Roots of Yablonskii–Vorob’ev Polynomial $Q_{25}(z)$



Theorem

(Fukutani, Okamoto & Umemura [2000])

- *The polynomial* $Q_n(z)$ *has* $\frac{1}{2}n(n+1)$ *simple roots.*
- *The polynomials* $Q_n(z)$ *and* $Q_{n+1}(z)$ *have no common roots.*

Theorem

(PAC & Joshi)

- *The polynomials* $Q_{2n-1}(z)$ *and* $Q_{2n}(z)$ *have* n *<u>real</u> roots.*
- *The <u>real</u> roots of* $Q_{n-1}(z)$ *and* $Q_{n+1}(z)$ *and of* $Q_n(z)$ *and* $Q_{n+1}(z)$ *interlace.*

Remarks

- Since $Q_n(z)$ has only simple roots then

$$Q_n(z) = \prod_{j=1}^{n(n+1)/2} (z - a_{n,j})$$

where $a_{n,j}$, for $j = 1, 2, \ldots, \frac{1}{2}n(n+1)$, are the roots. These roots satisfy

$$\sum_{j=1, j\neq k}^{n(n+1)/2} \frac{1}{(a_{n,j} - a_{n,k})^3} = 0, \qquad j = 1, 2, \ldots, \tfrac{1}{2}n(n+1)$$

- If $A_n = \max_{1\leq j\leq n(n+1)/2} \{|a_{n,j}|\}$ then $n^{2/3} \leq A_{n+2} \leq 4n^{2/3}$ (Kametaka [1983]).

Determinantal Form of Rational Solutions of $\mathrm{P_{II}}$

Theorem

(Kajiwara & Ohta [1996])

Let $\varphi_k(z)$ *be the polynomial defined by*

$$\sum_{j=0}^{\infty} \varphi_j(z)\lambda^j = \exp\left(z\lambda - \tfrac{4}{3}\lambda^3\right), \qquad \varphi_j(z) = {}_1F_2(a; b_1, b_2; z^3/36)$$

and $\tau_n(z)$ *be the* $n \times n$ *determinant given by*

$$\tau_n(z) = \begin{vmatrix} \varphi_n & \varphi_{n+1} & \cdots & \varphi_{2n-1} \\ \varphi_{n-2} & \varphi_{n-1} & \cdots & \varphi_{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{-n+2} & \varphi_{-n+3} & \cdots & \varphi_1 \end{vmatrix} \equiv \begin{vmatrix} \varphi_1 & \varphi_3 & \cdots & \varphi_{2n-1} \\ \varphi_1' & \varphi_3' & \cdots & \varphi_{2n-1}' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_3^{(n-1)} & \cdots & \varphi_{2n-1}^{(n-1)} \end{vmatrix}$$

then

$$w_n(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln\left\{\frac{\tau_{n-1}(z)}{\tau_n(z)}\right\}$$

satisfies $\mathrm{P_{II}}$ *with* $\alpha = n$*.*

Remarks

- Flaschka and Newell [1980], following the earlier work of Airault [1979], expressed the rational solutions of $\mathrm{P_{II}}$ as the logarithmic derivatives of determinants.
- The Yablonskii–Vorob'ev polynomials can be expressed as Schur polynomials.

Discriminants of Yablonskii–Vorob'ev Polynomials

- Let $f(z) = z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0$ be a monic polynomial of degree m with roots $\alpha_1, \alpha_2, \ldots, \alpha_m$, so $f(z) = \prod_{j=1}^{m}(z-\alpha_j)$.
- The **discriminant** of $f(z)$ is $\mathrm{Dis}(f) = \prod_{1\le j<k\le m}(\alpha_j - \alpha_k)^2$.
- For the Yablonskii–Vorob'ev polynomials

$$
\begin{aligned}
\mathrm{Dis}(Q_2(z)) &= -2^4\,3^3\\
\mathrm{Dis}(Q_3(z)) &= 2^{20}\,3^{12}\,5^5\\
\mathrm{Dis}(Q_4(z)) &= 2^{60}\,3^{27}\,5^{20}\,7^7\\
\mathrm{Dis}(Q_5(z)) &= 2^{140}\,3^{66}\,5^{45}\,7^{28}\\
\mathrm{Dis}(Q_6(z)) &= -2^{280}\,3^{147}\,5^{80}\,7^{63}\,11^{11}\\
\mathrm{Dis}(Q_7(z)) &= 2^{504}\,3^{270}\,5^{125}\,7^{112}\,11^{44}\,13^{13}\\
\mathrm{Dis}(Q_8(z)) &= 2^{840}\,3^{450}\,5^{195}\,7^{175}\,11^{99}\,13^{52}\\
\mathrm{Dis}(Q_9(z)) &= 2^{1320}\,3^{702}\,5^{305}\,7^{252}\,11^{176}\,13^{117}\,17^{17}\\
\mathrm{Dis}(Q_{10}(z)) &= -2^{1980}\,3^{1026}\,5^{455}\,7^{343}\,11^{275}\,13^{208}\,17^{68}\,19^{19}\\
\mathrm{Dis}(Q_{11}(z)) &= 2^{2860}\,3^{1443}\,5^{645}\,7^{469}\,11^{396}\,13^{325}\,17^{153}\,19^{76}\\
\mathrm{Dis}(Q_{12}(z)) &= 2^{4004}\,3^{1974}\,5^{875}\,7^{651}\,11^{539}\,13^{468}\,17^{272}\,19^{171}\,23^{23}
\end{aligned}
$$

Hamiltonian Representation

$\mathrm{P_{II}}$ can be written as the Hamiltonian system

$$\begin{cases} q' = \dfrac{\partial \mathcal{H}_{\mathrm{II}}}{\partial p} = p - q^2 - \frac{1}{2}z \\ p' = -\dfrac{\partial \mathcal{H}_{\mathrm{II}}}{\partial q} = 2qp + \alpha + \frac{1}{2} \end{cases} \tag{II}$$

where $\mathcal{H}_{\mathrm{II}}(q,p,\alpha)$ is the Hamiltonian defined by

$$\mathcal{H}_{\mathrm{II}}(q,p,\alpha) = \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q$$

Eliminating p then $q = w$ satisfies $\mathrm{P_{II}}$ whilst eliminating q yields

$$pp'' = \tfrac{1}{2}(p')^2 + 2p^3 - zp^2 - \tfrac{1}{2}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{P}_{34}$$

Theorem (Okamoto [1986])

The function $\sigma(z) = \mathcal{H}_{\mathrm{II}} \equiv \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q$ *satisfies*

$$(\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(z\sigma' - \sigma) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2$$

and conversely

$$q(z) = \frac{2\sigma''(z) + \alpha + \frac{1}{2}}{4\sigma'(z)}, \qquad p(z) = -2\sigma'(z)$$

is a solution of (II).

Remarks on Hamiltonians for Painlevé Equations

1. Each Hamiltonian function $\sigma = \mathcal{H}_{\mathrm{J}}$ satisfies a second-order second-degree ordinary differential equation whose solutions are in a 1-1 correspondence with solutions of the associated Painlevé equation through

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\mathrm{J}}}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\mathrm{J}}}{\partial q}$$

   since

$$q = \mathrm{F}_{\mathrm{J}}(\sigma, \sigma', \sigma'', z), \qquad p = \mathrm{G}_{\mathrm{J}}(\sigma, \sigma', \sigma'', z)$$

   for suitable functions $\mathrm{F}_{\mathrm{J}}(\sigma, \sigma', \sigma'', z)$ and $\mathrm{G}_{\mathrm{J}}(\sigma, \sigma', \sigma'', z)$. Thus given q and p one can determine σ and conversely, given σ one can determine q and p.

2. The ordinary differential equations which the σ functions satisfy are part of the classification of second-order, second-degree equations of Painlevé type by Cosgrove and Scoufis [1993]. They were first derived by Chazy [1911] and later rederived by Bureau [1964].

3. The Hamiltonian functions $\sigma = \mathcal{H}_{\mathrm{J}}$ frequently arise in applications, e.g.
   - Random Matrix Theory (Tracy and Widom [1994–1996]; see also Forrester and Witte [2001, 2002])
   - Statistical Physics (Jimbo, Miwa, Mori and Sato [1980])

Affine Weyl Groups

Okamoto [1986–1987] has shown that P_{II}–P_{VI} admit the action of **affine Weyl groups** of types $A_1^{(1)}$, $C_2^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$ and $D_4^{(1)}$, respectively, which are related to the associated Bäcklund transformations.

Example Suppose that $w(z;\alpha)$ is a solution of P_{II}

$$w'' = 2w^3 + zw + \alpha$$

Then

$$\begin{array}{ll} \boldsymbol{\mathcal{S}} & w(z;-\alpha) = -w(z;\alpha) \\ \boldsymbol{\mathcal{T}}_{\pm} & w(z;\alpha \pm 1) = -w(z;\alpha) - \dfrac{2\alpha \pm 1}{2w^2(z;\alpha) \pm 2w'(z;\alpha) + z} \end{array}$$

are also solutions of P_{II}. Since the composition of two Bäcklund transformations is a Bäcklund transformation, consider the group of Bäcklund transformations.

- The Bäcklund transformations $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{T}}_+$ (or $\boldsymbol{\mathcal{T}}_-$) generate the group $\mathcal{W} = \langle \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{T}}_+ \rangle$, which is isomorphic to the **affine Weyl group** of type $A_1^{(1)}$, with
$$\boldsymbol{\mathcal{S}}^2 = \boldsymbol{\mathcal{T}}_+\boldsymbol{\mathcal{T}}_- = \boldsymbol{\mathcal{T}}_-\boldsymbol{\mathcal{T}}_+ = \boldsymbol{\mathcal{I}}$$
where $\boldsymbol{\mathcal{I}}$ is the identity transformation.
- On the space of the parameter α, the group is generated by a reflection $\boldsymbol{\mathcal{S}}$ and a translation $\boldsymbol{\mathcal{T}}_+$ (or $\boldsymbol{\mathcal{T}}_-$), with
$$\boldsymbol{\mathcal{S}}(\alpha) = -\alpha, \qquad \boldsymbol{\mathcal{T}}_{\pm}(\alpha) = \alpha \pm 1$$

Classical Solutions of $\mathrm{P_{IV}}$

$$\frac{d^2w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

Theorem

1. $\mathrm{P_{IV}}$ *has rational solutions if and only if*

$$\text{(i)} \quad (\alpha,\beta) = \big(m, -2(2n-m+1)^2\big) \quad \text{or} \quad \text{(ii)} \quad (\alpha,\beta) = \big(m, -2(2n-m+\tfrac{1}{3})^2\big)$$

with $m, n \in \mathbb{Z}$. *Further the rational solutions for these parameter values are unique.*

2. $\mathrm{P_{IV}}$ *has solutions expressible in terms of the Riccati equation*

$$zw' = \varepsilon(w^2 + 2zw) - 2(1+\varepsilon\alpha)$$

if and only if

$$\text{(i)} \quad \beta = -2(2n+1+\varepsilon\alpha)^2 \qquad \text{or} \qquad \text{(ii)} \quad \beta = -2n^2$$

with $n \in \mathbb{Z}$ *and* $\varepsilon = \pm 1$. *The Riccati equation has solution*

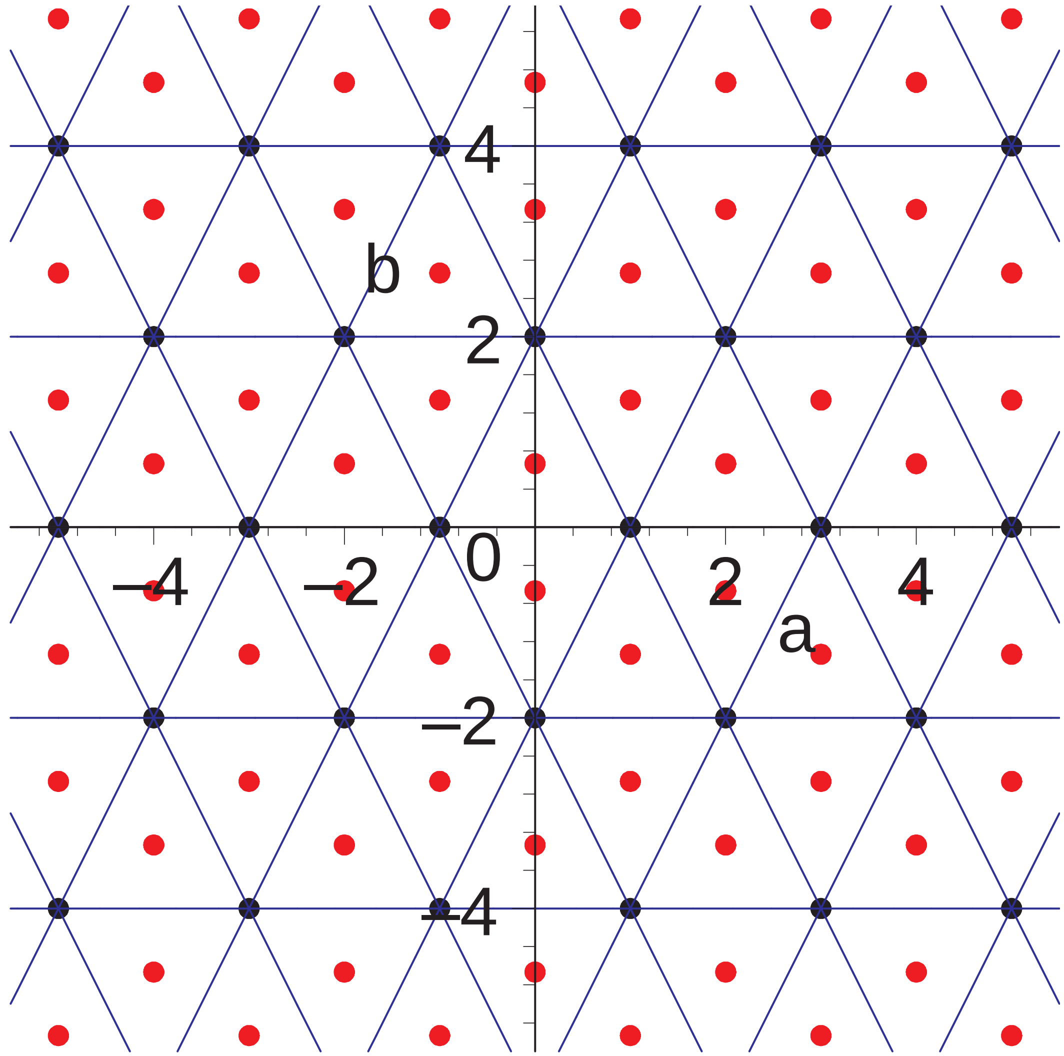
$$w(z) = -\varepsilon\varphi'(z)/\varphi(z)$$

where

$$\varphi(z) = \{C_1 D_\nu(\zeta) + C_2 D_{-\nu}(\zeta)\}\exp(\tfrac{1}{2}\varepsilon z^2), \qquad \nu = -\tfrac{1}{2}(1+2\alpha+\varepsilon), \quad \zeta = \sqrt{2}\, z$$

with $D_\nu(\zeta)$ *the parabolic cylinder function.*

Rational and Special Function Solutions of P_{IV}



$\mathrm{P_{IV}}$ — Generalized Hermite Polynomials

Theorem (Noumi & Yamada [1998])

Suppose that $H_{m,n}(z)$, *with* $m,n\geq 0$, *satisfies the recurrence relations*

$$2mH_{m+1,n}H_{m-1,n} = H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2 + 2mH^2_{m,n}$$
$$-2nH_{m,n+1}H_{m,n-1} = H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2 - 2nH^2_{m,n}$$

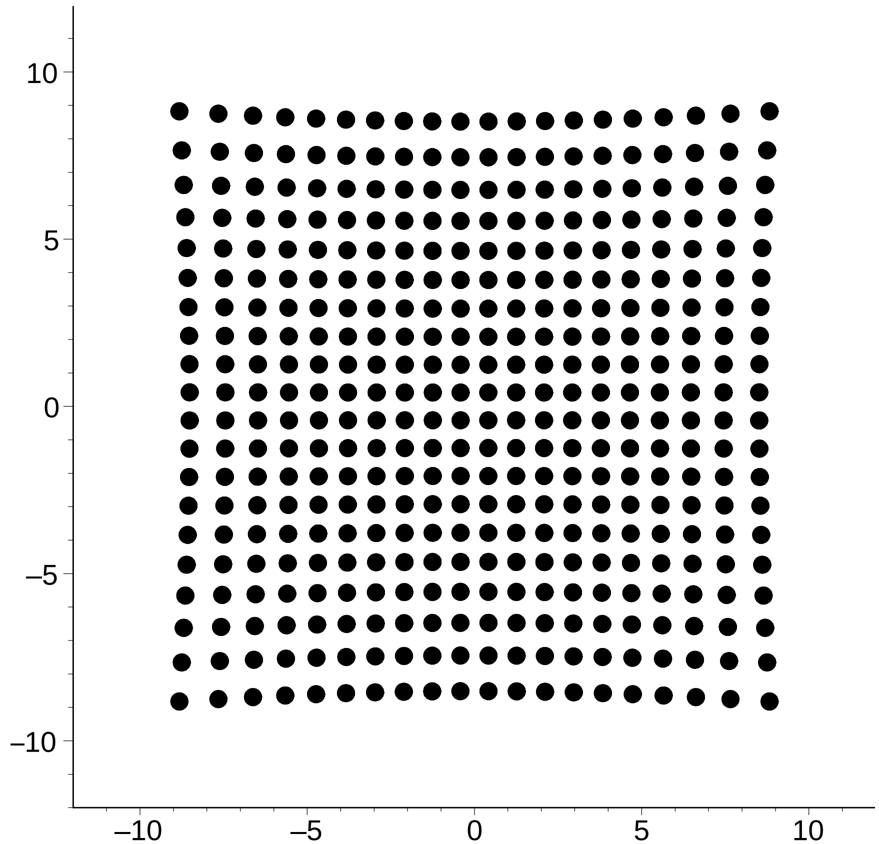
with $H_{0,0}(z) = H_{1,0}(z) = H_{0,1}(z) = 1$, $H_{1,1}(z) = 2z$ *then*

$$w^{(\mathrm{i})}_{m,n} = w(z;\alpha^{(\mathrm{i})}_{m,n},\beta^{(\mathrm{i})}_{m,n}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{H_{m+1,n}}{H_{m,n}}\right)$$
$$w^{(\mathrm{ii})}_{m,n} = w(z;\alpha^{(\mathrm{ii})}_{m,n},\beta^{(\mathrm{ii})}_{m,n}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{H_{m,n}}{H_{m,n+1}}\right)$$
$$w^{(\mathrm{iii})}_{m,n} = w(z;\alpha^{(\mathrm{iii})}_{m,n},\beta^{(\mathrm{iii})}_{m,n}) = -2z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{H_{m,n+1}}{H_{m+1,n}}\right)$$

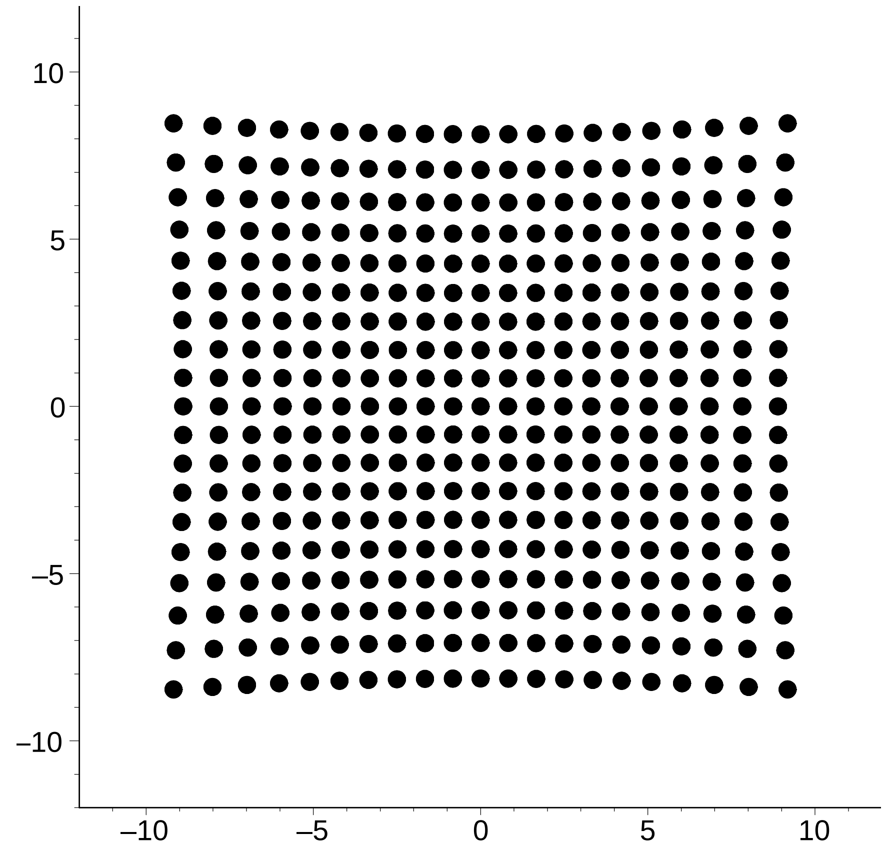
are respectively solutions of $\mathrm{P_{IV}}$ *for*

$$(\alpha^{(\mathrm{i})}_{m,n},\beta^{(\mathrm{i})}_{m,n}) = (2m+n+1, -2n^2)$$
$$(\alpha^{(\mathrm{ii})}_{m,n},\beta^{(\mathrm{ii})}_{m,n}) = (-m-2n-1, -2m^2)$$
$$(\alpha^{(\mathrm{iii})}_{m,n},\beta^{(\mathrm{iii})}_{m,n}) = (n-m, -2(m+n+1)^2)$$

Roots of the Generalized Hermite Polynomials $H_{m,n}(z)$



$H_{20,20}(z)$



$H_{21,19}(z)$

$m \times n$ "rectangles"

Properties of the Generalized Hermite Polynomials

- $H_{n,1}(z) = H_n(z)$, where $H_n(z)$ is the Hermite polynomial defined by

$$H_n(z) = (-1)^n \exp\left(z^2\right) \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left\{\exp\left(-z^2\right)\right\}$$

- The polynomial $H_{m,n}(z)$ can also be expressed as the multiple integral

$$H_{m,n}(z) = \frac{\pi^{m/2} \prod_{k=1}^{m} k!}{2^{m(m+2n-1)/2}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} \prod_{i=1}^{n} \prod_{j=i+1}^{n} (x_i - x_j)^2 \prod_{k=1}^{n} (z - x_k)^m$$
$$\times \exp\left(-x_1^2 - x_2^2 - \ldots - x_n^2\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

which arises in random matrix theory (Brézin & Hikami [2000], Chan & Feigen [2006], Forrester & Witte [2001])

- The monic polynomials orthogonal on the real line with respect to the weight

$$w(x; z, m) = (x - z)^m \exp(-x^2)$$

satisfy the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + a_n(z; m) p_n(x) + b_n(z; m) p_{n-1}(x)$$

where

$$a_n(z; m) = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}z} \ln\left(\frac{H_{n+1,m}}{H_{n,m}}\right), \qquad b_n(z; m) = \frac{n H_{n+1,m} H_{n-1,m}}{2 H_{n,m}^2}$$

(Chan & Feigen [2006])

$\mathrm{P_{IV}}$ — Generalized Okamoto Polynomials

Theorem

(Noumi & Yamada [1998])

Suppose that $Q_{m,n}(z)$, with $m,n\in\mathbb{Z}$, satisfies the recurrence relations

$$Q_{m+1,n}Q_{m-1,n}=\tfrac{9}{2}\Big[Q_{m,n}Q''_{m,n}-\big(Q'_{m,n}\big)^2\Big]+\big[2z^2+3(2m+n-1)\big]Q^2_{m,n}$$
$$Q_{m,n+1}Q_{m,n-1}=\tfrac{9}{2}\Big[Q_{m,n}Q''_{m,n}-\big(Q'_{m,n}\big)^2\Big]+\big[2z^2+3(1-m-2n)\big]Q^2_{m,n}$$

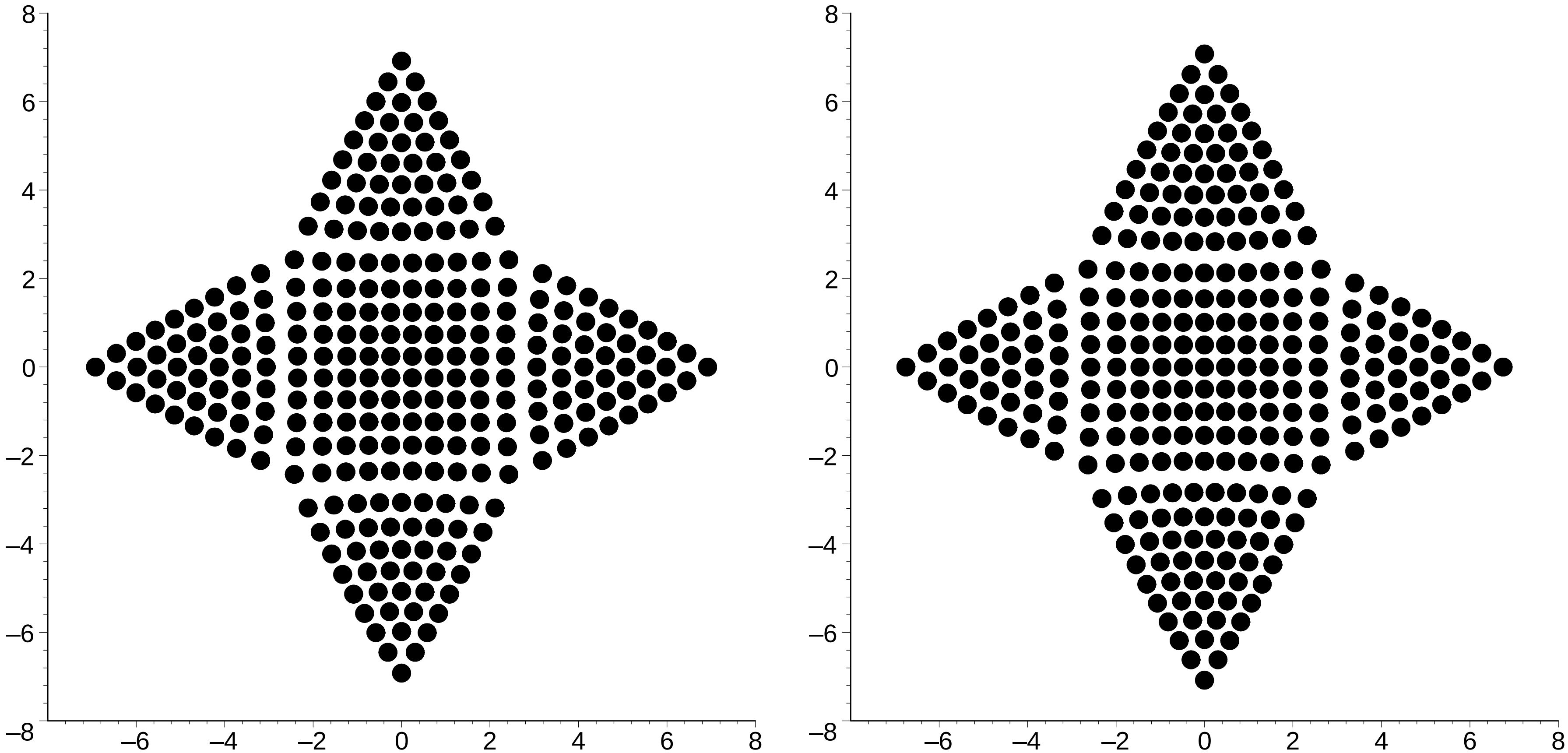
with $Q_{0,0}=Q_{1,0}=Q_{0,1}=1$ and $Q_{1,1}=\sqrt{2}\,z$ then

$$\widetilde{w}^{(\mathrm{i})}_{m,n}=w(z;\widetilde{\alpha}^{(\mathrm{i})}_{m,n},\widetilde{\beta}^{(\mathrm{i})}_{m,n})=-\tfrac{2}{3}z+\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{Q_{m+1,n}}{Q_{m,n}}\right)$$
$$\widetilde{w}^{(\mathrm{ii})}_{m,n}=w(z;\widetilde{\alpha}^{(\mathrm{ii})}_{m,n},\widetilde{\beta}^{(\mathrm{ii})}_{m,n})=-\tfrac{2}{3}z+\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{Q_{m,n}}{Q_{m,n+1}}\right)$$
$$\widetilde{w}^{(\mathrm{iii})}_{m,n}=w(z;\widetilde{\alpha}^{(\mathrm{iii})}_{m,n},\widetilde{\beta}^{(\mathrm{iii})}_{m,n})=-\tfrac{2}{3}z+\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{Q_{m,n+1}}{Q_{m+1,n}}\right)$$

are respectively solutions of $\mathrm{P_{IV}}$ for

$$(\widetilde{\alpha}^{(\mathrm{i})}_{m,n},\widetilde{\beta}^{(\mathrm{i})}_{m,n})=(2m+n,-2(n-\tfrac{1}{3})^2)$$
$$(\widetilde{\alpha}^{(\mathrm{ii})}_{m,n},\widetilde{\beta}^{(\mathrm{ii})}_{m,n})=(-m-2n,-2(m-\tfrac{1}{3})^2)$$
$$(\widetilde{\alpha}^{(\mathrm{iii})}_{m,n},\widetilde{\beta}^{(\mathrm{iii})}_{m,n})=(n-m,-2(m+n+\tfrac{1}{3})^2)$$

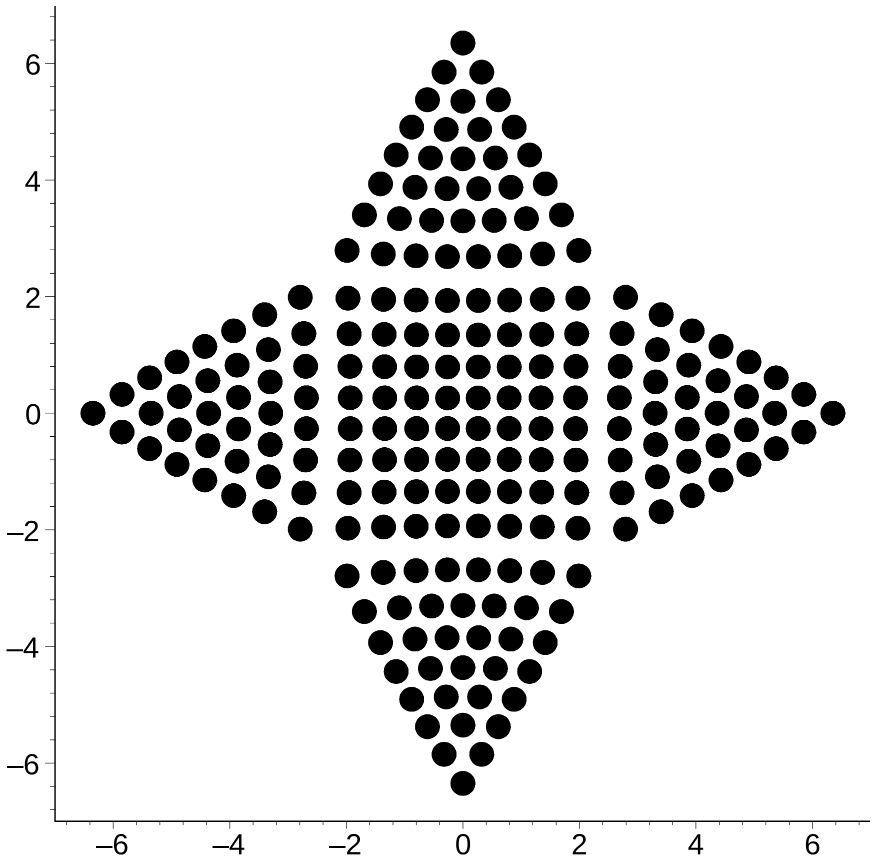
Roots of the Generalized Okamoto Polynomials $Q_{m,n}(z)$, $m,n > 0$



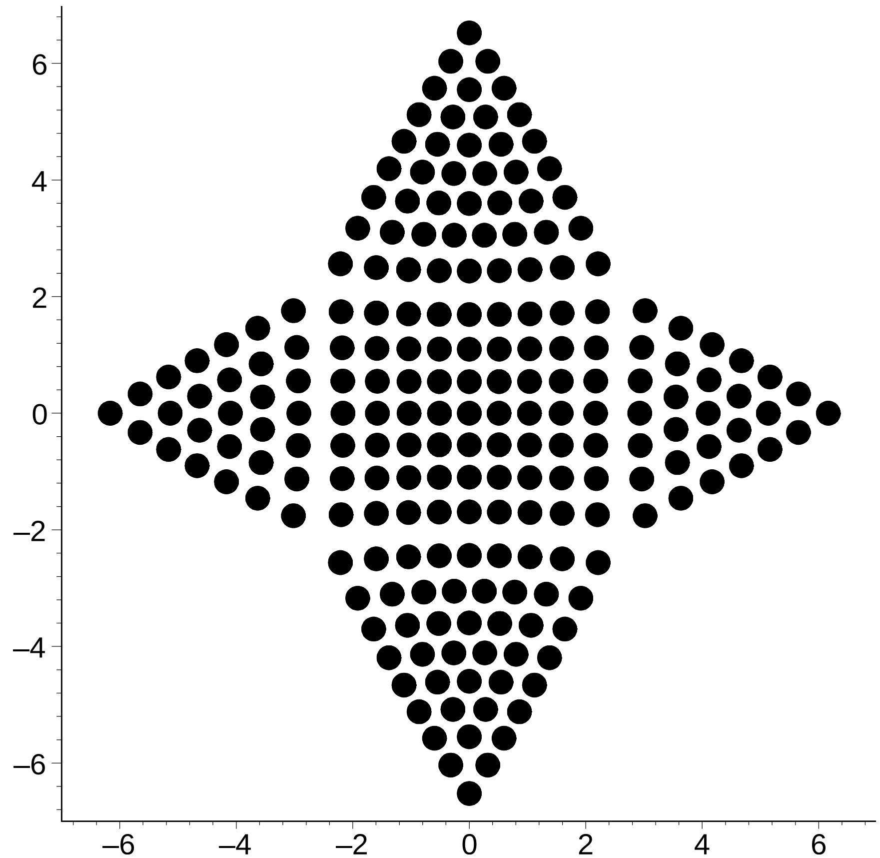
$Q_{10,10}(z)$ $\qquad$ $Q_{11,9}(z)$

$m \times n$ "rectangles" and "equilateral triangles" with sides $m-1$ and $n-1$

Roots of the Generalized Okamoto Polynomials $Q_{m,n}(z)$, $m,n < 0$



$Q_{-8,-8}(z)$



$Q_{-9,-7}(z)$

$m \times n$ "rectangles" and "equilateral triangles" with sides m and n

Asymptotics of $\mathrm{P_{IV}}$ — Nonlinear Harmonic Oscillator

Consider the special case of $\mathrm{P_{IV}}$ where $w(z) = 2\sqrt{2}\, y^2(x)$ and $x = \sqrt{2}\, z$, with $\alpha = 2\nu+1$ and $\beta = 0$, i.e.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 3y^5 + 2xy^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})y \tag{1}$$

and the boundary condition

$$y(x) \to 0, \qquad \text{as} \quad x \to +\infty \tag{2}$$

This equation has solutions have exponential decay as $x \to \pm\infty$ and so are nonlinear analogues of bound states for the linear harmonic oscillator.

Let $y_k(x)$ denote the unique solution of (1) which is asymptotic to $kD_\nu(x)$, i.e.

$$\frac{\mathrm{d}^2 y_k}{\mathrm{d}x^2} = 3y_k^5 + 2xy_k^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})y_k$$

with boundary condition

$$y_k(x) \sim kD_\nu(x), \qquad \text{as} \quad x \to +\infty$$

- If $0 \le k < k_*$, where
$$k_*^2 = \frac{1}{2\sqrt{2\pi}\,\Gamma(\nu+1)}$$
then this solution exists for all real z as $z \to -\infty$.

- If $\nu = n \in \mathbb{N}$
$$y_k(x) \sim \frac{k D_n(x)}{\sqrt{1 - 2\sqrt{2\pi}\, n!\, k^2}}, \qquad \text{as} \quad x \to -\infty$$

- If $\nu \notin \mathbb{Z}$, then for some d and $\theta_0 \in \mathbb{R}$,
$$y_k(x) = (-1)^\mu \left(-\tfrac{1}{6}x\right)^{1/2} + d|x|^{-1/2} \sin\left(\frac{x^2}{2\sqrt{3}} - \frac{4d^2}{\sqrt{3}} \ln|x| - \theta_0\right) + \mathcal{O}\left(|x|^{-3/2}\right), \qquad \text{as} \quad x \to -\infty$$
where $\mu = [\nu + 1]$, the integer part of $\nu + 1$. Its & Kapaev [1998] determined the connection formulae for $d(k;\nu)$ and $\theta_0(k;\nu)$.

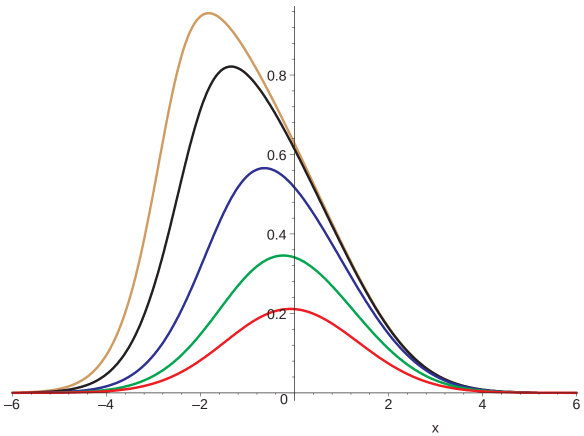
- If $k = k_*$, then
$$y_k(x) \sim \operatorname{sgn}(k) \left(-\tfrac{1}{2}x\right)^{1/2}, \qquad \text{as} \quad x \to -\infty$$

- If $k > k_*$ then $y_k(x)$ has a pole at a finite x_0 depending on k, so
$$y_k(x) \sim \operatorname{sgn}(k)(x - x_0)^{-1/2}, \qquad \text{as} \quad x \downarrow x_0$$

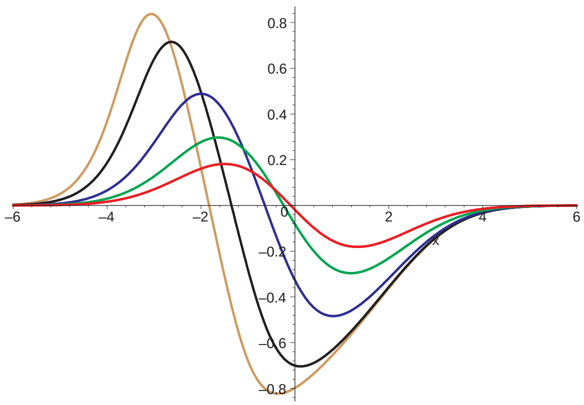
The first two bound state solutions are

$$y_k(x;0) = \frac{k\exp(-\frac{1}{4}x^2)}{\sqrt{1-k^2\psi(x)}} \equiv \Psi_k(x), \qquad y_k(x;1) = \frac{(x+2\Psi_k^2)\Psi_k}{\sqrt{1-2x\Psi_k^2-4\Psi_k^4}}$$

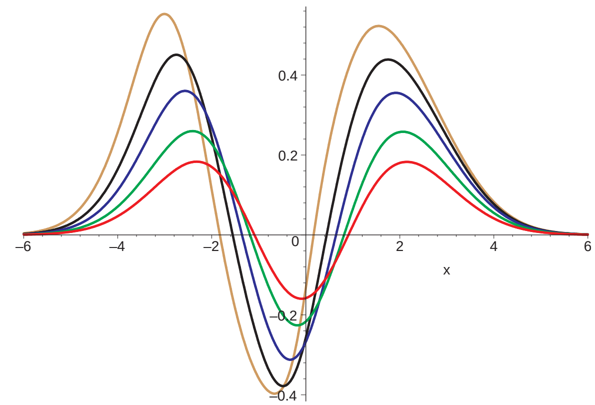
where $\psi(x) = \sqrt{2\pi}\,\operatorname{erfc}\left(\frac{1}{2}\sqrt{2}\,x\right)$ [note that $\psi(\infty)=0$ and $\psi(-\infty)=2\sqrt{2\pi}$].



$y_k(x;0)$
$0.2, 0.3, 0.4, 0.44, 0.445$



$y_k(x;1)$
$0.2, 0.3, 0.4, 0.44, 0.445$

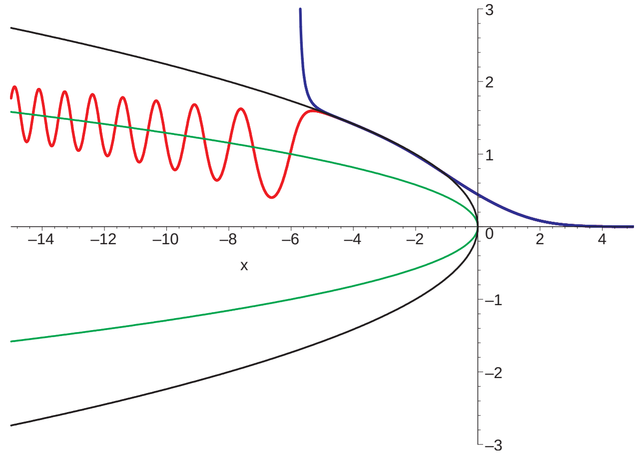


$y_k(x;2)$
$0.15, 0.2, 0.25, 0.28, 0.3$

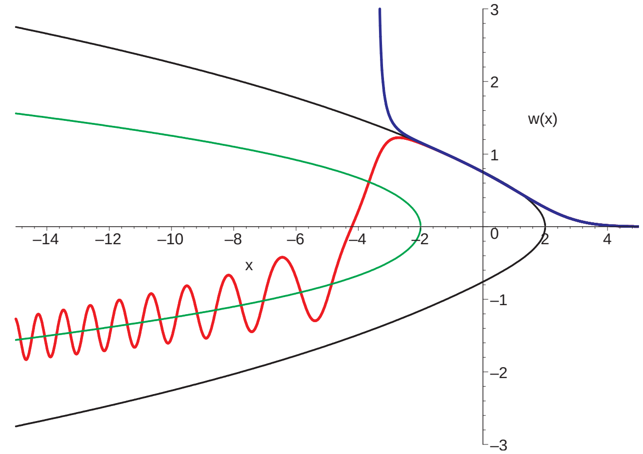
For $n \in \mathbb{Z}^+$, $y_k(x;n)$ exists for all x, has n distinct zeros and decays exponentially to zero as $x \to \pm\infty$ with asymptotic behaviour

$$y_k(x;n) \sim \begin{cases} k\exp(-\frac{1}{4}x^2), & \text{as} \quad x\to\infty \\ \dfrac{k\exp(-\frac{1}{4}x^2)}{\sqrt{1-2\sqrt{2\pi}\,n!\,k^2}}, & \text{as} \quad x\to-\infty \end{cases} \qquad k^2 < \frac{1}{2\sqrt{2\pi}\,n!}$$

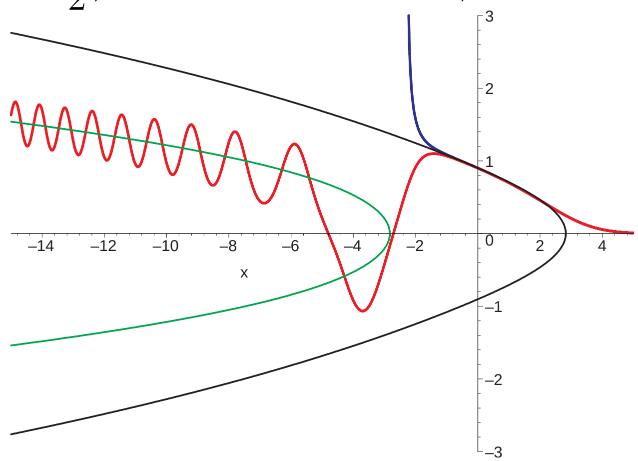
$$y_k'' = 3y_k^5 + 2xy_k^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})y_k, \qquad y_k(x) \sim kD_\nu(x), \quad \text{as} \quad x \to +\infty$$



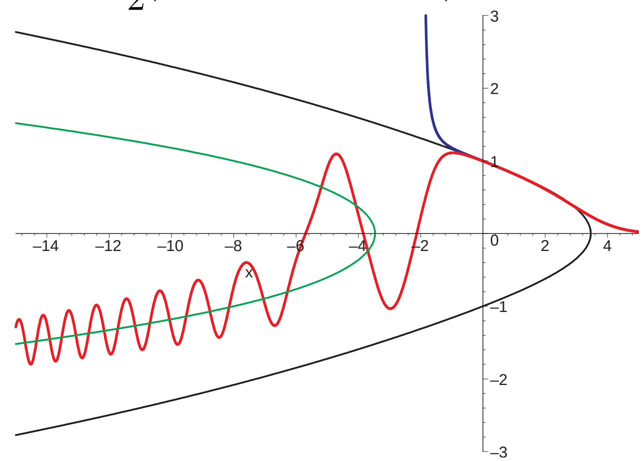
$\nu = -\frac{1}{2}, k = 0.33554691, 0.33554692$



$\nu = \frac{1}{2}, k = 0.47442, 0.47443$



$\nu = \frac{3}{2}, k = 0.38736, 0.38737$



$\nu = \frac{5}{2}, k = 0.244992, 0.244993$

Symmetric Form of $\mathrm{P_{IV}}$

(Bureau [1980], Veselov and Shabat [1993], Adler [1994], Noumi & Yamada [1998])

Consider the symmetric $\mathrm{P_{IV}}$ system

$$\varphi_0' + \varphi_0(\varphi_1 - \varphi_2) = 2\mu_0 \tag{1a}$$

$$\varphi_1' + \varphi_1(\varphi_2 - \varphi_0) = 2\mu_1 \tag{1b}$$

$$\varphi_2' + \varphi_2(\varphi_0 - \varphi_1) = 2\mu_2 \tag{1c}$$

where μ_0, μ_1 and μ_2 are constants, φ_0, φ_1 and φ_2 are functions of z, with

$$\mu_0 + \mu_1 + \mu_2 + 1 = 0, \qquad \varphi_0 + \varphi_1 + \varphi_2 + 2z = 0$$

Eliminating φ_1 and φ_2, then φ_0 satisfies $\mathrm{P_{IV}}$

$$\varphi_0\varphi_0'' = \tfrac{1}{2}(\varphi_0')^2 + \tfrac{3}{2}\varphi_0^4 + 4z\varphi_0^3 + 2(z^2 - \alpha)\varphi_0^2 + \beta$$

with

$$\alpha = \mu_2 - \mu_0, \qquad \beta = -2\mu_0^2$$

The system (1) is associated with the affine Weyl group $A_2^{(1)}$. Note that solving (1a) and (2) for φ_1 and φ_2 yields

$$\varphi_1 = -\frac{\varphi_0' + \varphi_0^2 + 2z\varphi_0 - 2\mu_1}{2\varphi_0}, \qquad \varphi_2 = \frac{\varphi_0' - \varphi_0^2 - 2z\varphi_0 - 2\mu_1}{2\varphi_0}$$

which are Bäcklund transformations for $\mathrm{P_{IV}}$ (Lukashevich [1967], Gromak [1975]).

Open Problems

- Study asymptotics and connection formulae for the Painlevé equations using the isomondromic deformation method. The uniform approximation procedure should apply to all the Painlevé equations. The ultimate objective would be to produce a sufficiently general theorem on uniform asymptotics for linear systems to cover all the linear systems which arise as isomonodromy problems of the Painlevé equations. Then application to different connection problems would always appeal to the same analytical theorem and so reduce to a relatively routine calculation.
- Continue the study of the relationship between Bäcklund transformations and exact (rational, algebraic and special functions) solutions of Painlevé equations and the associated isomondromy problems. The aim is to **algorithmically** derive all these special properties directly from the isomondromy problems.

Objective

- To provide a complete classification and unified structure for classical solutions, Bäcklund transformations and other properties of the Painlevé equations and the discrete Painlevé equations — the presently known results are rather fragmentary and non-systematic.