Conformally flat hypersurfaces with cyclic Guichard net

(Udo Hertrich-Jeromin, 12 August 2006)

Joint work with Y. Suyama

A geometrical Problem

Classify conformally flat hypersurfaces $f : M^{n-1} \to S^n$.

Def. $f : M^{n-1} \to S^n$ is conformally flat if there are (local) functions so that $e^{2u}\langle df, df\rangle$ is flat (or, equivalently, there are (local) conformal coordinates).

Known results.

$n = 3.$ Every $f : M^2 \to S^3$ is conformally flat (Gauss' Theorem).

$n > 4.$ f is conformally flat $\Leftrightarrow$ f is a branched channel hypersurface (Cartan 1917).

$n = 4.$ Branched channel hypersurfaces are conformally flat;

there are hypersurfaces that are not conformally flat (e.g., Veronese tubes);

there are generic conformally flat hypersurfaces, i.e., with 3 distinct principal curvatures (e.g., cones, cylinders, hypersurfaces of revolution over K-surfaces).

The problem: *Classify generic conformally flat hypersurfaces $f : M^3 \to S^4$.*

Observation: There is an intimate relation

conformally flat hypersurfaces in $S^4 \longleftrightarrow$ *curved flats* in the space of circles in S^4.

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The Program

1. Conformally flat hypersurfaces
2. Curved flats
3. Isothermic surfaces
4. Conformally flat hypersurfaces revisited

Conformally flat hypersurfaces

Cartan's Thm. If $f : M^{n-1} \to S^n$, $n \geq 5$, is conformally flat then f is a branched channel hypersurface.

Def. Write $I = \sum_{i=1}^3 \eta_i^2$ and $I\!I = \sum_{i=1}^3 k_i \eta_i^2$; then

$$\gamma_1 := \sqrt{(k_3 - k_1)(k_1 - k_2)}\,\eta_1,$$
$$\gamma_2 := \sqrt{(k_1 - k_2)(k_2 - k_3)}\,\eta_2,$$
$$\gamma_3 := \sqrt{(k_2 - k_3)(k_3 - k_1)}\,\eta_3$$

are the conformal fundamental forms of $f : M^3 \to S^4$.

Lemma. $f : M^3 \to S^4$ is conformally flat $\Leftrightarrow d\gamma_i = 0$.

Cor. If f is conformally flat then there are curvature line coordinates $(x_1, x_2, x_3) : M^3 \to \mathbb{R}^3_2$ so that $dx_i = \gamma_i$.

Observe: $I = \sum_{i=1}^3 l_i^2 dx_i^2$, where $\sum_{i=1}^3 l_i^2 = 0$.

Def. $x : (M^3, I) \to \mathbb{R}^3_2$ is called a Guichard net if $I = \sum_{i=1}^3 l_i^2 dx_i^2$ with $\sum_{i=1}^3 l_i^2 = 0$.

Remark. A generic conformally flat $f : M^3 \to S^4$ gives a Guichard net $x \circ y^{-1} : \mathbb{R}^3 \to \mathbb{R}^3_2$ ($x : M^3 \to \mathbb{R}^3_2$ canonical Guichard net and $y : M^3 \to \mathbb{R}^3$ conformal coordinates).

Thm. A Guichard net $x : \mathbb{R}^3 \to \mathbb{R}^3_2$ gives a conformally flat $f : \mathbb{R}^3 \to S^4$ with $\gamma_i = dx_i$.

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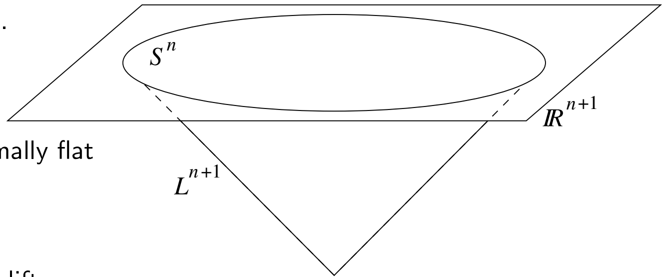
How to prove all this?



Consider: $f : M^{n-1} \to S^n \subset L^{n+1}$.

Observe: If $u \in C^\infty(M^{n-1})$ then

$$\langle d(e^u f), d(e^u f)\rangle = e^{2u}\langle df, df\rangle.$$

Thus: If $f : M^{n-1} \to S^n$ is conformally flat we may choose (locally) a flat lift

$$e^u f : M^{n-1} \to L^{n+1}.$$

Then: The tangent bundle of a flat lift

$$f : M^3 \to L^5 \subset \mathbb{R}^6_1 \text{ is flat.}$$

Lemma. In this situation, the normal bundle of

$$f : M^3 \to \mathbb{R}^6_1 \text{ is also flat.}$$

Cor. If $f : M^3 \to L^5$ is a flat lift of a conformally flat hypersurface then its Gauss map

$$\gamma : M^3 \to \frac{O_1(6)}{O(3)\times O_1(3)},\ p \mapsto \gamma(p) = d_p f(T_pM)$$ is a "curved flat".

Note. Curved flats come with special coordinates:

$\rightsquigarrow$ integrability of conformal fundamental forms and of Cartan's umbilic distributons;

$\rightsquigarrow$ conformally flat hypersurfaces come with principal Guichard nets.

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Curved flats

Setup: Let G/K be a symmetric (or reductive homogeneous) space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding symmetric decomposition of the Lie algebra, i.e.,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

For a map $\gamma : M^m \to G/K$ we consider any lift $F : M^m \to G$ and decompose its connection form $F^{-1}dF = \Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p}$.

Def. $\gamma : M^m \to G/K$ is called a curved flat if $[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \equiv 0$.

Observation: $\gamma : M^m \to G/K$ is a curved flat $\Leftrightarrow$ $\Phi_\lambda := \Phi_{\mathfrak{k}} + \lambda \Phi_{\mathfrak{p}}$ is integrable for all λ, i.e., the Gauss-Ricci equations split:

$$0 = d\Phi_\lambda + \tfrac{1}{2}[\Phi_\lambda \wedge \Phi_\lambda] \quad \Longleftrightarrow \quad \begin{cases} 0 = d\Phi_{\mathfrak{k}} + \frac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}] + \frac{1}{2}[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \\ 0 = d\Phi_{\mathfrak{p}} + [\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{p}}] \\ 0 = [\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \end{cases}$$

Consequences:

- curved flats come in "associated families"; and...
- the curved flat equations become a 0-curvature condition for the family;
- hence integrable systems methods (e.g., finite gap integration etc) can be applied.

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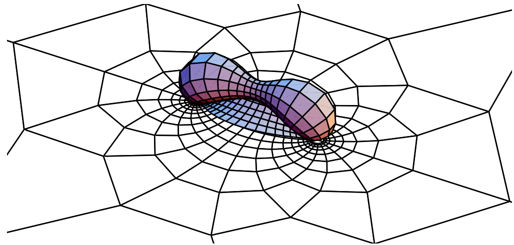
Isothermic surfaces

Def. $f : M^2 \to S^3$ is isothermic if there are (local) conformal curvature line parameters.

Well understood:

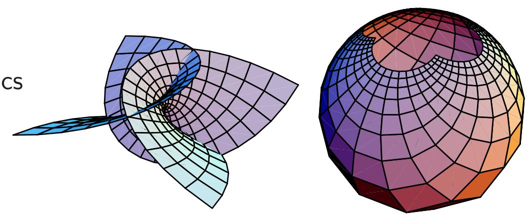


- Darboux pairs of isothermic surfaces in S^3:
  - (i) envlope a Ribaucour sphere congruence
  - (ii) induce conformally equivalent metrics

  $\leftrightarrow$ curved flats in $\frac{O_1(5)}{O(3)\times O_1(2)}$ (in the space of point-pairs)

- Christoffel pairs of isothermic surfaces in $\mathbb{R}^3$ ("limiting case of Darboux pairs"):
  - (i) parallel curvature directions
  - (ii) induce conformally equivalent metrics

  $\rightsquigarrow$ curved flats in $\frac{O_1(5)}{O(3)\times O_1(2)}$



Note: Special coordinates are already "built in".

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Curved flats come in associated families!

The associated family of curved flats yields:

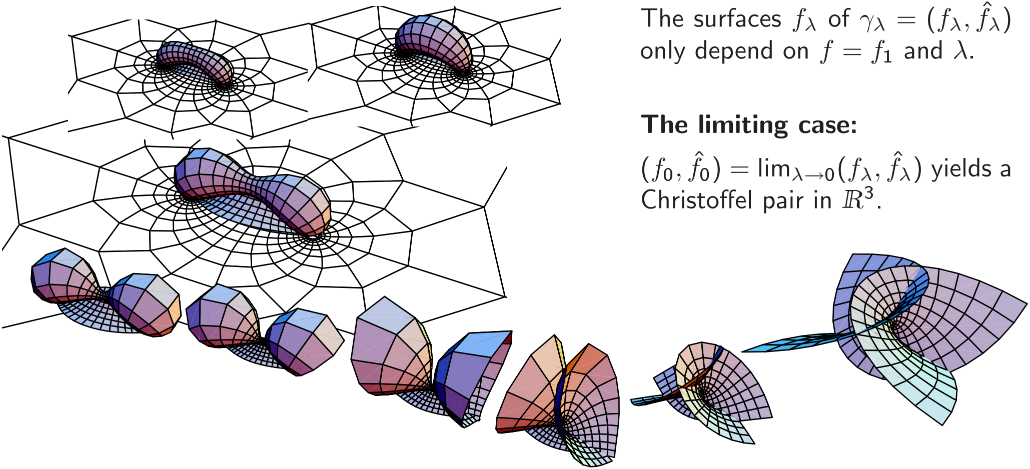
- the classical Calapso transformation (T-transformation)
- the conformal deformation for isothermic surfaces.

Small miracle:

The surfaces f_λ of $\gamma_\lambda = (f_\lambda, \hat{f}_\lambda)$ only depend on $f = f_1$ and λ.

The limiting case:

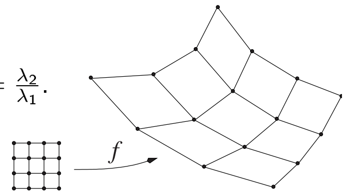
$(f_0, \hat{f}_0) = \lim_{\lambda\to 0}(f_\lambda, \hat{f}_\lambda)$ yields a Christoffel pair in $\mathbb{R}^3$.



Discrete isothermic and cmc nets.

Bianchi permutability:

$$\mathcal{D}_{\lambda_1}\mathcal{D}_{\lambda_2}f = \mathcal{D}_{\lambda_2}\mathcal{D}_{\lambda_1}f \text{ and } [f; \mathcal{D}_{\lambda_1}f; \mathcal{D}_{\lambda_1}\mathcal{D}_{\lambda_2}f; \mathcal{D}_{\lambda_2}f] = \frac{\lambda_2}{\lambda_1}.$$



Def. $f : \mathbb{Z}^2 \to S^3$ is isothermic if $q_{m,n} = \frac{a(m)}{b(n)}$.

This yields a completely analogous discrete theory:

- Christoffel transformation;
- Darboux transformation;
- Calapso transformation;
- Bianchi permutability theorems;

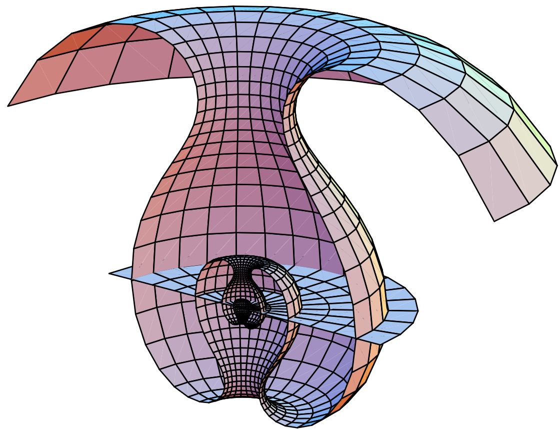
⇝ discrete minimal & cmc surfaces;

⇝ Weierstrass representation;

⇝ Bryant type representation;

⇝ Bonnet's theorem;

- Polynomial conserved quantities by Burstall/Calderbank/Santos...



Conformally flat hypersurfaces with cyclic Guichard net

We saw: From a conformally flat $f : M^3 \to S^4$ we get

$\rightsquigarrow \gamma : M^3 \to \frac{O_1(6)}{O(3)\times O_1(3)}$, $p \mapsto \gamma(p) = d_p f(T_p M)$ curved flat (non-unique),

$\rightsquigarrow x : (M^3, I) \to \mathbb{R}^3_2$ Guichard net (unique), and

$\rightsquigarrow x \circ y^{-1} : \mathbb{R}^3 \to \mathbb{R}^3_2$ Guichard net (unique up to Möbius transformation).

Conversely:

- A curved flat $\gamma : M^3 \to \frac{O_1(6)}{O(3)\times O_1(3)}$ is a "cyclic system" with conformally flat orthogonal hypersurfaces (analogue of the Darboux transformation);
- A Guichard net $x : \mathbb{R}^3 \to \mathbb{R}^3_2$ gives rise to a conformally flat hypersurface (unique up to Möbius transformation).

Questions:

1. How are the hypersurfaces of a curved flat related ("Darboux transformation")?
2. What is the geometry of the associated family ("Calapso transformation")?
3. How are the geometry of a conformally flat hypersurface and a Guichard net related?
4. How to define a suitable discrete theory?
5. ...

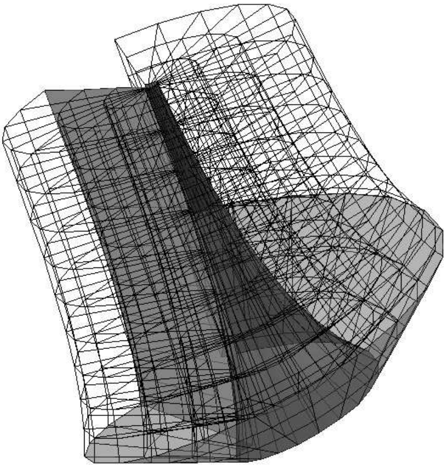
Partial answers to the 3rd question.

Thm. Cones, cylinders and hypersurfaces of revolution over K-surfaces in S^3, $\mathbb{R}^3$ and H^3, respectively, correspond to cyclic Guichard nets with totally umbilic orthogonal surfaces.

Def. A cyclic system is a smooth 2-parameter family of circles in S^3 with a 1-parameter family of orthogonal surfaces, i.e., a smooth map

$$\gamma : M^2 \to \frac{O_1(5)}{O(2)\times O_1(3)}$$

so that the bundle $\gamma^\perp$ of Minkowski spaces is flat.



Example. The normal line congruence of a surface in a space form $\mathcal{Q}^3_\kappa$ is a cyclic system.

Thm. A cyclic Guichard net is a normal line congruence in some $\mathcal{Q}^3_\kappa$ with all orthogonal surfaces linear Weingart

Question: What are the corresponding hypersurfaces?

Classification result: They "live" in some $\mathcal{Q}^4_\kappa$, where the orthogonal surfaces of the cyclic system are (extrinsically) linear Weingarten surfaces in a family of (parallel) hyperspheres in $\mathcal{Q}^4_\kappa$.

Conversely, conformally flat hypersurfaces with cyclic Guichard net can be constructed starting from suitable linear Weingarten surfaces in any space form in a unique way.

How to prove this?

Recall: If $f : M^3 \to S^4$ is conformally flat then there are curvature line coordinates

$$(x, y, z) : M^3 \to \mathbb{R}^3 \text{ so that } I = e^{2u}\{\cos^2\varphi\, dx^2 + \sin^2\varphi\, dy^2 + dz^2\}.$$

Lemma. φ satisfies

$$d\alpha = 0, \text{ where } \alpha := -\varphi_{xz}\cot\varphi\, dx + \varphi_{yz}\tan\varphi\, dy + \frac{\varphi_{xx}-\varphi_{yy}-\varphi_{zz}\cos 2\varphi}{\sin 2\varphi}\, dz, \text{ and}$$

$$0 = \frac{\varphi_{xxz}+\varphi_{yyz}+\varphi_{zzz}}{2} + \frac{\varphi_z(\varphi_{xx}-\varphi_{yy}-\varphi_{zz}\cos 2\varphi)}{\sin 2\varphi} - \varphi_x\varphi_{xz}\cot\varphi + \varphi_y\varphi_{yz}\tan\varphi.$$

Conversely, f can be reconstructed from φ.

Lemma. The z-lines are circular arcs if and only if

$$\varphi_{xz} = \varphi_{yz} \equiv 0.$$

Cor. Conformally flat hypersurfaces with cyclic principal Guichard net correspond to φ's satisfying:

$$\varphi(x, y, z) = u(x, y) + g(z) \text{ with } u_{xx} - u_{yy} = A\sin 2u \text{ and } g'^2 = C + A\cos 2g;$$

or: similar formulas with $\cosh\varphi$ and $\sinh\varphi$ (then, more cases occur).

Observation: Separation of variables *considerably* simplifies the PDE's for φ.

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Symmetry breaking.

From the structure equations, define $T = T(z) \in S^5_1$ and $Q = Q(z) \in \mathbb{R}^6_1 \setminus \{0\}$ with

$$T' = \frac{1}{1+g'^2}\, Q \text{ and } Q' = \frac{\kappa}{1+g'^2}\, T, \text{ where } \kappa := -|Q|^2 \equiv (1+C)^2 - A^2.$$

In particular, with $\zeta(z) = \int_0^z \frac{dz}{1+g'^2(z)}$,

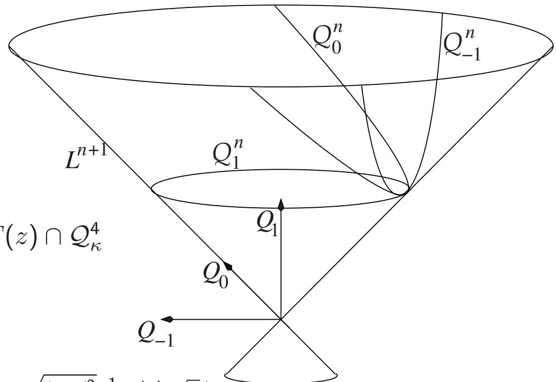
$$T = \cosh\sqrt{\kappa}\zeta\, T_{z=0} + \frac{1}{\sqrt{\kappa}} \sinh\sqrt{\kappa}\zeta\, Q_{z=0} \text{ and } Q = \kappa\frac{1}{\sqrt{\kappa}} \sinh\sqrt{\kappa}\zeta\, T_{z=0} + \cosh\sqrt{\kappa}\zeta\, Q_{z=0}.$$

Consequences:

- span$\{T, Q\}$ is a fixed sphere pencil;
- $Q(0)$ defines a space form $\mathcal{Q}^4_\kappa$;
- $T(z)$ are parallel hyperspheres in $\mathcal{Q}^4_\kappa$;
- each surface

$$(x,y) \mapsto \frac{f(x,y,z)}{\langle T(z), T(0)\rangle\sqrt{1+g'^2(z)}} \in T(z) \cap \mathcal{Q}^4_\kappa$$

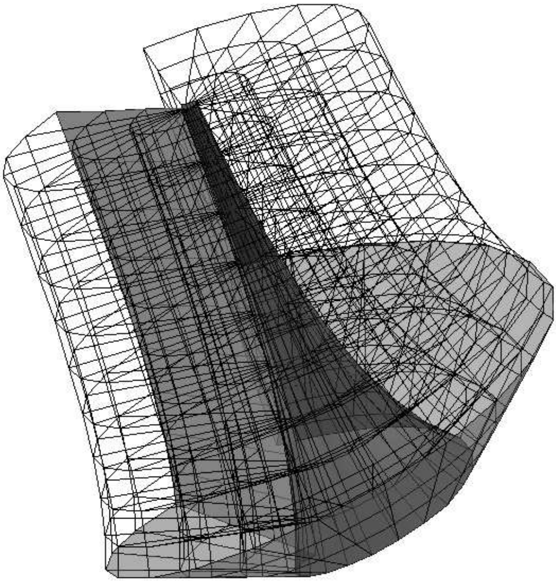
  is a linear Weingarten surface.



Explicitely:

$$f = \frac{\sqrt{1+A+C}\cos g}{\sqrt{1+g'^2}\cosh\sqrt{\kappa}\zeta}\Big\{f_0 + \frac{\tan g}{1+A+C}\cdot n + \frac{\sqrt{1+g'^2}\,\frac{1}{\sqrt{\kappa}}\sinh\sqrt{\kappa}\zeta}{\sqrt{1+A+C}\cos g}\cdot t\Big\},$$

where $f_0 = f(.,.,0)$, with Gauss map n in $T(0) \subset \mathcal{Q}^4_\kappa$, and t the unit normal of $T(0) \subset \mathcal{Q}^4_\kappa$.



Thank you!

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