Integrable systems for real time simulation of fluid flow

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Durham, August 2006

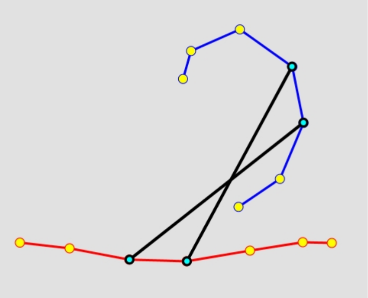
Joint work with S. Weissmann

Vortex rings in fluids

Darboux transforms of polygons (Tim Hoffmann)

A polygon $\eta_1, \ldots, \eta_n$ in $\mathbb{R}^3$ is called a Darboux transform of a polygon $\gamma_1, \ldots, \gamma_n$ if

- corresponding edges of γ and η have the same length.
- the distance d between corresponding points of γ and η is constant.
- the twist angle α of the quadrilaterals $\gamma_j, \gamma_{j+1}, \eta_{j+1}, \eta_j$ is constant.



Evolution of closed polygons

- For generic distance *d* and twist angle α every closed polygon has exactly two closed Darboux transforms
- Iterate Darboux transforms to obtain a (discrete time) flow on polygons $\rightsquigarrow$ integrable system.
- For $\alpha = \pi$ this flow is a discrete version of the mKdV-flow for smooth curves:

$$\dot{\gamma} = \gamma''' - \frac{|\gamma''|^2}{2}\gamma'$$

- A suitable combination of two Darboux transforms (same *d*, opposite twist) gives a discrete version of the smoke ring flow:

$$\dot{\gamma} = \gamma' \times \gamma''$$

Incompressible ideal fluids

Let M be a compact Riemannian 3-manifold with boundary.

- $SDiff(M) =$ $\{$volume-preserving diffeomorphisms $g : M \to M\}$
- $sDiff(M) =$ $\{$divergence-free vector fields on M tangent to $\partial M\}$
- L^2-norm of vector fields defines a right invariant Riemannian metric on $SDiff(M)$.
- geodesic on $SDiff(M) \leftrightarrow$ motion of ideal incompressible fluid in M

Similar statements if $M = \mathbb{R}^3$ for fluids at rest near infinity.

Fluid motion: velocity in terms of vorticity

- for every vector field ω on $\mathbb{R}^3$ with compact support and

$$\operatorname{div} \omega = 0$$

there is a unique L^2-vector field v on $\mathbb{R}^3$ with

$$\operatorname{div} v = 0$$

$$\operatorname{curl} v = \omega$$

- v is given by the Biot-Savart formula:

$$v(x) = \int_{\mathbb{R}^3} \frac{\omega(y) \times (x - y)}{|x - y|^3} dy$$

Euler equations for ideal fluids

- a single equations governs the evolution of ω:

$$\dot{\omega} = [\omega, v]$$

- vorticity "flows with the fluid"
- topology of $\operatorname{supp} \omega$ is invariant

Vortex filaments

Suppose ω is supported in a tubular neighborhood of an oriented link $\gamma_1, \ldots, \gamma_n$.

$\rightsquigarrow$ total vorticities $K_1, \ldots, K_n$

$$K_j = \int_\eta v$$

η a small loop around γ_j

K_j is the flux of ω through the tube around γ_j

Limit of thin tubes

Look at a single vortex tube. Suppose within the tube ω looks like

$$\omega(s, r, \phi) = K/R^2 \; f(r/R) \; \gamma'(s)$$

- f = a fixed function ("vorticity profile")
- s = arclength along γ
- r = distance to γ
- R = tube radius

Then in the limit $R \to 0$ the velocity field v generated by γ becomes

$$v(x) = \frac{K}{4\pi} \oint \frac{\gamma' \times (x - \gamma)}{|x - \gamma|^3}$$

Smoke ring flow

- Evolution of γ: Evaluate velocity v on $\gamma \rightsquigarrow$

$$\dot{\gamma} \approx C_f \; K \; \log(R) \; \gamma' \times \gamma''$$

- Scale down K as $R \to 0 \rightsquigarrow$ smoke ring flow

$$\dot{\gamma} = \gamma' \times \gamma''$$

  (da Rios and Levi-Civita 1906)

- Integrable system equivalent to the non-linear Schroedinger equation (Hashimoto 1972)

Hamiltonian formulation

- Symplectic form on the space of (weighted) links:

$$\sigma(\dot{\gamma},\mathring{\gamma}) = \sum_j K_j \oint_{\gamma_j} \det(\gamma', \dot{\gamma}, \mathring{\gamma})$$

- Hamiltonian:

$$H = \sum_j K_j \,\mathrm{Length}(\gamma_j)$$

- Renormalized version of

$$H = \sum_{i,j} \frac{K_i K_j}{8\pi} \iint \frac{\langle \gamma_i'(s), \gamma_j'(t) \rangle}{|\gamma_i(s) - \gamma_j(t)|} ds\ dt$$

Turning on interactions between filaments

Problem: In the smoke ring limit

- Fluid is at rest, vortex filaments just cut through
- No interaction between different components of a link

Solution:

- Keep symplectic form
- Replace Hamiltonian by a smoothed version

$$H = \sum_{i,j} \frac{K_i K_j}{8\pi} \iint \frac{\langle \gamma_i'(s), \gamma_j'(t) \rangle}{\sqrt{R^2 + |\gamma_i(s) - \gamma_j(t)|^2}} ds\ dt$$

Resulting evolution equation

$$\dot{\gamma}_k(s) = \sum_j \frac{K_j}{4\pi} \int \frac{\gamma_j'(t) \times (\gamma_k(s) - \gamma_j(t))}{\sqrt{R^2 + |\gamma_k(s) - \gamma_j(t)|^2}^3} dt$$

- Still Hamiltonian but not anymore integrable
- Conserved quantitity: Sum of (weighted) areas of orthogonal projection to planes, encoded by the area vector

$$A = \sum_j K_j \oint \gamma_j \times \gamma_j'$$

Flow generated by $\gamma_1, \ldots, \gamma_n$ on $\mathbb{R}^3$

- $\gamma_1, \ldots, \gamma_n$ flow according to some divergence-free vector field v on $\mathbb{R}^3$:

$$v(x) = \sum_j \frac{K_j}{4\pi} \int \frac{\gamma_j' \times (x - \gamma_j)}{\sqrt{R^2 + |x - \gamma_j|^2}^3}$$

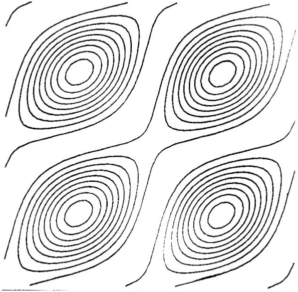
- Vorticity $\omega = \operatorname{curl} v$ concentrated within distance R of $\gamma_1, \ldots, \gamma_n$

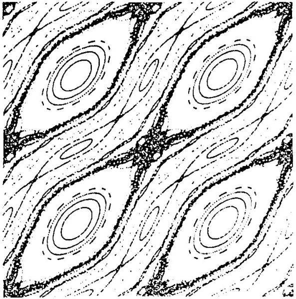
- δ-function like vorticity ω_0 smoothed by a convolution kernel:

$$\omega(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{3R^2}{\sqrt{R^2 + |x - y|^2}^5} \, \omega_0(y) dy$$

Perturbed integrable system

- Approximation to the Euler equations that ignores distortions of the cross section of the vortex tubes.
- View above evolution of $\gamma_1, \dots, \gamma_n$ as a perturbation of the smoke ring flow $\rightsquigarrow$ KAM picture.





Polygonal vortex filaments

- Perturb the discrete smoke ring dynamics of a polygon by the long range interactions via Biot-Savart.
- Biot-Savart alone would ignore the influence of the adjacent edges on the motion of a vertex.
- Biot-Savart alone would always model vortex filaments of thickness $\approx$ edgelengths $\rightsquigarrow$ too thick, too slow.
- Discrete smoke ring dynamics is therefore needed.