Constant mean curvature tori in $\mathbb{S}^3$

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joint work with M. Kilian

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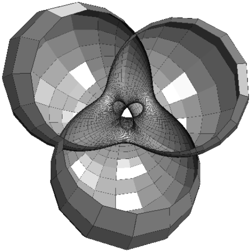

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- non–zero holomorphic functions μ_1 and μ_2 on $X \setminus \{x^+, x^-\}$ with

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- 4 points with $\mu_1(x_j) = \mu_2(x_j) = \pm 1$
  $x_1, x_2 = \sigma x_1 = \eta x_1, x_3, x_4 = \sigma x_3 = \eta x_3$.

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Alexandrov embedded:

Immersion $f : \mathbb{T}^2 \to \mathbb{S}^3$ extends to
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Then f_t 1–sided Alexandrov emb. for all $t \in (0,1)$.

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ODE: $a, b_1, b_2, \kappa_1, \kappa_3 \implies c_1, c_2, \dot{a}, \dot{b}_1, \dot{b}_2, \dot{\kappa}_1, \dot{\kappa}_3$.

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Problem: determine connected components
of spectral curves of 1–sided A.e. cmc tori in $\mathbb{S}^3$.

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For every $L \in \mathbb{N}$ there exists a family
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The L–wrapped family of flat tori has
for every $K > L$ one double point and others.

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Theorem

Every $g = 1$ family has as a limit the spectral curve of a flat torus.

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$\Im(\ln \mu_1)$–$\Im(\ln \mu_2)$
Diagram of
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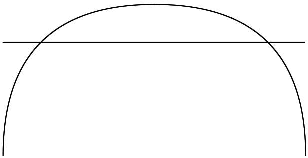

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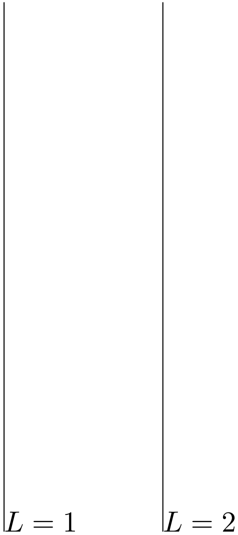
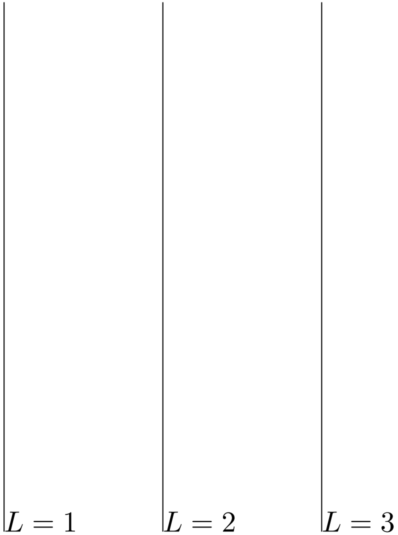
Moduli space

$L = 1$

Moduli space



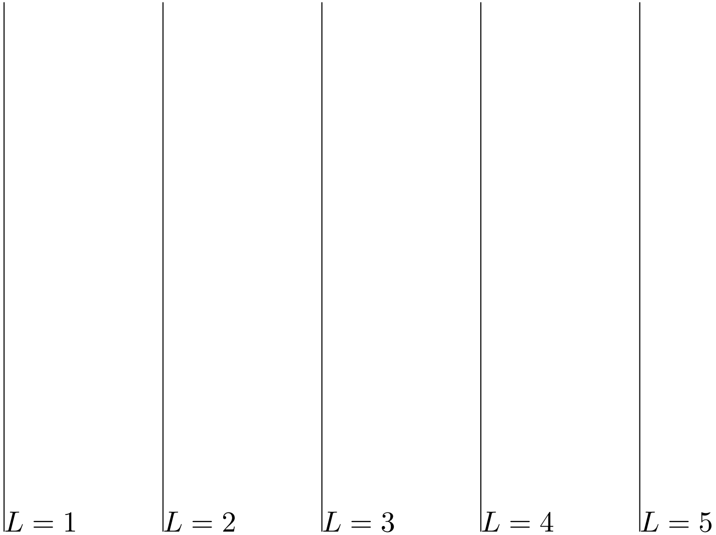
Moduli space



Moduli space

$L = 1$ $L = 2$ $L = 3$ $L = 4$

Moduli space



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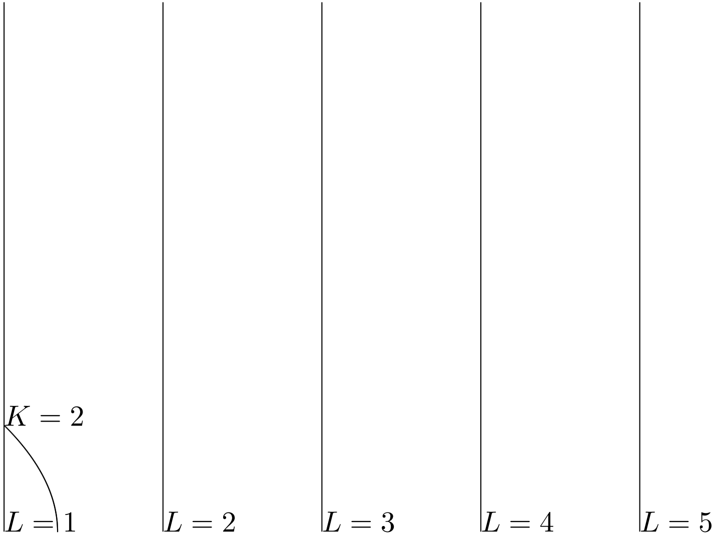

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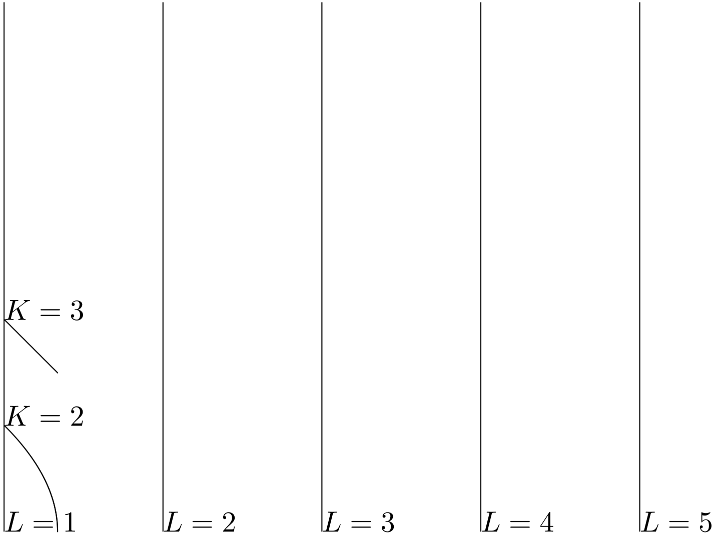

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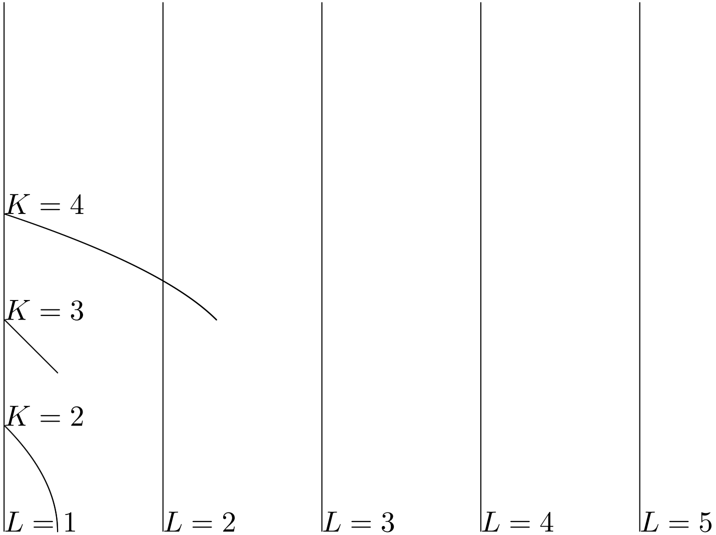

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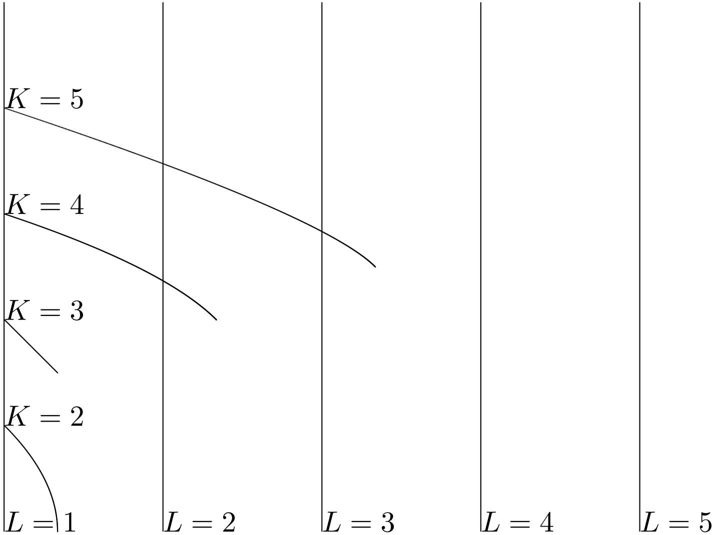

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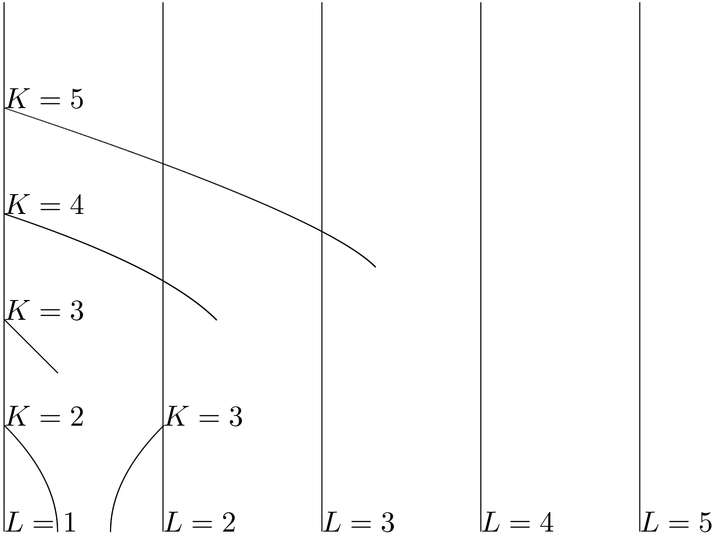

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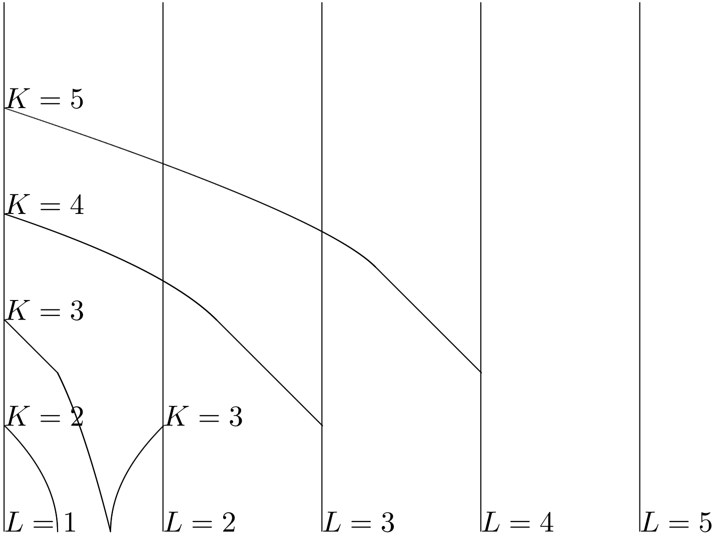

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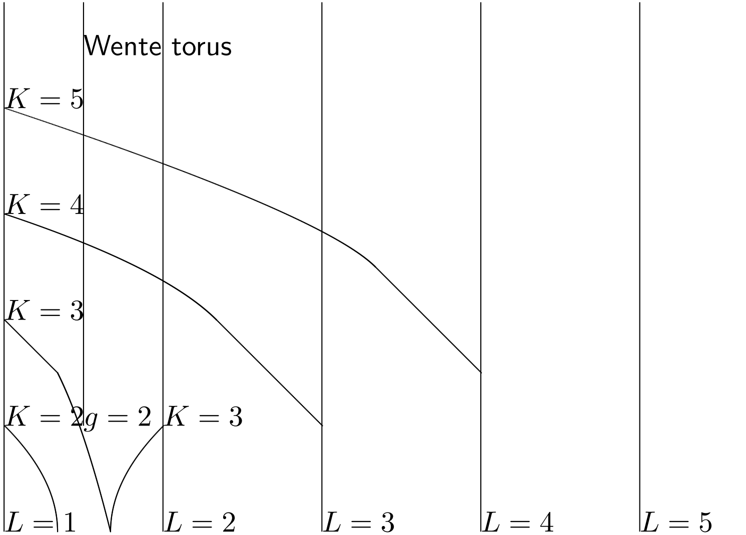

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