An expansion for self-interacting random walks

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Self-interacting random processes and mean-field behaviour

In the past two decades, many examples of self-interacting random processes have been introduced and studied.

In general, these processes are

highly non-Markovian,

which makes them hard to analyse.

Goal Find perturbation technique that allows to study behaviour for such models.

Edge-reinforced random walk with drift

Stochastic process is denoted by

$$\vec{\omega}_i = (\omega_0, \dots, \omega_i).$$

Given its history, the walker takes a nearest-neighbour step from $\omega_n = x$ to y with probability

$$p^{\vec{\omega}_n}(x,y) = \frac{w^{\vec{\omega}_n}(x,y)}{\sum_{z \sim x} w^{\vec{\omega}_n}(x,z)},$$

where the weight $w^{\vec{\omega}_n}(x,y)$ is given by

$$w^{\vec{\omega}_n}(x,y) = w_0(x,y) + \beta I[\exists t \le n : \{\omega(t-1), \omega(t)\} = \{x,y\}].$$

Key assumption is that initial weight $w_0(b)$ is translation invariant, and

$$\sum_x x w_0(0,x) \neq 0.$$

Main Theorem

Fix $d \geq 1$. Under the above assumptions, there exist $\beta_0 = \beta_0(d, w_0) > 0$ and $\theta = \theta(\beta, w_0) > 0, \Sigma = \Sigma(\beta, w_0)$ such that for $\beta \leq \beta_0$

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{1}{n})].$$

$$\text{Var}_\beta(\omega_n) = \Sigma n[1 + O(\frac{1}{n})].$$

$$\frac{\omega_n - \theta n}{\sqrt{n}} \stackrel{d}{\Longrightarrow} \mathcal{N}(0, \Sigma).$$

Durett-Kesten-Limic (2002): Similar result on regular tree.

Challenge Extension to ordinary OERRW.

Excited random walk

A random walker gets excited each time he/she visits a new site, and, when random walk is excited, it has a positive drift of size $\frac{\beta}{d}$ in direction of first coordinate.

Key question Does excited random walk have a positive drift?

Answer

YES (if $d > 1$)

$d \geq 4$: Benjamini+Wilson (2003).
$d = 3$: Kozma (2003).
$d = 2$: Kozma (2005).

Proofs make use of tan points for simple random walk, but does not give more insight into properties of speed (e.g., monotonicity in the excitement).

Main Theorem

For $d > 8$, there exist $\beta_0 = \beta_0(d) > 0$ and $\theta = \theta_\beta > 0, \Sigma = \Sigma_\beta$ such that for $\beta \leq \beta_0$

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{1}{n})].$$

$$\text{Var}_\beta(\omega_n) = \Sigma n[1 + O(\frac{1}{n})].$$

$$\frac{\omega_n - \theta n}{\sqrt{n}} \stackrel{d}{\Longrightarrow} \mathcal{N}(0, \Sigma).$$

Able to prove LLN for $d > 5$.

Advantage We obtain a formula for speed.

Extension: Functional CLT by Bernard and Ramirez (2007) using regeneration times, more implicit description speed/variance.

Monotonicity for Excited Random Walk

In many models, there are obvious conjectures of monotonicity of speed and/or variance in parameters model.

Traditional techniques yield implicit formulae, from which it is not clear how to deduce monotonicity.

Theorem. For $d \gg 1$, $\beta \mapsto \theta_\beta$ is monotonically increasing on $[0, l]$.

[Question: What is speed when $\beta = 1$?]

General self-interacting random walks: Formal Definition

Stochastic process is denoted by

$$\vec{\omega}_i = (\omega_0, \ldots, \omega_i).$$

Measure on $\vec{\omega}_n$ is

$$\mathbb{Q}(\vec{\omega}_n = (x_0, x_1, \ldots, x_n)) = \prod_{i=0}^{n-1} p^{\vec{\omega}_i}(x_i, x_{i+1}),$$

where $p^{\vec{\omega}_i}(x_i, x_{i+1})$ is the

conditional probability of the walker to jump from x_i to x_{i+1} given that the history of the walk equals $\vec{\omega}_i$.

Investigate

$$c_n(x) = \mathbb{Q}(\omega_n = x).$$

Examples

For excited random walk, $p^{\vec{\omega}_i}(x_i, x_{i+1})$ equals either $p_0(x_{i+1} - x_i)$ or $p_\beta(x_{i+1} - x_i)$, depending on whether the walk visits a new site, where, for $\beta \in [0, 1]$,

$$p_\beta(x) = \frac{1 + \beta x_1}{2d} I[|x| = 1].$$

For edge-reinforced random walk with drift,

$$p^{\vec{\omega}_i}(x_i, x_{i+1}) = \frac{w_i(x_i, x_{i+1})}{\sum_y w_i(x_i, y)},$$

where $w_t(x, y)$ is eight of edge (x, y) at time t, which equals $w_0(x, y) + \beta$ or $w_0(x, y)$, depending on whether edge (x, y) was previously visited or not.
Assume that

$$\sum_x x w_0(0, x) \neq 0.$$

Proof: A perturbation expansion

The proof relies on a perturbation or lace expansion, which reads

$$c_{n+1}(x) = \sum_y D(y) c_n(x-y) + \sum_y \sum_{m=2}^{\infty} \pi_m(y) c_{n+1-m}(x-y)$$

for certain expansion coefficients $\{\pi_m\}_{m=2}^{\infty}$, and where

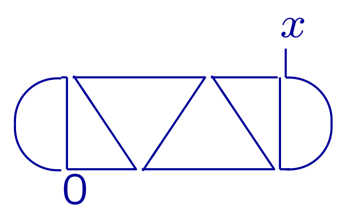
$$D(x) = p^{\varnothing}(0,x)$$

is transition probability of first step of walk.

Expansion coefficient $\pi_m(x)$ has diagrammatic expansion

$$\pi_m(x) = \sum_{N=1}^{\infty} \pi_m^{(N)}(x),$$

with

$\pi_m^{(N)}(x) =$ 

Consequences of the expansion

Expansion yields

$$c_{n+1}(x) = \sum_y D(y) c_n(x-y) + \sum_y \sum_{m=2}^{\infty} \pi_m(y) c_{n+1-m}(x-y).$$

Using that $\sum_y \pi_m(y) = 0, \sum_x c_{n+1-m}(x) = 1$, we can identify

$$\theta_\beta = \theta_\varnothing + \sum_{m=2}^{\infty} \sum_x x \pi_m(x),$$

where $\theta_\varnothing = \sum_x x D(x)$.

Similarly,

$$\Sigma_\beta = \Sigma_\varnothing - \theta_\beta \theta_\beta^t - \sum_{m=2}^{\infty} \nabla^2 \Big[e^{-i\theta_\beta (m-1)k} \widehat{\pi}_m(k) \Big]_{k=0}.$$

Proof using the expansion

Analyses using perturbation expansion contains three main steps:

1. Derivation of the expansion. Uses inclusion-exclusion, and will be described in more detail later.
Model independent.

2. Bounds on the expansion coefficients.
Model dependent.

3. Asymptotic analysis of the perturbation expansion.
Uses model independent induction, using bounds on coefficients.
Will be omitted here.

The expansion

If η and ω are two paths of length at least j and m respectively and such that $\eta(j) = \omega(0)$, then the concatenation is defined by

$$(\vec{\eta}_j \circ \vec{\omega}_m)(i) = \begin{cases} \eta(i) & \text{when } 0 \le i \le j, \\ \omega(i-j) & \text{when } j \le i \le m + j. \end{cases}$$

In terms of the above notation, we can rewrite

$$c_{n+1}(x) = \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec{\omega}_n^{(1)}:\omega_1^{(0)} \to x} \prod_{i=0}^{n-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}),$$

where $\vec{\omega}_1^{(0)}$ denotes first step of walk, and $\vec{\omega}_n^{(1)}$ the subsequent n steps.

We wish to treat two walks as being independent.

The expansion (Cont.)

We write

$$p^{\vec{\eta}_m \circ \vec{\omega}_i}(\omega_i, \omega_{i+1}) = p^{\vec{\omega}_i}(\omega_i, \omega_{i+1}) + \left(p^{\vec{\eta}_m \circ \vec{\omega}_i} - p^{\vec{\omega}_i}\right)(\omega_i, \omega_{i+1}).$$

First term has ‘forgotten’ first step, while second term makes up mistake. We wish to expand out product in

$$\prod_{i=0}^{n-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}).$$

For this, we note that

$$\prod_{i=0}^{n-1}(a_i + b_i) = \prod_{i=0}^{n-1} a_i + \sum_{j=0}^{n-1}\Big(\prod_{i=0}^{j-1}(a_i + b_i)\Big) b_j \Big(\prod_{i=j+1}^{n-1} a_i\Big),$$

where, by convention, empty product is defined to be equal to 1.

The expansion (Cont.)

We apply this with

$$a_i = p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}), \qquad b_i = \left(p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}} - p^{\vec{\omega}_i^{(1)}}\right)(\omega_i^{(1)}, \omega_{i+1}^{(1)}),$$

to arrive at

$$\begin{aligned} c_{n+1}(x) &= (D * c_n)(x) \\ &\quad + \text{correction term due to } \left(p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}} - p^{\vec{\omega}_i^{(1)}}\right)(\omega_i^{(1)}, \omega_{i+1}^{(1)}). \end{aligned}$$

Apply similar arguments to correction term.

The expansion (Cont.)

We abbreviate

$$\Delta_{j+1}^{(1)} = \frac{p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_j^{(1)}} - p^{\vec{\omega}_j^{(1)}}(\omega_j^{(1)}, \omega_{j+1}^{(1)})}{p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_j^{(1)}}(\omega_j^{(1)}, \omega_{j+1}^{(1)})},$$

and

$$\mathbb{Q}^{\vec{\eta}_m}(\vec{\omega}_n = (x_0, x_1, \ldots, x_n)) \equiv \prod_{i=0}^{n-1} p^{\vec{\eta}_m \circ \vec{\omega}_i}(x_i, x_{i+1}),$$

where $x_0 = \eta_m$.

We write $\mathbb{E}^{\vec{\eta}_m}$ for the expected value with respect to $\mathbb{Q}^{\vec{\eta}_m}$. Then, we have that

$$\prod_{i=0}^{j-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\eta}_i}(\eta_i, \eta_{i+1}) = \mathbb{Q}^{\vec{\omega}_1^{(0)}}(\vec{\omega}_j^{(1)} = \vec{\eta}_j).$$

The expansion (Cont.)

We further write

$$c_n^{\vec{\eta}_m}(\eta_m, x) = \mathbb{Q}^{\vec{\eta}_m}(\omega_n = x).$$

Then

$$\sum_{(\omega_{j+2}^{(1)},\ldots,\omega_n^{(1)}):\omega_n^{(1)}=x} \Big[\prod_{i=j+1}^{n-1} p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)})\Big] = c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)}, x),$$

so that

$$c_{n+1}(x) = (D * c_n)(x) + \sum_{j=0}^{n-1} \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \mathbb{E}^{\vec{\omega}_1^{(0)}} \Big[\Delta_{j+1}^{(1)} \; c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)}, x)\Big].$$

Expansion follows by applying similar techniques on $c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)}, x)$.

Bounds on the expansion

We note that factors $\left(p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_i^{(1)}} - p^{\vec{\omega}_i^{(1)}}\right)(\omega_i^{(1)}, \omega_{i+1}^{(1)})$ enforce loops.

For some constant C_p, we can bound the above as follows,

$$|\left(p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_i^{(1)}} - p^{\vec{\omega}_i^{(1)}}\right)(\omega_i^{(1)}, \omega_{i+1}^{(1)})| \leq C_p I[\omega_i^{(1)} \in \vec{\omega}_1^{(0)}].$$

Similar bound apply to $N \geq 2$. For β small, bound on $\pi_m^{(N)}(y)$ contains factor β^N.

- Bounds OERRW with drift rely on fact that it is exponentially unlikely to return to starting point. Persists when interaction small.

- Bounds ERW rely on fact that second up to last coordinate are equal in distribution to the ones of SRW.

Conclusion and questions

- Derived a general expansion for self-interacting random walks.

- Bounds on coefficients are model dependent, and can so far only be derived for certain examples.

- We have not made use of detailed properties of models involved.

Can the bounds be derived for other models, for example for OERRW, using more detailed properties of these models?