Random walks and the lace expansion I

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Durham: July, 2007

Abstract

e an introduction to the lace expansion with emphasis on recent
s to:
neration of self-avoiding walks, and
lysis of random walks on the incipient infinite cluster for oriented

s at http://www.math.ubc.ca/∼slade.
n supported in part by NSERC.

The lace expansion

ydges–Spencer (1985) to analyse weakly SAW for $d > 4$.

xtended by several people to analyse the critical behaviour of:

> 4,

s and lattice animals for $d > 8$,

for $d > 6$,

= directed) percolation for $d > 4$,

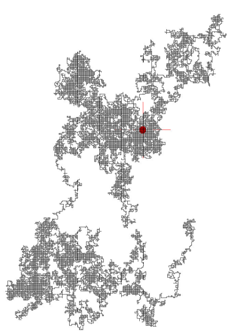
ocess, Ising model for $d > 4$.

ed to enumerate SAWs in all dimensions.

Slade, The Lace Expansion and its Applications, Springer LNM 1879, (2006).

Simple random walk

$\mathbb{Z}^d$. Choose one of the $2d$ neighbours at random and step to it. Continue
nt steps to a neighbour of current position.



he position after n steps. Let $s_n(x)$ be the number of n-step SRWs with

number of n-step SRWs.

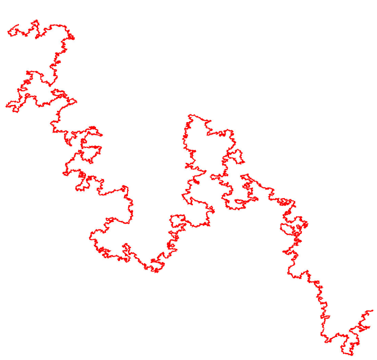
ion: $s_n(x) = \sum_{y \in \mathbb{Z}^d} s_1(y) s_{n-1}(x-y)$, which can easily be solved.
$_n = 2d s_{n-1}$ which has solution $s_n = (2d)^n$.

isplacement: $E|\omega(n)|^2 = n$.

Self-avoiding walk

the set of $\omega : \{0, 1, \ldots, n\} \to \mathbb{Z}^d$ with:
$n) = x$, $|\omega(i+1) - \omega(i)| = 1$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$.
$_{\mathbb{Z}^d} \mathcal{S}_n(x)$

$\mathcal{S}_n(x)|$. Let $c_n = \sum_x c_n(x) = |\mathcal{S}_n|$.
ks in $\mathcal{S}_n$ to be equally likely: each has probability c_n^{-1}.



$_n$ and $E|\omega(n)|^2 = c_n^{-1} \sum_{\omega \in \mathcal{S}_n} |\omega(n)|^2 = c_n^{-1} \sum_{x \in \mathbb{Z}^d} |x|^2 c_n(x)$.

Critical exponents

nstant $\mu = \lim_{n\to\infty} c_n^{1/n}$ exists because $c_{n+m} \leq c_n c_m$.
$\leq 2d - 1$.

ymptotic behaviour:

$$c_n \sim A\mu^n n^{\gamma-1}, \quad E|\omega(n)|^2 \sim Dn^{2\nu}$$

critical exponents γ and ν (and $\log$ corrections for $d = 4$).

$= \frac{43}{32}$ and $\nu = \frac{3}{4}$ will follow if scaling limit is $\mathrm{SLE}_{8/3}$ (Lawler–Schramm–

rigorous results.
$\gamma = 1$ and $\nu = \frac{1}{2}$ with $(\log n)^{1/4}$ corrections for hierarchical lattice
e 2003).
$= 1$ and $\nu = \frac{1}{2}$ (Hara–Slade 1992).

or $d = 2, 3, 4$? Best bound is $\mu^n \leq c_n \leq \mu^n e^{Cn^{2/(d+2)}\log n}$.
t $cn \leq E|\omega(n)|^2 \leq Cn^{2-\epsilon}$.

Critical exponents for $d = 3$

… to compute the exponents:

…eory (physics)

…Carlo (walks of length 640,000 have been simulated)

…numeration plus series analysis: determine c_n exactly for $n = 1, 2, \ldots, N$
…e sequence to determine μ, A, γ.
… back to Domb (1949) and was later developed at King's College London;
…tbed is Melbourne.

SAW enumeration using the lace expansion

n Nathan Clisby (Melbourne) and Richard Liang (Berkeley).

569 905 525 454 674 614
$492.3\ldots$

852 857 467 211 187 784
$14450.8\ldots$

742 525 570 299 581 210 090
3.3×10^8

39 265 092 167 904 101 731 532
2.3×10^{10}

wn values due to MacDonald et al 2000 ($d=3$),
($d=4,5,6$).

SAW enumeration using the lace expansion

called the "two-step method" was also crucial.)

ok 15000 CPU hours;
took 4400 CPU hours.

"finite lattice method" is remarkable (Jensen 2004):
93 020 903 935 054 619 120 005 916
xpansion cannot compete.

Series analysis

ritical parameters for $d = 3$: $c_n \sim A\mu^n n^{\gamma-1}$, $E[|\omega(n)|^2] \sim Dn^{2\nu}$.

$4043(12)$

$68(8)$ [Caracciolo et al 1998, MC: 1.1575(6)]

$76(5)$ [Prellberg 2001, MC: 0.5874(2)]

$6(5)$, $D = 1.220(12)$.

obtained for μ, A, D in dimensions $4 \leq d \leq 8$; accuracy improves as $d \uparrow$.

The lace expansion: Recursion relation

ction $\pi_m(x)$ such that for $n \geq 1$,

$$c) = \sum_{y\in\mathbb{Z}^d} c_1(y)c_{n-1}(x-y) + \sum_{m=2}^{n}\sum_{y\in\mathbb{Z}^d} \pi_m(y)c_{n-m}(x-y).$$

$_{\in\mathbb{Z}^d}\, \pi_m(y)$ and sum over $x \in \mathbb{Z}^d$ to get:

$$c_n = 2dc_{n-1} + \sum_{m=2}^{n} \pi_m c_{n-m}.$$

$(\pi_m)_{2\leq m\leq n}$ is equivalent to knowledge of $(c_m)_{0\leq m\leq n}$.

The lace expansion: graphs

$_n(x)$ = set of m-step simple random walks that start at the origin and end

$$U_{st}(\omega) = \begin{cases} -1 & \text{if } \omega(s) = \omega(t) \\ 0 & \text{if } \omega(s) \neq \omega(t). \end{cases}$$

, let

$$K[a,b] = K_\omega[a,b] = \prod_{a \le s < t \le b} (1 + U_{st}).$$

$$c_n(x) = \sum_{\omega \in \mathcal{W}_n(x)} K_\omega[0,n].$$

t of pairs st with $s < t$. Let $\mathcal{B}_{[a,b]}$ denote the set of all graphs on $[a,b]$.

The lace expansion: connected graphs

$$K[0,n] = \prod_{0 \le s < t \le n} (1 + U_{st}) = \sum_{\Gamma \in \mathcal{B}_{[0,n]}} \prod_{st \in \Gamma} U_{st}.$$

onnected on $[a,b]$ if, as intervals of real numbers, $\cup_{st \in \Gamma}(s,t) = (a,b)$.
connected graphs on $[a,b]$ is denoted $\mathcal{G}_{[a,b]}$. Let

$$J[0,n] = \sum_{\Gamma \in \mathcal{G}_{[0,n]}} \prod_{st \in \Gamma} U_{st}.$$

$$K[0,n] = K[1,n] + \sum_{m=2}^{n} J[0,m] K[m,n].$$

$$c_n(x) = \sum_{\omega \in \mathcal{W}_n(x)} K_\omega[0,n].$$

s

$$\sum_{\omega \in \mathcal{W}_n(x)} K_\omega[1,n] = \sum_{y \in \mathbb{Z}^d} c_1(y) c_{n-1}(x-y).$$

The lace expansion: factorisation

$$\ldots_\omega[0,m]K_\omega[m,n] = \sum_y \sum_{m=2}^{n} \sum_{\omega_1\in\mathcal{W}_m(y)} J_{\omega_1}[0,m] \sum_{\omega_2\in\mathcal{W}_{n-m}(x-y)} K_{\omega_2}[0,n-m].$$

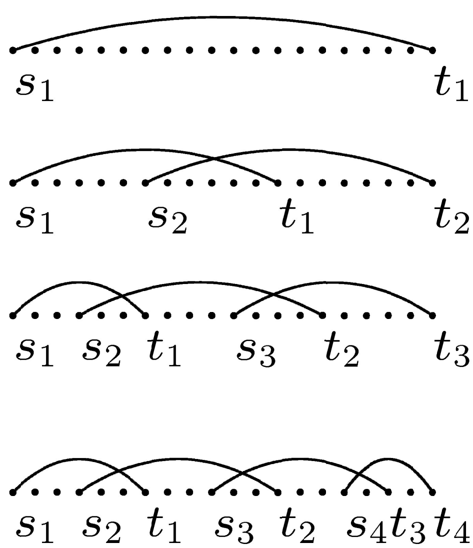
$$\sum_y \sum_{m=2}^{n} \pi_m(y)c_{n-m}(x-y)$$

$$\pi_m(y) = \sum_{\omega\in\mathcal{W}_m(y)} J_\omega[0,m].$$

$$\ldots x) = \sum_y c_1(y)c_{n-1}(x-y) + \sum_y \sum_{m=2}^{n} \pi_m(y)c_{n-m}(x-y).$$

The lace expansion: laces

$_{n]}$, choose a ‘minimal’ connected $L \subset \Gamma$,
denote the edges which are *compatible* with L in the sense that L remains
oice for $\Gamma = L \cup \{st\}$.
ces L with $N = 1, 2, 3, 4$ edges:



The lace expansion: resummation

$$0, m] = \sum_{L \in \mathcal{L}_{[0,m]}} \prod_{st \in L} U_{st} \sum_{\Gamma \in \mathcal{G}_{[0,m]}(L)} \prod_{s't' \in \Gamma \setminus L} U_{s't'}$$

$$= \sum_{L \in \mathcal{L}_{[0,m]}} \prod_{st \in L} U_{st} \prod_{s't' \in \mathcal{C}(L)} (1 + U_{s't'})$$

$$= \sum_{N=1}^{\infty} (-1)^N \sum_{L \in \mathcal{L}^{(N)}_{[0,m]}} \prod_{st \in L} [-U_{st}] \prod_{s't' \in \mathcal{C}(L)} (1 + U_{s't'}),$$

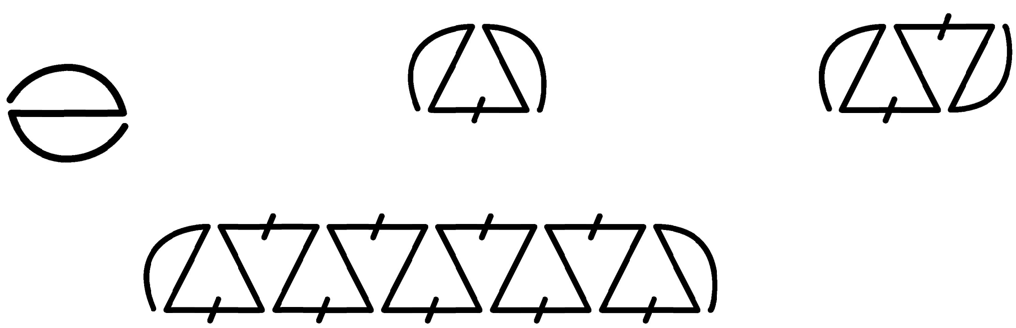
$$\pi_m(x) = \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(x)$$

$$(x) = \sum_{\omega \in \mathcal{W}_m(x)} \sum_{L \in \mathcal{L}^{(N)}_{[0,m]}} \prod_{st \in L} [-U_{st}(\omega)] \prod_{s't' \in \mathcal{C}(L)} (1 + U_{s't'}(\omega)).$$

The lace expansion: lace graphs

$$(x) = \sum_{\omega \in \mathcal{W}_m(x)} \sum_{L \in \mathcal{L}^{(N)}_{[0,m]}} \prod_{st \in L} [-U_{st}(\omega)] \prod_{s't' \in \mathcal{C}(L)} (1 + U_{s't'}(\omega)).$$

hs are the walks that give nonzero products in the above sum, and this is what we enumerate.



Lace graphs for $N = 1, 2, 3, 4, 11.$

The lace expansion: smaller enumeration task

find that the ratio of SAWs to lace graphs is approximately

$$d = 2, n = 30 : \quad 36$$

$$d = 3, n = 30 : \quad 525$$

$$d = 4, n = 24 : \quad 1700$$

$$d = 5, n = 24 : \quad 6200$$

$$d = 6, n = 24 : \quad 20000$$

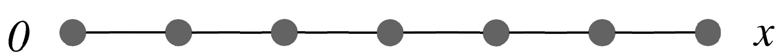
of $(\pi_m)_{m \le M}$ in dimensions $d \le \frac{M}{2}$ gives $(\pi_m)_{m \le M}$ in *all* dimensions d.
$_{24}$ in dimensions $d \le 12$ gives $(c_n)_{n \le 24}$ in *all* dimensions d.

:

2	$\delta = 3$	$\delta = 4$	$\delta = 5$	$\delta = 6$
1	0	0	0	0
3	0	0	0	0
3	-4	0	0	0
9	15	0	0	0
0	-86	-27	0	0
1	300	106	0	0
5	-1 511	-1 340	- 248	0
6	5 297	5 333	966	0
3	-25 566	-52 252	-25 020	-2 830
1	91 234	211 403	100 988	10 755
4	- 435 330	-1 907 566	-1 850 364	- 515 509
4	1 586 306	7 854 601	7 635 822	2 029 500
3	-7 568 792	-68 777 498	- 123 248 980	-64 816 437
6	28 105 857	288 074 727	517 006 517	260 695 401
3	- 134 512 520	-2 498 227 824	-7 899 351 270	-7 074 329 136
7	507 675 751	10 626 960 167	33 569 520 427	28 860 719 280
6	-2 438 375 322	-92 047 793 514	- 500 752 577 733	- 724 291 034 691
7	9 330 924 963	396 919 882 288	2 150 581 793 271	2 984 307 507 943
1	-44 965 008 206	-3 445 692 397 195	-31 789 616 257 271	-72 005 867 458 629
2	174 103 216 625	15 035 569 992 917	137 713 940 393 321	298 797 296 949 195
9	- 841 380 441 626	- 130 974 140 581 412	-2 032 548 406 479 564	-7 072 798 632 884 530
5	3 290 830 791 268			
3	-15 941 476 401 251			
5	62 897 919 980 935			
7	- 305 298 415 550 796			
2	1 213 812 491 872 081			
3	-5 901 490 794 431 276			

Inclusion-exclusion

contributing to the two-point function are regarded as a string of mutually-
(= vertices):



ually avoiding. The problem would be easy if the beads were independent

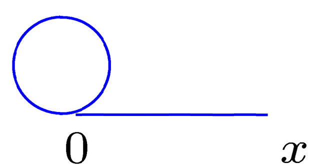
ernate view: The lace expansion is an inclusion-exclusion argument that
s as independent to first order, with explicit higher order corrections.

ns give rise to lace graphs.

The lace expansion via inclusion-exclusion

$$c_n(x) = \sum_y c_1(y) c_{n-1}(x-y) - R_n^{(1)}(x)$$

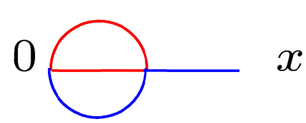
$$R_n^{(1)}(x) =$$



sion again:

$$R_n^{(1)}(x) = \sum_{m=2}^{n} u_m c_{n-m}(x) - R_n^{(2)}(x)$$

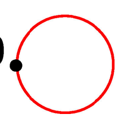
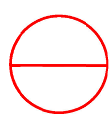
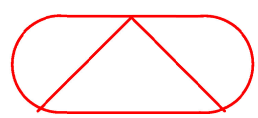
$$R_n^{(2)}(x) =$$



The lace expansion via inclusion-exclusion

s to

$$x) = \sum_y c_1(y)c_{n-1}(x-y) + \sum_{m=2}^{n} \sum_y \pi_m(y)c_{n-m}(x-y)$$

$\pi_m(y) = -\delta_{0,y}$ 0  $+$ 0  y $-$  $+\cdots$

0 y

$1/d$ **expansions**

rsion relation

$$c_n = 2dc_{n-1} + \sum_{m=2}^{n} \pi_m c_{n-m}.$$

erating functions

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \Pi(z) = \sum_{m=2}^{\infty} \pi_m z^m.$$

relation gives

$$\chi(z) = \frac{1}{1 - 2dz - \Pi(z)}.$$

convergence of $\chi(z)$ is $z_c = \mu^{-1}$, and $\chi(z_c) = \infty$, so

$$1 - 2dz_c - \Pi(z_c) = 0.$$

$1/d$ **expansions: truncation**

int is given implicitly by

$$_c = \frac{1}{2d}[1 - \Pi(z_c)] = \frac{1}{2d}\left[1 - \sum_{m=2}^{\infty}\sum_{M=1}^{\infty}(-1)^M \pi_m^{(M)} z_c^m\right].$$

ed this to prove that there exist $a_i \in \mathbb{Z}$ such that

$$z_c \sim \sum_{i=1}^{\infty} \frac{a_i}{(2d)^i} \quad \text{as } d \to \infty.$$

e gives (in high d)

$$\sum_{m=2}^{\infty}\sum_{M=N}^{\infty} \pi_m^{(M)} z_c^m \le C_N d^{-N}$$

in high d)

$$\sum_{m=j}^{\infty} \pi_m^{(M)} z_c^m \le C_{M,j} d^{-j/2}.$$

$1/d$ expansions: results

edge of $\pi_m^{(M)}$ for $m \leq 2N$ and $M \leq N$ permits the recursive calculation
$, \ldots, N+1$.

$m \leq 24$ and $M \leq 12$ gives

$$- 1 - \frac{1}{2d} - \frac{3}{(2d)^2} - \frac{16}{(2d)^3} - \frac{102}{(2d)^4} - \frac{729}{(2d)^5} - \frac{5533}{(2d)^6} - \frac{42229}{(2d)^7}$$

$$\frac{88761}{(2d)^8} - \frac{1026328}{(2d)^9} + \frac{21070667}{(2d)^{10}} + \frac{780280468}{(2d)^{11}} + O\left(\frac{1}{(2d)^{12}}\right).$$

e full asymptotic series is divergent. Note sign change at order $(2d)^{-10}$.

ons result for the amplitudes A and D, using e.g.,

$$\frac{1}{A} = 2dz_c + \sum_{m=2}^{\infty} m\pi_m z_c^m$$