Random walks and the lace expansion II

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Abstract

e an introduction to the lace expansion with emphasis on recent
s to:
neration of self-avoiding walks, and
lysis of random walks on the incipient infinite cluster for oriented

s at http://www.math.ubc.ca/∼slade.

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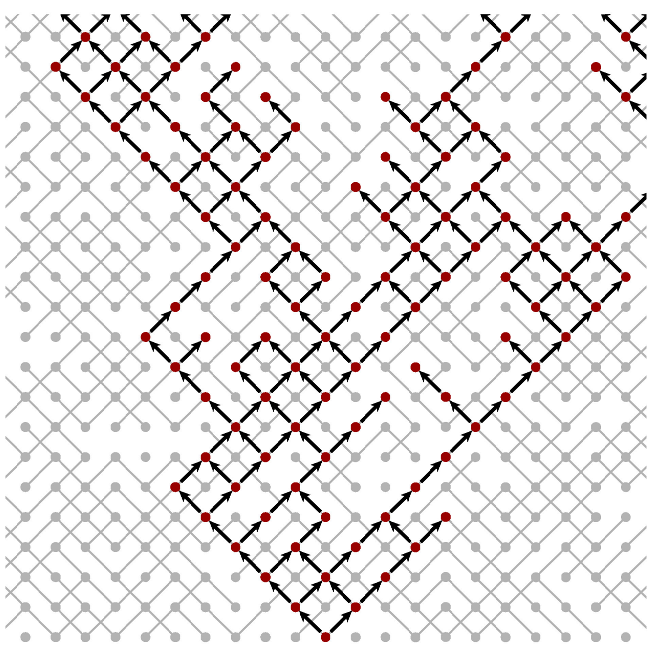
Spread-out oriented percolation

on on the directed graph with vertices (x, n), $x \in \mathbb{Z}^d$, $n = 0, 1, 2, \ldots$
onds $((x, n), (y, n + 1))$ with $\|x - y\| \leq L$ (later: L large).

pendently "occupied" with probability p and otherwise "vacant."



$p = 0.7 > p_c \approx 0.645$

Oriented percolation has a helpful Markov property.

Phase transition for oriented percolation

$, 0) \longrightarrow (x, n)$ if there is an occupied oriented path from $(0, 0)$ to (x, n),

$$C(0,0) = \{(x,n) : (0,0) \longrightarrow (x,n)\}.$$

$(|C(0,0)| = \infty)$. Phase transition: $\exists\, p_c = p_c(d, L) \in (0,1)$ s.t.

$$\theta(p) = 0 \quad \text{if } p < p_c,$$
$$\theta(p) > 0 \quad \text{if } p > p_c.$$

ezuidenhout–Grimmett (1990) proved $\theta(p_c) = 0$.

our for $p \approx p_c$ well understood for $d > 4$ (when $L \gg 1$).

Oriented percolation: r-point functions

he r-point functions ($r \geq 2$) are given by:

$$\vec{x}) \quad = \quad \tau^{(r)}_{n_1,\ldots,n_{r-1}}(x_1. \ldots, x_{r-1})$$

$$= \quad \mathbb{P}\Big((0,0) \to (x_1,n_1), \ldots, (0,0) \to (x_{r-1},n_{r-1})\Big).$$

ransforms are

$$\hat{\tau}^{(r)}_{\vec{n}}(\vec{k}) = \sum_{\vec{x}\in\mathbb{Z}^{d(r-1)}} \tau^{(r)}_{\vec{n}}(\vec{x}) e^{i\vec{k}\cdot\vec{x}}.$$

function $\tau = \tau^{(2)}$ obeys

$$0) = \sum_{x\in\mathbb{Z}^d} \mathbb{E} I[(0,0) \to (x,n)] = \mathbb{E}|C(0,0) \cap (\mathbb{Z}^d \times \{n\})|$$

c,

$$\hat{\tau}_n(0;p) \to \begin{cases} 0 & (p < p_c) \\ \infty & (p > p_c). \end{cases}$$

The two-point function: Results

der Hofstad – S 2003 Let $d > 4$, $p = p_c$, $\delta \in (0, 1 \wedge \frac{d-4}{2})$.
$, L)$, $D(d, L)$, $C_i(d)$ such that for $L \geq L_0$ we have

$$\hat{\tau}_n\Big(\frac{k}{\sqrt{Dn}}\Big) = Ae^{-k^2/2d}\left[1 + O\Big(\frac{k^2}{n^\delta}\Big) + O\Big(\frac{1}{n^{(d-4)/2}}\Big)\right]$$

$$C_1 L^{-d} n^{-d/2} \leq \sup_x \tau_n(x) \leq C_2 L^{-d} n^{-d/2}.$$

roved previously by Nguyen and Yang (1995), with a weaker error estimate.
fferent and also yields the second line.

$\mathrm{m}_{n\to\infty} \hat{\tau}_n(0) = A$
ber of sites in cluster of origin at time n).

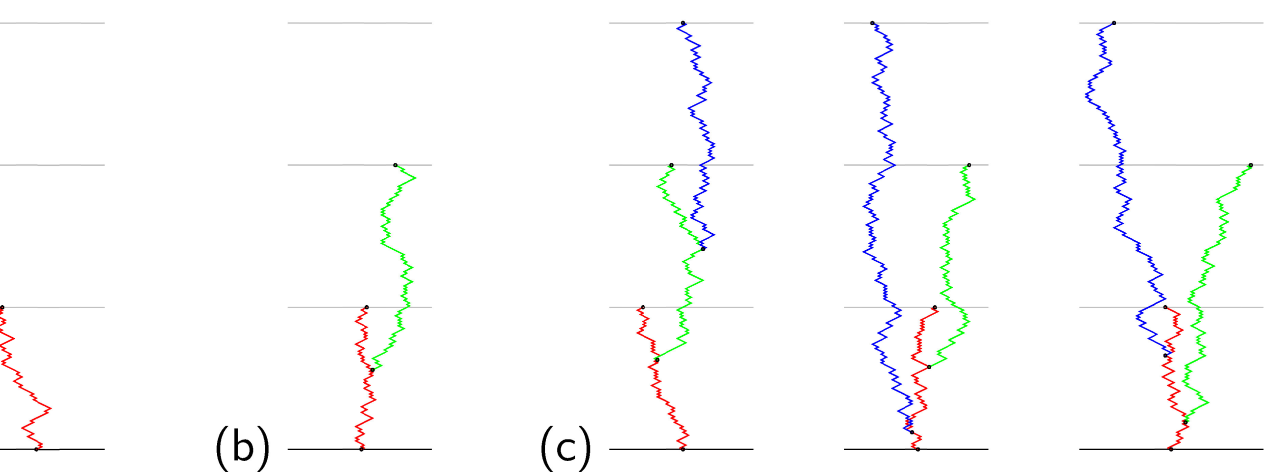
The r-point functions: Results

$-k^2 t/2d$.

der Hofstad – S 2003 Let $d > 4$, $p = p_c$, $\delta \in (0, 1 \wedge \frac{d-4}{2})$, $t_i \in (0, \infty)$.
ch that for $L \geq L_0$ we have

$$\vec{k} \over \overline{Dn} \Big) = nVA^3 \left[\int_0^{t_1 \wedge t_2} \hat{p}_s(k_1 + k_2) \hat{p}_{t_1 - s}(k_1) \hat{p}_{t_2 - s}(k_2)\, ds + O\Big(\frac{1}{n^\delta}\Big) \right]$$

result for all r-point functions, $r \geq 4$ (convergence to super-Brownian



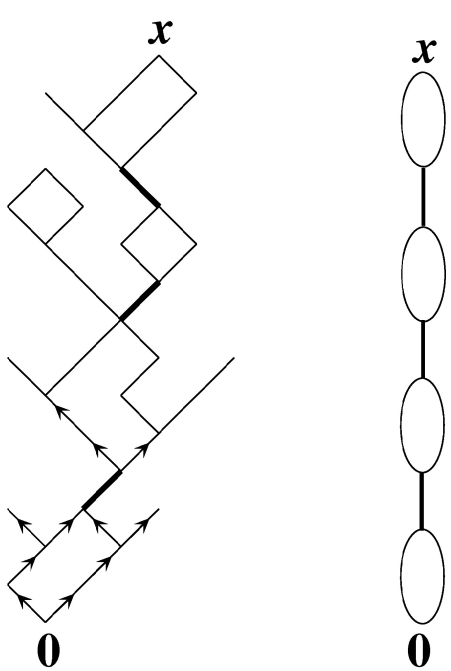
Notation

$$(x, n) = \boldsymbol{x}$$

$$\tau_n(x) = \tau(\boldsymbol{x})$$

$$\tau)(\boldsymbol{x}) = \sum_{\boldsymbol{y}} \sigma(\boldsymbol{y})\tau(\boldsymbol{x} - \boldsymbol{y}) = \sum_{y \in \mathbb{Z}^d} \sum_{m=0}^{n} \sigma_m(y)\tau_{n-m}(x - y)$$

Lace expansion for two-point function



uration, and its schematic representation as a “string of sausages.”

onsists of the pivotal bonds.

sion is an inclusion-exclusion expansion in which the “sausages” are treated
, to leading order.

Lace expansion for two-point function

ercolation, $\exists$ three different versions of the expansion for the two-point

Nguyen–Yang, (iii) Sakai.

e following:

$$\Rightarrow \boldsymbol{x}) + \mathbb{P}(\mathbf{0} \longrightarrow \boldsymbol{x},\ \mathbf{0} \not\Rightarrow \boldsymbol{x})$$

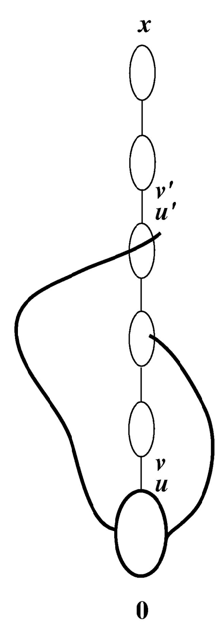
$$\Rightarrow \boldsymbol{x}) + \sum_{(\boldsymbol{u},\boldsymbol{v})} \mathbb{P}(\mathbf{0} \Rightarrow \boldsymbol{u} \text{ and } (\boldsymbol{u},\boldsymbol{v}) \text{ is occupied and pivotal for } \mathbf{0} \longrightarrow \boldsymbol{x})$$

$$\Rightarrow \boldsymbol{x}) + \sum_{(\boldsymbol{u},\boldsymbol{v})} \mathbb{P}(\mathbf{0} \Rightarrow \boldsymbol{u})\tau_1(\boldsymbol{v}-\boldsymbol{u})\tau(\boldsymbol{x}-\boldsymbol{v}) - R(\boldsymbol{x}).$$

$\boldsymbol{x}) = \delta_{\mathbf{0},\boldsymbol{x}} + \pi^{(0)}(\boldsymbol{x})$. For $\boldsymbol{x} \neq \mathbf{0}$,

$$\boldsymbol{x}) = \pi^{(0)}(\boldsymbol{x}) + (\tau_1 * \tau)(\boldsymbol{x}) + (\pi^{(0)} * \tau_1 * \tau)(\boldsymbol{x}) - R(\boldsymbol{x}).$$

Lace expansion for two-point function



ne remainder term. The heavy and light lines correspond to percolation
on distinct copies of the same lattice.
nsion gives

$$\tau(\boldsymbol{x}) = \pi(\boldsymbol{x}) + (\tau_1 * \tau)(\boldsymbol{x}) + (\pi * \tau_1 * \tau)(\boldsymbol{x}),$$

$$\pi^{(0)}(\boldsymbol{x}) - \pi^{(1)}(\boldsymbol{x}) + \pi^{(2)}(\boldsymbol{x}) - \cdots .$$

Induction

ion as:

$$\sum \tau_1(y)\tau_n(x-y) + \pi_{n+1}(x) + \sum_{m=2}^{n}\sum_{u,v}\pi_m(u)\tau_1(v-u)\tau_{n-m}(x-v)$$

er transform:

$$= \hat{\tau}_1(k)\hat{\tau}_n(k) + \hat{\pi}_{n+1}(k) + \hat{\tau}_1(k)\sum_{m=2}^{n}\hat{\pi}_m(k)\hat{\tau}_{n-m}(k). \qquad (*)$$

bution to π_m is $\pi_m^{(0)}$ and, by the BK inequality,

$$\sum_x \mathbb{P}((0,0) \Rightarrow (x,m))e^{ik\cdot x}\Bigg| \le \sum_x \tau_m(x)^2 \le \|\tau_m\|_\infty \|\tau_m\|_1 \le \|\hat{\tau}_m\|_1 \hat{\tau}_m(0).$$

ctive approach to recursions like (*) is given in van der Hofstad – S '02, and nclusion that $\hat{\tau}_n(k/\sqrt{n})$ behaves like a Gaussian, if $d > 4$ and $L \gg 1$.

ritical branching process vs oriented percolation

Difference: multiple occupancy for branching process

vs

single occupancy for oriented percolation.

noticeable until two oriented percolation paths join, e.g., $(0,0) \Rightarrow (x,n)$.

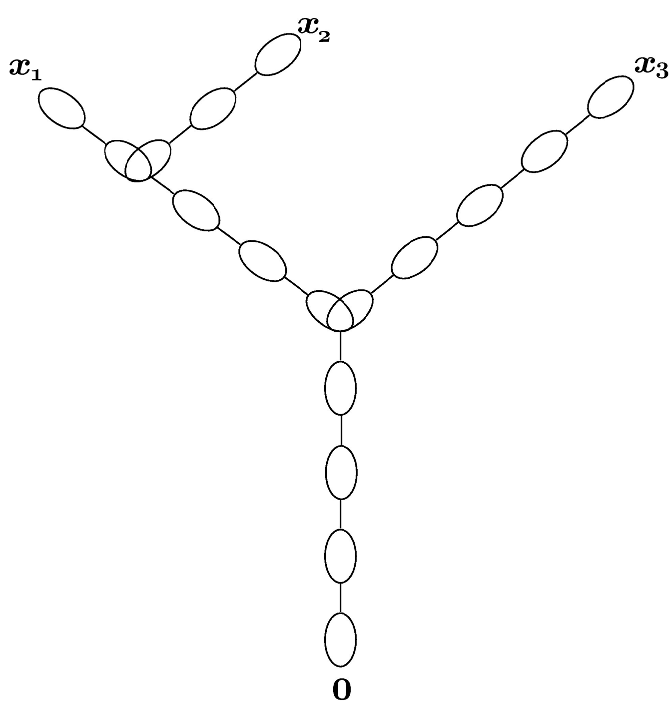
$\geq L_0$, $n \geq 1$,

$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_{p_c}((0,0) \Rightarrow (x,n)) \leq CL^{-d} n^{-d/2},$$

$$\sum_{n=1}^{\infty} n \sum_{x \in \mathbb{Z}^d} \mathbb{P}_{p_c}((0,0) \Rightarrow (x,n)) \leq CL^{-d}.$$

ne effect of closed loops is small when $d > 4$, and the difference is small.

General r-point functions



ticated expansion shows that the general r-point function, which can be a "tree of sausages," asymptotically decomposes into a product of two-point a vertex factor V at each branch point.

The survival probability

$_p(\exists x \in \mathbb{Z}^d : (0,0) \to (x,n)) \equiv \mathbb{P}_p(\mathbf{0} \to n)$.

aviour of $\theta_n(p)$ as $n \to \infty$? Clearly $\theta_n(p) \downarrow \theta(p)$ as $n \uparrow$,

(exponentially fast) for $p < p_c$, and $\theta_n(p) \to \theta(p) > 0$ for $p > p_c$.

lation at p_c, $\theta_n(p_c) \to 0$ as $n \to \infty$. In what manner?

$_n(p_c) \approx n^{-1/\rho}$.

lton–Watson branching process with offspring distribution having mean 1 ^{2},

$$\hat{\theta}_n \sim \frac{2}{\sigma^2 n}.$$

eld value of ρ is 1.

The survival probability: main result

bability: Define

$$\begin{aligned}\Delta\theta_n(p) &= \theta_n(p) - \theta_{n+1}(p)\\ &= \mathbb{P}_p((0,0) \to n,\ (0,0) \not\to n+1).\end{aligned}$$

der Hofstad – den Hollander – S 2007. For $d > 4$, $\exists L_0(d)$ such that, for

$\to \infty$,

$$\Delta\theta_n(p_c) = \frac{2}{AVn^2}\left[1 + O(n^{-1}\log n) + L^{-d}O(\delta_n)\right]$$

$$\delta_n = \begin{cases} n^{-(d-4)/2}\log n & (4 < d < 6)\\ n^{-1}\log^2 n & (d = 6)\\ n^{-1}\log n & (d > 6),\end{cases}$$

$\to \infty$,

$$= \sum_{m=n}^{\infty} \Delta\theta_m(p_c) = \frac{2}{AVn}\left[1 + O(n^{-1}\log n) + L^{-d}O(\delta_n)\right].$$

The survival probability: constants

$$\theta_n(p_c) \sim \frac{2}{AVn}$$

A, V are those seen before. In particular,

$$\sum_{x\in\mathbb{Z}^d} \tau_n(x) = A[1 + o(1)],$$

$$\tau_{n_1,n_2}(x_1, x_2) = A^3 V(n_1 \wedge n_2)[1 + o(1)] \qquad \text{as } n_1 \wedge n_2 \to \infty,$$

$O(L^{-d})$ and $V = 1 + O(L^{-d})$ as $L \to \infty$.

Survival is rare but vigorous when it occurs

r $d > 4$ and $L \geq L_0(d)$, conditional on survival to time n, $n^{-1}N_n$ kly to an exponential random variable with parameter $\lambda = 2/(A^2 V)$.

≥ 1,

$$(n^{-1}N_n)^l | N_n > 0] = \lim_{n\to\infty} \frac{1}{\theta_n}\frac{1}{n^l} \sum_{y_1,\ldots,y_l} \mathbb{P}((0,0) \longrightarrow (y_j, n))$$

$$= \lim_{n\to\infty} \frac{AVn}{2}\frac{1}{n^l} \hat{\tau}^{(l+1)}_{n,\ldots,n}(\vec{0})$$

$$= (A^2V)^l 2^{-l} l!.$$

that $\mathbb{E}[N_n] = \sum_x \tau_n(x) \longrightarrow A$ can now be understood to correspond to ents

$$\mathbb{P}(N_n > 0) \sim \frac{2}{AVn}, \qquad \mathbb{E}[N_n | N_n > 0] \sim \frac{A^2V}{2} n.$$

clusters rarely survive to time n, but when they do, they are large.

urvival probability for critical branching process

ical branching process with offspring distribution q_m such that:

$$\sum_{m=0}^{\infty} m q_m = 1, \quad \text{variance } \sigma^2 = \sum_{m=0}^{\infty} m(m-1) q_m, \quad \text{survival prob. } \hat{\theta}_n.$$

n number of offspring of initial particle that survive to time $n+1$ leads to:

$$\hat{\theta}_{n+1} = \hat{\theta}_n - \frac{\sigma^2}{2}\hat{\theta}_n^2 + O(\hat{\theta}_n^3).$$

$_n$, so that $\hat{v}_{n+1} = \hat{v}_n + \frac{1}{2}\sigma^2 + O(\hat{v}_n^{-1})$.

e that $\hat{v}_n = \frac{1}{2}\sigma^2 n + O(\log n)$, so that

$$\hat{\theta}_n = \frac{2}{\sigma^2 n}[1 + O(n^{-1}\log n)].$$

Survival probability for oriented percolation

ace expansion for survival probability (a point-to-plane expansion) gives

$$= \sum_{m=0}^{n-1} \pi_m p \theta_{n-1-m} - \sum_{m_1=1}^{\lfloor n/2 \rfloor} \sum_{m_2=m_1}^{n} \phi_{m_1,m_2} \theta_{n-m_1} \theta_{n-m_2} + e_n,$$

agrammatic estimates are proved for π_m, ϕ_{m_1,m_2} and e_n, valid for $p = p_c$ $\sim 1)$, $d > 4$ and $L \geq L_0(d)$, e.g.,

$= 0$, $|\pi_m| \leq C_\pi L^{-d} m^{-d/2}$ $(m \geq 2)$, and $p_c \sum_{m=0}^{\infty} \pi_m = 1$.

$+ O(L^{-d})]$ and

$$_\phi L^{-d} m_1^{-(d-2)/2} (m_2 - m_1)^{-(d-2)/2} \quad (m_2 \geq m_1 \geq 1,\ m_1 + m_2 \geq 3).$$

s then analysed via induction, with induction hypothesis on $v_j = 1/\theta_j$.

The incipient infinite cluster

he IIC is a natural measure $\mathbb{P}_\infty$ such that $\mathbb{P}_\infty(|C(0,0)| = \infty) = 1$.

uct IIC on a tree: Kesten, Barlow-Kumagai.

percolation constructed by Kesten for $d = 2$:
$0 \to \partial B_n\}$ at p_c and let $n \to \infty$,
$\{0 \to \infty\}$ at $p > p_c$ and let $p \downarrow p_c$.

other constructions for $d = 2$.

d – Járai: construction of IIC for $d > 6$ (spread-out model).

The IIC for oriented percolation

$\mathbb{P}_{p_c}((0,0) \to (x,n))$ and $\tau_n = \sum_x \tau_n(x)$. Let

$$\mathbb{P}_n(E) = \frac{1}{\tau_n} \sum_x \mathbb{P}(E \cap \{(0,0) \to (x,n)\}),$$

$$\mathbb{Q}_n(E) = \mathbb{P}_{p_c}(E \mid (0,0) \to n).$$

H–dH–S 02,07. For $d > 4$, $p = p_c$, $L \geq L_0(d)$,

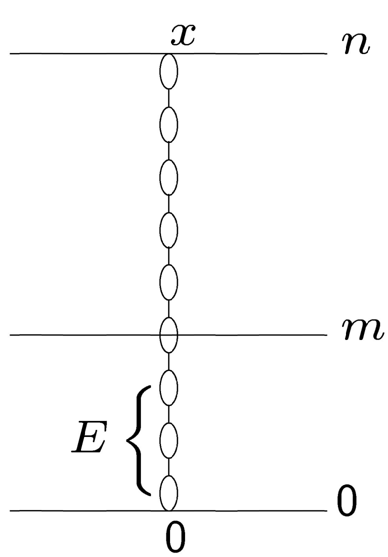
$$\lim_{n\to\infty} \mathbb{P}_n(E) \quad \text{exists,} \quad \mathbb{Q}_\infty(E) = \lim_{n\to\infty} \mathbb{Q}_n(E) \quad \text{exists,} \quad \mathbb{Q}_\infty = \mathbb{P}_\infty,$$

$$) \to \infty) = 1.$$

Idea of proof of existence of $\mathbb{P}_\infty$

$$\mathbb{P}_n(E) = \frac{1}{\tau_n} \sum_{x \in \mathbb{Z}^d} \mathbb{P}(E \cap \{(0,0) \to (x,n)\}).$$

guration contributing to $E \cap \{(0,0) \to (x,n)\}$ as a "string of sausages:"



$_n$, i.e., E depends only on bonds below level m.

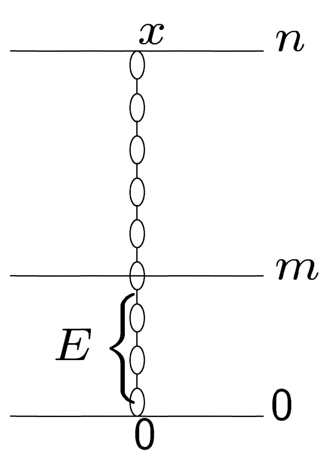
Idea of proof of existence of $\mathbb{P}_\infty$

e lace expansion, which to leading order treats as independent the sausages ve m, gives

$$\mathbb{P}_n(E) = \frac{1}{\tau_n}\left[\sum_{l=m}^{n-1} \varphi_l(E)\tau_{n-l-1} + \varphi_n(E)\right]$$

$\leq C(l-m+1)^{-(d-2)/2}$ (for $d > 4$, L large). Can then take limit to get

$$\mathbb{P}_\infty(E) = \sum_{l=m}^{\infty} \varphi_l(E) \quad (E \in \mathcal{E}_m).$$



$_\infty$ is similar but uses asymptotics of survival probability.

Geometry of IIC

t $N_n = \#\{y \in \mathbb{Z}^d : (0,0) \to (y,n)\}$. Under $\mathbb{P}_\infty$, $n^{-1}N_n$ converges
ze-biased exponential random variable (density $\lambda^2 x e^{-\lambda x}$) with parameter
).

t computation of moments as before.

Let $B_n = \sum_{m=0}^{n} N_n$. Under $\mathbb{P}_\infty$, $n^{-2}B_n$ converges weakly to a
random variable.

row.

is 4-dimensional: Under $\mathbb{P}_\infty$, (in fact more can be said)

$$c_1 R^4 \leq \mathbb{E}_\infty\big[\#\{(y,m) \in C(0,0) : |y| \leq R\}\big] \leq c_2 R^4.$$

s of the IIC two-point function

$$_m(y) = \mathbb{P}_\infty((0,0) \to (y,m)) = \lim_{n\to\infty} \frac{1}{\tau_n} \sum_x \tau^{(3)}_{n,m}(x,y).$$