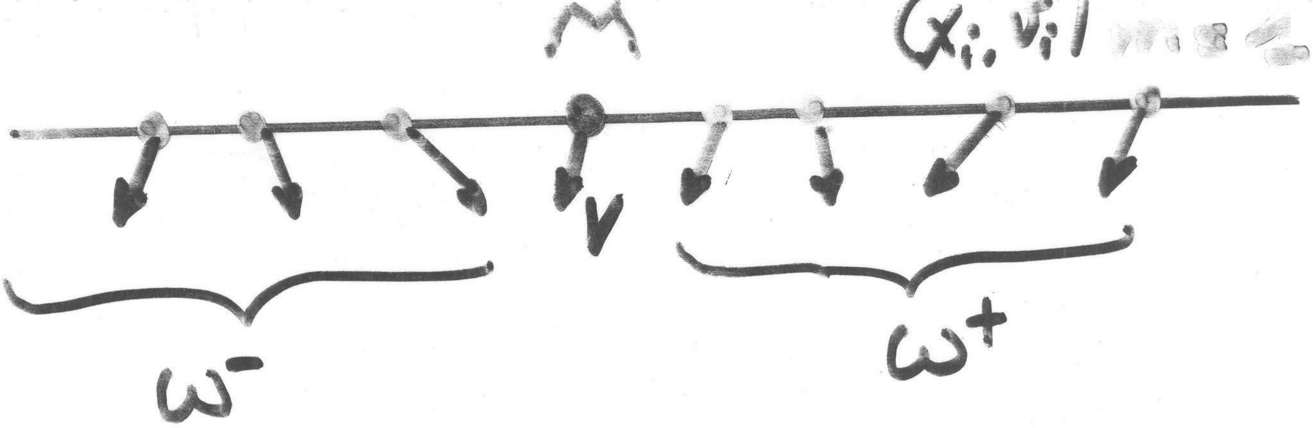


$$\mathcal{X} = (\omega^+, \omega^-, V) \in \mathcal{X}$$

Dynamics: free motion + elastic collisions

$$V^{out} = \frac{M-1}{M+1} V^{in} + \frac{2}{M+1} v^{in}$$

$$v^{out} = \frac{2M}{M+1} V^{in} - \frac{M-1}{M+1} v^{in}$$

Gibbs measure:

$\omega^\pm$: PPP on $R^\pm \times R$ with intensity $dx \cdot \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$

V: $N(0, \sigma^2 = \frac{1}{M})$

$(\mathcal{X}, \mu^M, S_t^M)$ dynamical flow

$$V_t^M := V(S_t^M \mathcal{X}) \; ; \; Q_t^M = \int_0^t V_s^M \, ds$$

$$\frac{Q_{At}^M}{\sqrt{A}} \Longrightarrow ? \quad \text{as } A \to \infty$$

$$\boxed{\begin{aligned} \underline{\sigma}^2 &= \sqrt{\pi/8} \\ \overline{\sigma}^2 &= \sqrt{2/\pi} \end{aligned}}$$

Summary of old results:

① $M = 1$, all masses equal

F. Spitzer (1969)

T. Harris (1965)

$$A^{-1/2} Q_{At}^M \Longrightarrow \overline{\sigma} W_t$$

② Ornstein-Uhlenbeck limit

R. Holley (1971)

fix $m > 0$

$$\gamma(m) = \frac{4}{m}\sqrt{\frac{2}{\pi}} \; ; \; D(m) = \frac{8}{m^2}\sqrt{\frac{2}{\pi}}$$

$$d\eta_t^m = -\gamma(m)\,\eta_t^m\,dt + \sqrt{D(m)}\,dW_t$$

$$d\xi_t^m = \eta_t^m\,dt$$

$M = m.A$:

$$A^{1/2}\,V_{At}^M \Rightarrow \eta_t^m, \quad A^{-1/2}\,Q_{At}^M \Rightarrow \xi_t^m$$

Remark:

$$\xi_t^m \Rightarrow \underline{\sigma}\,W_t \quad \text{as} \quad m \to 0$$

③ Bounds on the limiting variance

Sinai, Soloveichik (1986)

Szász, Tóth (1986)

$M \ll A$:

$$\underline{\sigma}^2 t \le \overline{\lim_{A\to\infty}}\,\mathrm{Var}\left(\frac{Q_{At}^M}{\sqrt{A}}\right) \le \bar{\sigma}^2 t$$

(4) Large mass Wiener limit

Szász, Tóth (1987)

$1 \ll M \ll A$: $\qquad A^{-1/2} Q^{M}_{At} \Rightarrow \sigma W_t$

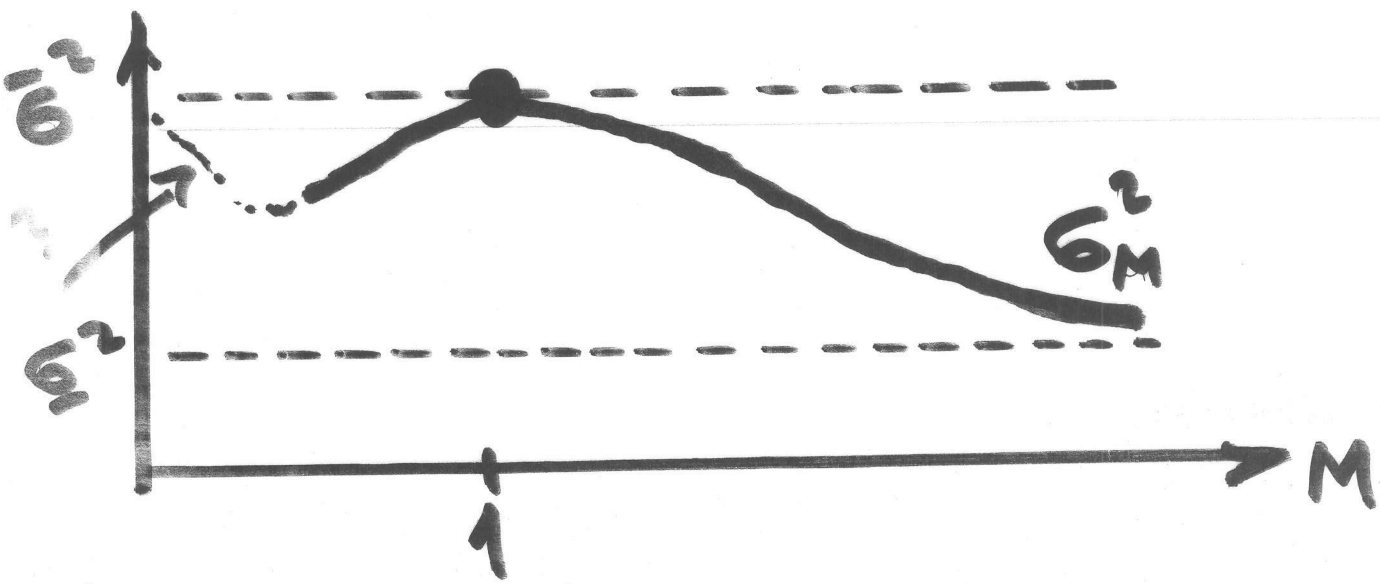
Numerics, simulations:

Omerti, Ronchetti, Dürr (1986)

Khazin (1987)

Boldrighini, Cosimi, Frigio (1990)

Fernandez, Marro (1993)



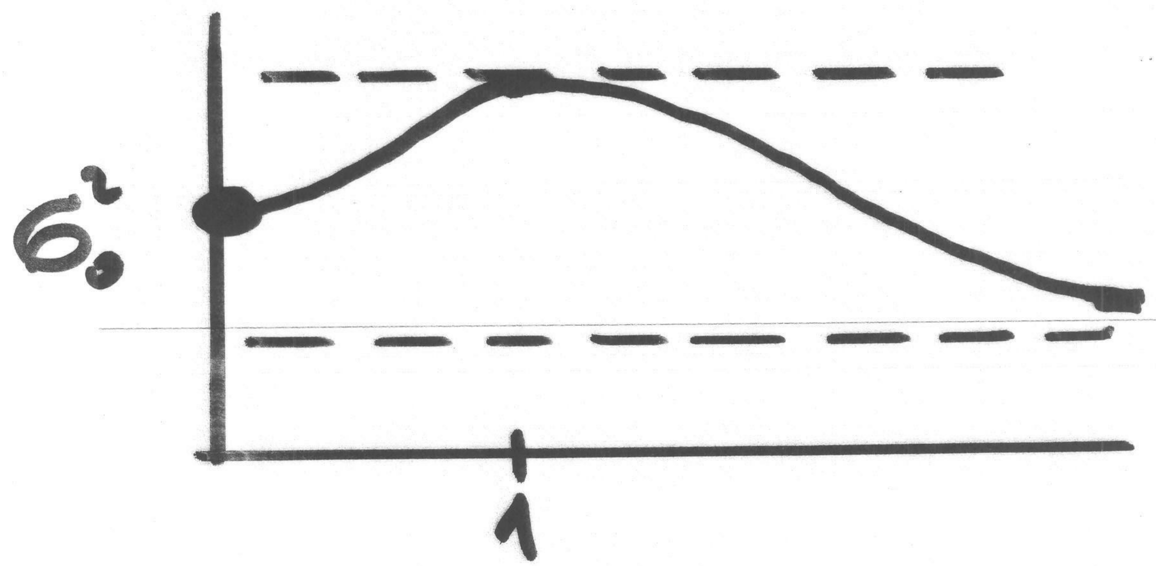
Questions:

$M \mapsto \sigma^2_M$? , $\lim_{M\to\infty} \sigma^2_M = \sigma^2$? , $\lim_{M\to 0} \sigma^2_M = \bar{\sigma}^2$?

$$\text{weak-lim} \frac{Q^M_{At}}{\sqrt{A}} = \begin{matrix}\text{Wiener ?} \\ \text{Gauss ?}\end{matrix}$$

More recent numerical work:

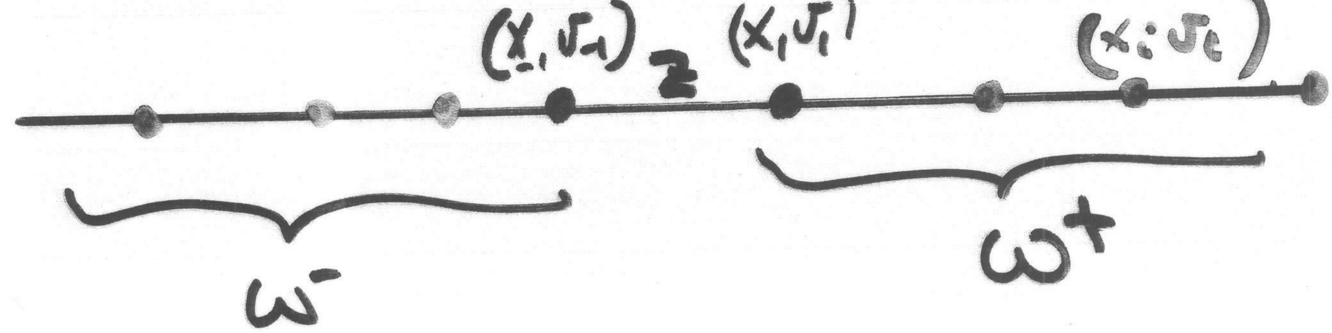
Boldrighini, Frigio, Tognetti (2002)



$$\underline{\sigma}^2 < \sigma_0^2 < \overline{\sigma}^2$$

Model II.

all masses equal



$$x = (\omega^+, \omega^-, z) \in \Omega^+ \times \Omega^- \times \mathbb{R}^+ = \mathcal{X}$$

free motion + elastic collisions + pair potential between the two central particles

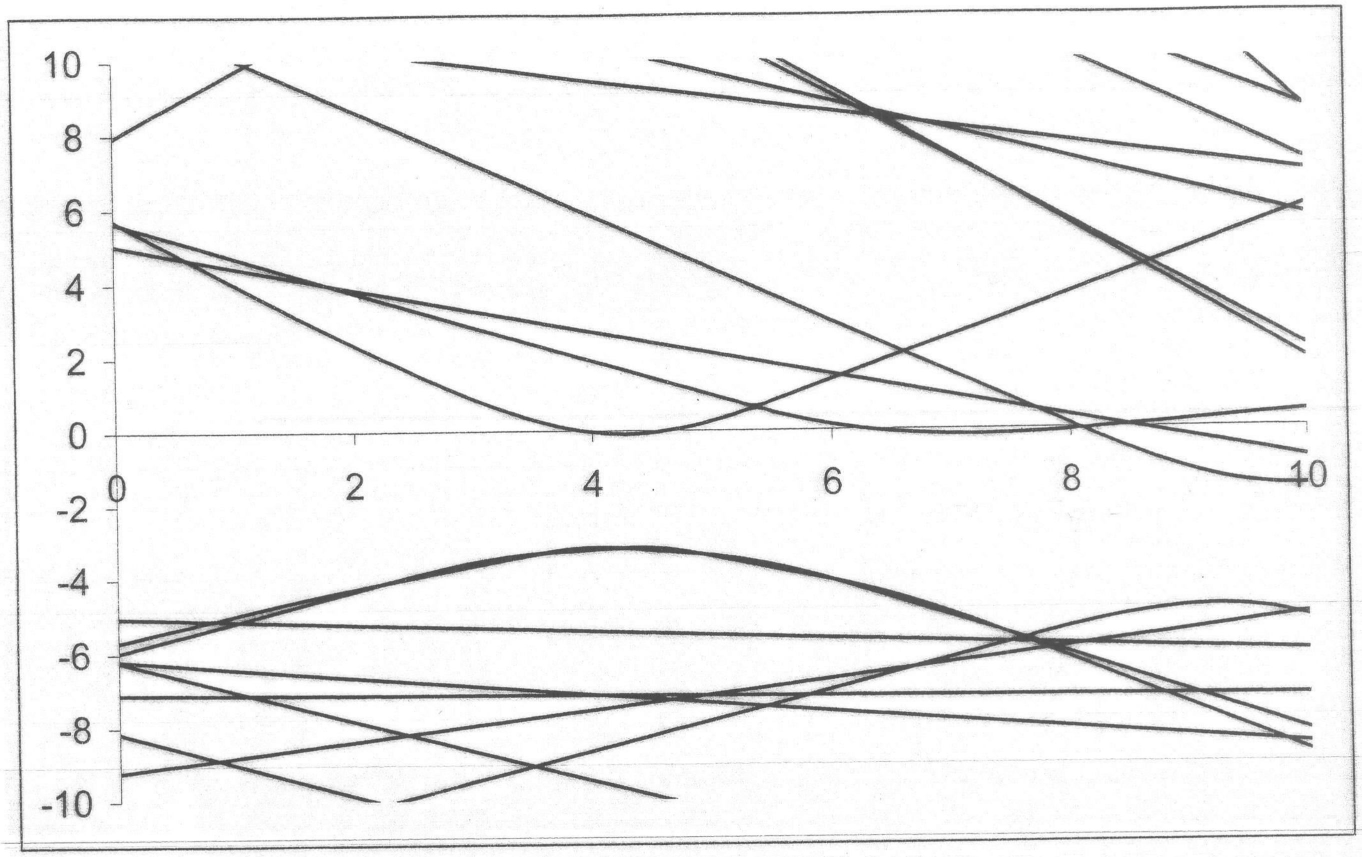
$$U(z) = \frac{c^2}{2z^2}, \quad F(z) = \frac{c^2}{z^3}$$

Gibbs measure:

$\mu^\pm$ PPP $\qquad dx \cdot \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv$

$$d\mu^c = d\mu^+ \times d\mu^- \times f^c(z)\,dz$$

$$f^c(z) = \frac{1}{Z(c)} \exp\left(-z - \frac{c^2}{2z^2}\right)$$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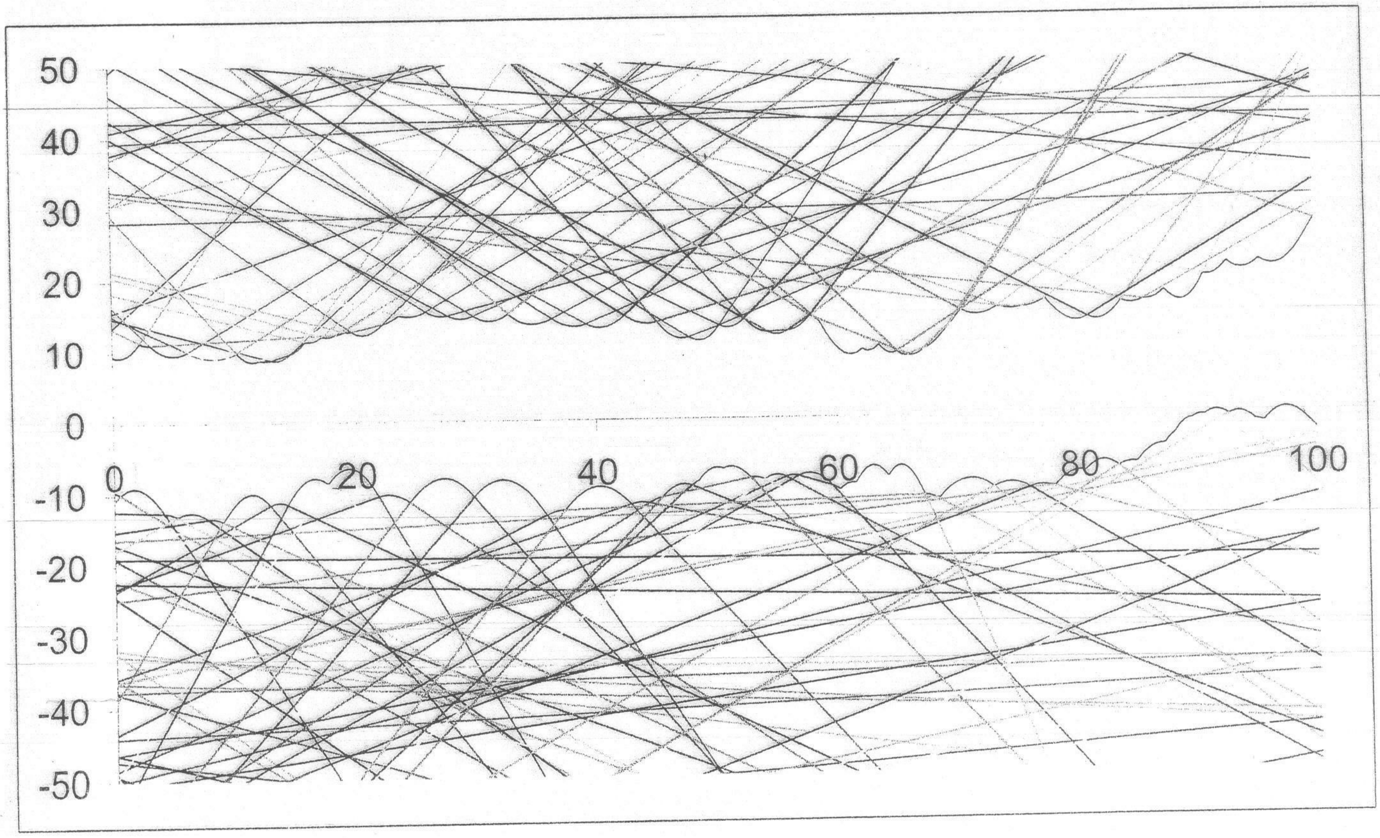
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60
80
100

$(\mathcal{X}, \mu^c, S^c_t)$ dynamical flow

$$V^c_t := \frac{1}{2}\left\{ v_{-1}(S^c_t x) + v_{+1}(S^c_t x) \right)$$

$$Q^c_t := \int_0^t V^c_s \, ds$$

<u>Thm</u>: Fix $z \in \mathbb{R}^+$, $W \in \mathbb{R}$, $c := \sqrt{z} \cdot W$

Let $M_n \to 0$, $V_n = W M_n^{-1/2} + o(M_n^{-1/2})$

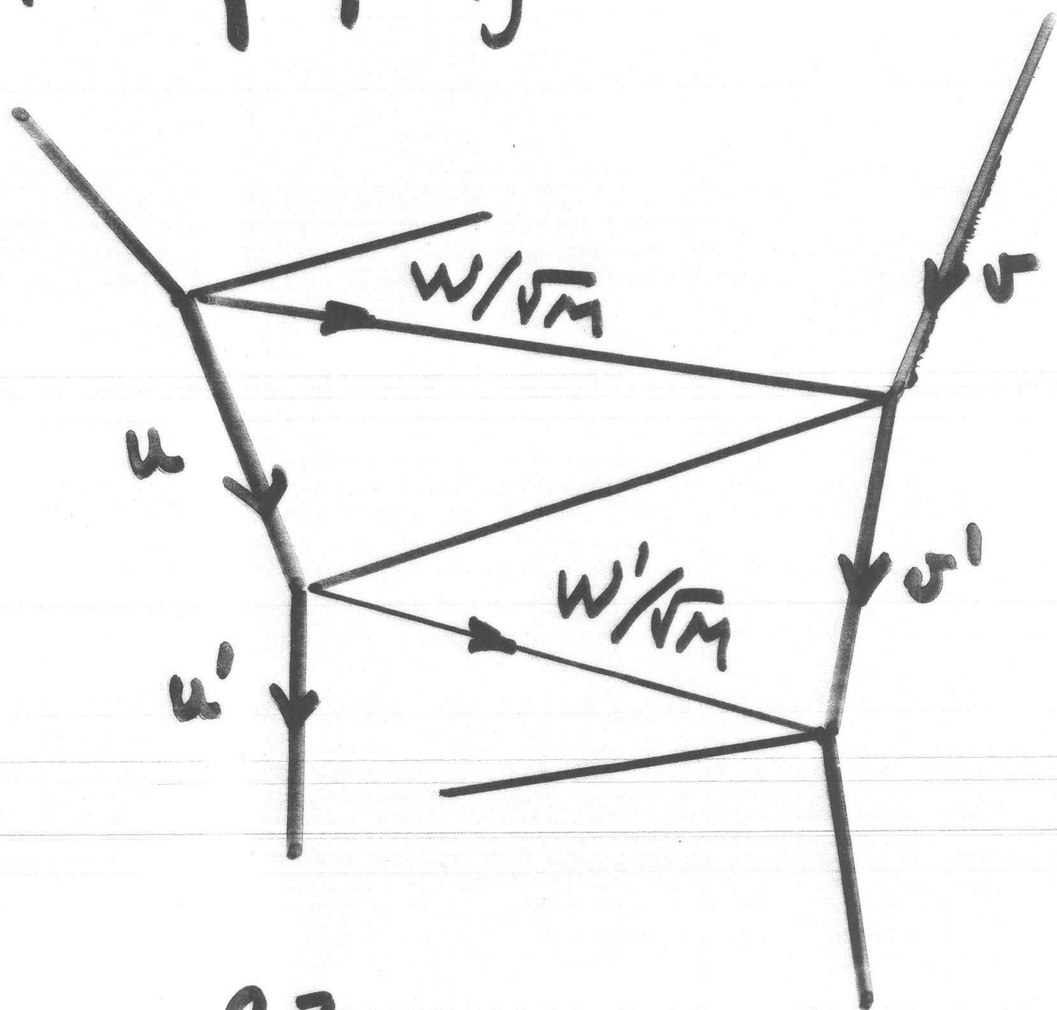
(ω^+, ω^-, u) such that

$$\left.\begin{array}{l} S_t^{I, M_n}(\omega^+, \omega^-, z, u, V_n) \\ S_t^{II, c}(\omega^+, \omega^-, z) \end{array}\right\} \text{well defined}$$

Then: $\forall t$:

$$\lim_{n\to\infty} \Pi^{II, I} S_t^{I, M_n}(\omega^+, \omega^-, z, u, V_n) = S^{II, c}(\omega^+, \omega^-, z)$$

Sketch of proof:



$$dt = \frac{2z}{W}\sqrt{M} + o(\sqrt{M})$$

$$W' - W = 2(u-v)\sqrt{M} + o(\sqrt{M})$$

$$u' - u = -2W\sqrt{M} + o(\sqrt{M})$$

$$v' - v = 2W\sqrt{M} + o(\sqrt{M})$$

$$\dot{W} = \frac{W(u-v)}{z}, \quad \dot{z} = v - u$$

$$\dot{v} - \dot{u} = \frac{2W^2}{z}$$

$$\frac{\dot{W}}{W} + \frac{\dot{z}}{z} = 0 : \qquad W \cdot z = c$$

$$\ddot{z} = \frac{2c^2}{z^3} : \quad U(z) = \frac{c^2}{2z^2}$$

$$F(z) = \frac{c^2}{z^3}$$

Remark on measures:

Under μ^M: $\xi \overset{\text{distr}}{=\!=} \Gamma(2)$

$z e^{-z} \, dz$

Under μ^c: $\xi \overset{\text{distr}}{=\!=} f^c(z) \, dz$

Fact: Let $\underset{\Gamma(2)}{\xi}, \underset{N(0,1)}{W}$ be indep.

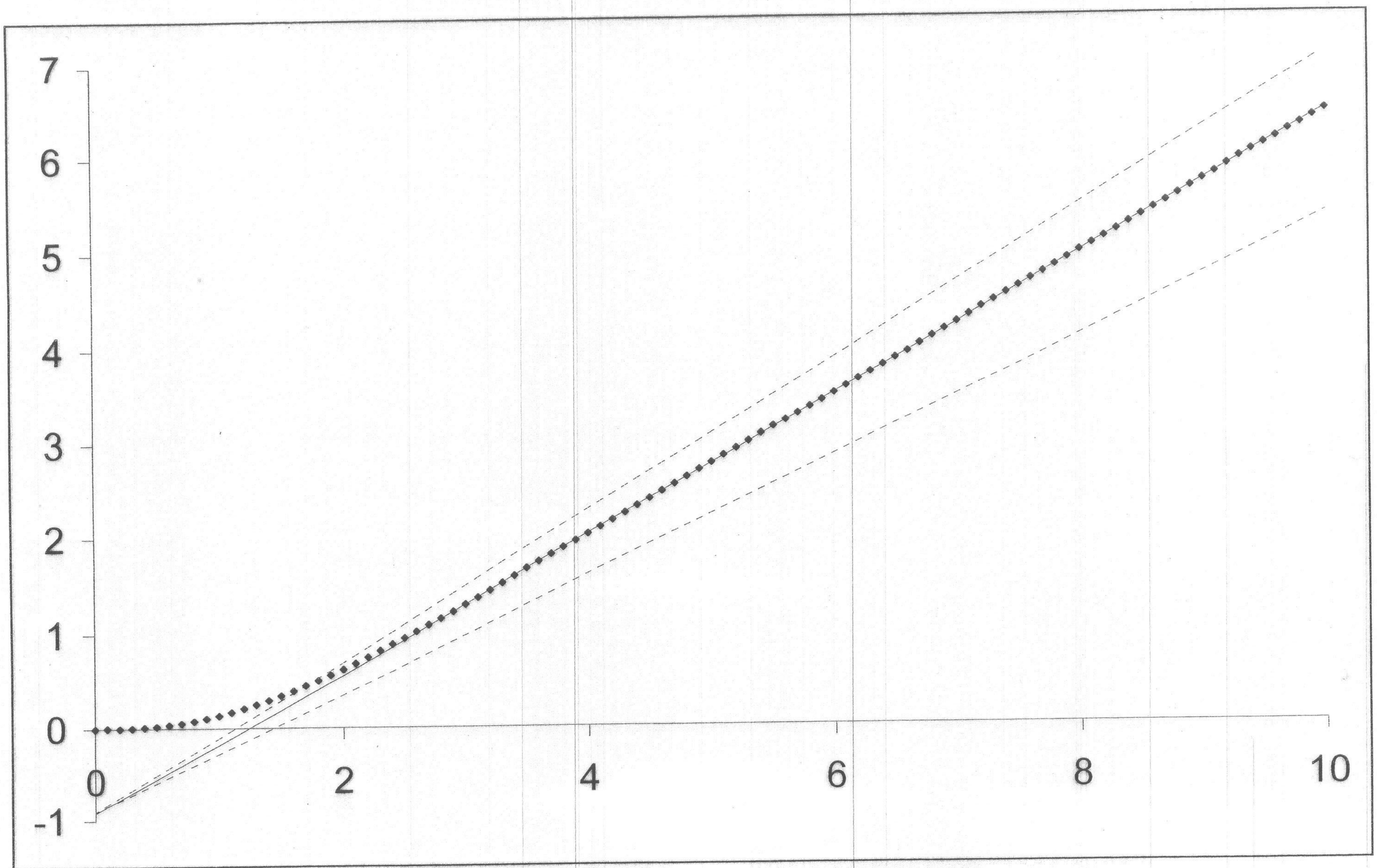
$$(\xi \mid |\xi \cdot W| = c) \overset{\text{distr}}{=\!=} f^c(z) \, dz$$

or $c \overset{\text{distr}}{=\!=} |\Gamma(2) \otimes N(0,1)|$

given c, $\xi \overset{\text{distr}}{=\!=} f^c(z) \, dz$

then $\xi \overset{\text{distr}}{=\!=} \Gamma(2)$.

interpret: (we get mixture.)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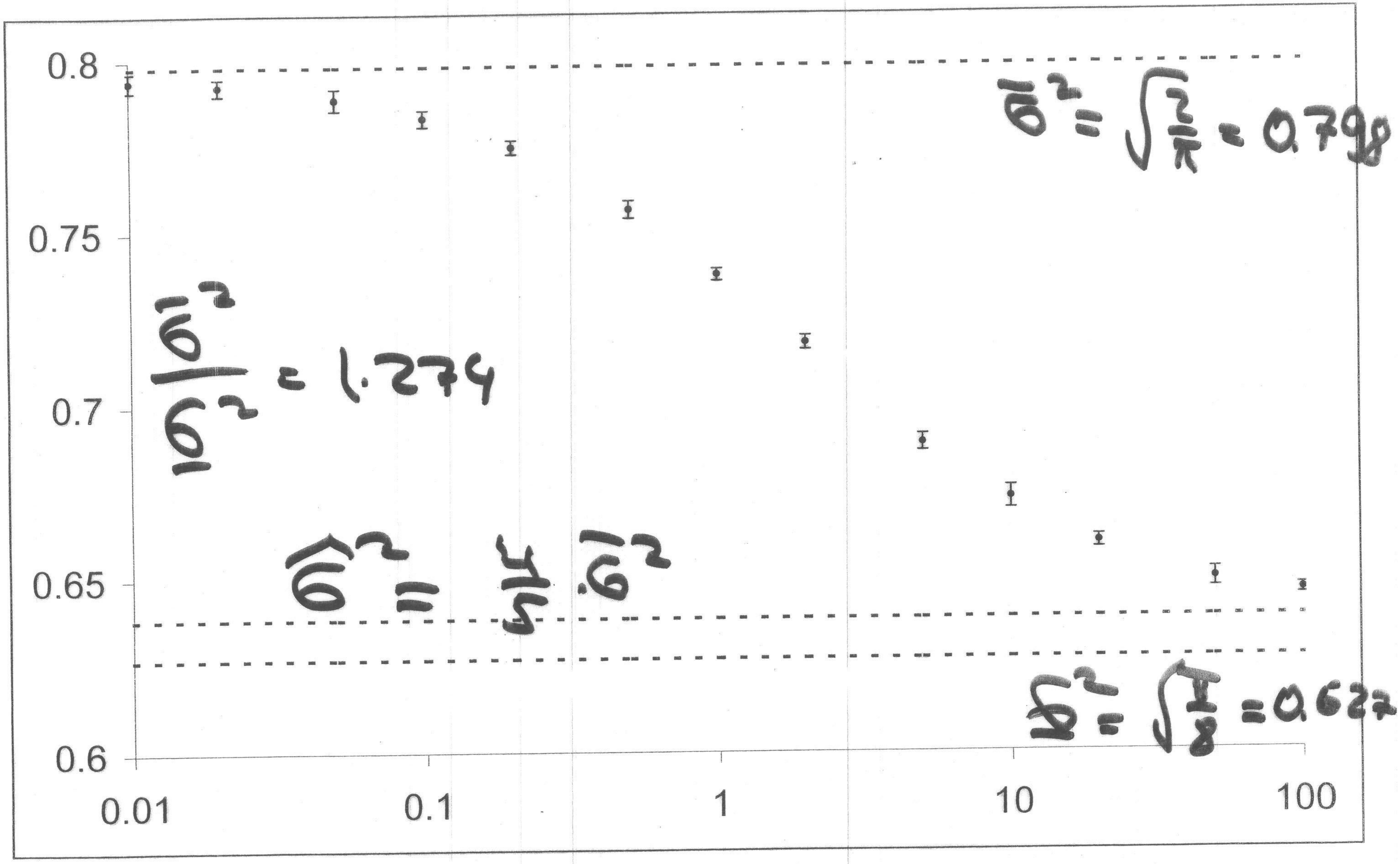

0.8
0.75
0.7
0.65
0.6
0.01
0.1
1
10
100
$\bar{\sigma}^2 = \sqrt{\frac{2}{\pi}} = 0.798$
$\frac{\bar{\sigma}^2}{\hat{\sigma}^2} = 1.274$
$\hat{\sigma}^2 = \frac{\pi}{4} \cdot \bar{\sigma}^2$
$\hat{\sigma}^2 = \sqrt{\frac{\pi}{8}} = 0.627$

Bounds on the variance

$\mathcal{C}_t^{\pm} := \{ (x_i, v_i) \in \omega^{\pm}(0) :$ particle i is frontal sometime in $[0, t] \}$

$\mathcal{N}_t^{\pm} := \{ (x_i, v_i) \in \omega^{\pm} :$ particle i would cross the origin in free dynamics, within $[0, t] \}$

$$C_t^{(k)} := \sum_{i \in \mathcal{C}_t^-} |v_i|^k - \sum_{i \in \mathcal{C}_t^+} |v_i|^k$$

$$N_t^{(k)} := \sum_{i \in \mathcal{N}_t^-} |v_i|^k - \sum_{i \in \mathcal{N}_t^+} |v_i|^k$$

__Lemma__ (Sz, T '86 with inspiration from S, S '86)

$$C_t^{(k)} + m_k Q_t = N_t^{(k)} + o_{L_p}(t^{1/4 + \varepsilon})$$

$k=0: \quad M_t + Q_t = N_t$ ← these are compound Poisson

$k=1: \quad P_t + m_1 Q_t = R_t$ ←

$k=2: \quad K_t + Q_t = L_t$ ←

time reversal:

for $X_t = X_t(x)$

$$\bar{X}_t = X_t(\mathcal{R} S_t x)$$

same variable observed on the reversed trajectory

$$X_t^s = \frac{1}{2}(X_t + \bar{X}_t)$$

$$X_t^a = \frac{1}{2}(X_t - \bar{X}_t)$$

$Q_t = Q_t^a$

$M_t = M_t^s$ conserv. of particle n

$P_t = P_t^a + o(t^{1/4+\varepsilon})$ cons. of mom.

$$K_t^a = \Delta_t E_{kin}^+ - \Delta_t E_{kin}^- = \cdots$$

$$= 2\int_0^t V_s \frac{c^2}{Z_s^3}\, ds$$

$$= 2\Big[\gamma Q_t + \underbrace{\int_0^t V_s \Big(\frac{c^2}{Z_s^3} - \gamma\Big) ds}_{Y_t}\Big]$$

$\gamma = \gamma(c)$ chosen so that

$$\lim_{t\to\infty} \mathbb{E}\Big(\frac{Q_t}{\sqrt{t}} \cdot \frac{Y_t}{\sqrt{t}}\Big) = 0$$

then: $K_t = K_t^s + \underbrace{2Y_t + 2\gamma Q_t}_{K_t^a}$

$$\left.\begin{aligned} M_t^s + Q_t &= N_t \\ K_t^s + 2Y_t + (2\gamma+1)Q_t &= L_t \end{aligned}\right\}$$

Hence

$$\overline{\lim}\, \mathbb{E}\, Q_t^2/t \le \inf_{a,b} \frac{E\left((aN_t + bL_t)^2\right)}{\left(a + b(1+2\gamma)\right)^2}$$

$$\lim_{c\to\infty} \gamma(c) = 1 \quad \text{almost sure}$$

Hence:

$$\overline{\lim_{c\to\infty}}\ \overline{\lim_{t\to\infty}}\ \mathbb{E}\, Q_t^2/t \le \frac{4}{5}\bar{\sigma}^2 =: \hat{\sigma}^2$$