On mixing for MP's (diffusions) & Poisson equations in Sobolev spaces

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(1a)

Some motivations

- Limit theorems
  [Ibragimov & Linnik book]
  [Freidlin & Wentzell]
  [Ethier & Kurtz]
- Poisson equations in $\mathbb{R}^d$
  [Pardoux, V] – '01, '03, '05

$\Rightarrow$ · functional CLT
· averaging in BSDE's

- Filtering with unspecified initial data
  [Kleptsina, V. – '07, CRAS Paris]
  [—"— preprints*]
- More generally, Statistics problems

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* www.maths.leeds.ac.uk/~veretenn/ → preprints
& www.math.univ-lemans.fr/ → statistiques → prépublications

β-mixing for homogeneous MP's

[TV - convergence]

- Doeblin*-Doob : - case $\mathcal{D}$ [ε & ν]
  - case C [$C^{-1}\nu(\cdot) \le P_x(X_1 \in \cdot) \le C\nu(\cdot)$]

  *(C$\mathcal{D}$/$\mathcal{D}\cup\mathcal{D}$)

  uniform bounds

  [Doob '53]

- $\mathcal{DDD}$: $\underbrace{\inf_{x,\tilde{x}} \int \left(\frac{P_x(dx')}{P_{\tilde{x}}(dx')} \wedge 1 \right) P_{\tilde{x}}(dx')}$ $=: 1 - \rho_{\mathcal{D}} > 0$

  [Dobrushin ~'57]

- rate of β-mixing is NOT a spectral property

- Local $\mathcal{DD}$ [case C] + recurrence

  vs

  Local $\mathcal{DDD}$ + recurrence

- Q: uniform bounds under any condition "between" case $\mathcal{D}$ and $\mathcal{DDD}$?

- For Markov diffusions, the role of $\mathcal{DDD}$ plays Harnack inequality [V-'87], et al. ["case C" is unnatural here]

Poisson equation with a parameter

(P) $\underbrace{L(x,y)\,u(x,y) = -f(x,y), \quad x \in \mathbb{R}^d}$, $y \in \mathbb{R}^l$,

where $L(x,y) = \sum_{i,j=1}^{d} a_{ij}(x,y) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x,y) \frac{\partial}{\partial x_i}$,

$$a = \tfrac{1}{2} \sigma \sigma^* .$$

[PSV-'76; Ethier & Kurtz-'86] [FW-'70s]

- L is an "ergodic generator" $\forall y$,
  $\mu_\infty^y(dx)$ - a unique invariant measure

- Centering condition on f:

$$\underbrace{\langle f, \mu_\infty \rangle = \int f(x,y)\, \mu_\infty^y(dx) = 0}, \ \forall y$$

- <u>Goal</u>: to have two derivatives* w.r.t. (x,y), for Itô's formula

[Pardoux & V - '01, '03, '05]

* Sobolev Ok

Assumptions

$$dX_t^y = b(X_t^y, y)\,dt + \sigma(X_t^y, y)\,dB_t, \quad X_0^y.$$

(H_b) $\qquad \overline{\lim}_{|x|\to\infty} \sup_y \langle b(x,y), x\rangle = -\infty$

[recurrence, may be relaxed]

(H_a) $\qquad a(\cdot)$ bounded & uni. elliptic (& uni. continuous in x)

(H_f) $\qquad f(\cdot)$ bounded [may be relaxed]

$(H^{0,j})$ $\quad f, a, b \in C_b^{0,j}$ $\qquad$ [f — " —]

Class of solutions:

$$u \in \bigcap_{p>1} W^2_{p,loc} \cap C,$$

$$\& \quad \langle u, \mu^y_\infty \rangle = 0.$$

$$\|g\|_{L_{p,loc}} := \sup_{x_0} \left(\int_{B_1} |g(x_0+x)|^p dx\right)^{\frac{1}{p}} \times (1+|x_0|)^{-\varepsilon}, \quad \varepsilon>0$$

Representations & Theorem

- $u(x,y) = \int_0^\infty E_x f(X_s^y, y)\,ds$

$$\equiv \int_0^\infty \underbrace{p_t(x, f(\cdot,y), y)}\,ds,$$

where

$$p_t(x, g(\cdot), y) := \underbrace{E_x g(X_t^y) - \langle g, \mu_\infty^y \rangle}.$$

[Thm] $u_y(x,y) = \int_0^\infty \overbrace{(\partial_y p_t)}^{q_t}(x, f(\cdot,y),y)\,ds$

$+ \int_0^\infty p_t(x, f_y(\cdot,y), y)\,ds;$ } under $(H^{0,1})$, $(H_{a,b,f})$

- u_{yx} - sim.
- u_{yy} - similarly; $\leftarrow$ } under $(H^{0,2})$ & $(H_{a,b,f})$

- $\partial_y p_t(x,g,y) \equiv q_t(x,g,y) \equiv q_t^g(x,y)$

$$:= \underbrace{\int_0^t E_x \frac{\partial L}{\partial y}(X_s^y, y)\, p_{t-s}(X_s^y, g, y)\,ds},$$

- q_t^g bounded, continuous in y

Preliminaries

<u>Pro 1</u> [V'99] : under $(H_{a,b})$,

- $\sup_y \| \mu^y_{x,t} - \mu^y_\infty \|_{TV} \leq C \dfrac{(1+|x|^m)}{(1+t^k)}$ $\quad (\forall k \, \exists m, C)$

- $\sup_y \sup_t \int |x'|^m \mu^y_{x,t}(dx') \leq C (1+|x|^{m'})$
  $(\forall m, \exists C, m')$

- $\sup_y \int |x'|^m \mu^y_\infty (dx') < \infty \quad (\forall m)$

=

<u>Pro 2</u> [Pardoux, V - '01] : under $(H_{a,b})$,

- $\exists!$ u - solution of (P);

- f - bdd $\Rightarrow$ $\forall \varepsilon > 0$, $\exists C$:
  $$\| u(x,y) \| + |\nabla_x u(x,y)| \leq C (1+|x|)^\varepsilon .$$

(· In addition, if $\sup_y \| f(x,y) \| \leq C(1+|x|)^{-2-\varepsilon}$,
then u & $\nabla_x u$ are <u>bounded</u>.)

$$\text{(q)}\quad \underbrace{q^g_t(x,y) := \int_0^t E_x \overbrace{\frac{\partial L}{\partial y}(X^y_s, y)\, p_{t-s}}^{f^1_{t-s}(\dots)}(X^y_s, g, y)\, ds}_{\equiv \int_0^t ds \int f^1_s(x'', g)\, \mu^y_{x,t-s}(dx'')}$$

- $p_{t-\cdot}(\cdot, g, y) \in W^{1,2}_{p,loc}$

  $\Rightarrow \frac{\partial L}{\partial x}\, p_{t-\cdot}(\cdot, g, y) \in L_{p,loc}$

  $\Rightarrow q^g_t$ exists, by Krylov's theorem.

- continuity in y follows

- to show that $\underbrace{q^g_t = \partial_y p_t(x,g,y)}$,

idea of (q)

consider finite differences

$$q^{h,g}_t(x,y) := \frac{p_t(x,g,y+he_i) - p_t(x,g,y)}{h},$$

and use standard semi-group representations, or Dynkin's formula,

$$q^{h,g}_t(x,y) = \int_0^t ds\; E_x\left(\underbrace{\frac{L(y+h)-L(y)}{h}}_{h \downarrow 0}\, p_{t-s}(X^y_s, g, y+h)\right).$$

$\boxed{q_t^g \text{ as } t \to \infty}$

- Denote
$$f^1_t(x,g,y) = \frac{\partial L}{\partial y} p_t(x,g,y).$$

<u>Lemma 1</u>

$$|f^1_s(x,g,y)| \le \frac{C(1+|x|^m)}{(1+t^k)}.$$

[Follows from Pro 1]

<u>Lemma 2</u> · $q_\infty(g,y) = \int_0^\infty ds \left(\int f^1_s(x'',g,y)\,\mu^y_\infty(dx'')\right).$

· $|q_t(x,g,y) - q_\infty(g,y)| \le C\frac{(1+|x|^m)}{(1+t^k)}$,

· $|q_\infty(g,y)| \le C$,

· $q_\infty(g,y) = \partial_y p_\infty(g,y).$

[Follows from Pro 1]

<u>Cor. 1</u> : u_y is well-defined

$\boxed{\partial_x p_t^g}$ & $\boxed{\partial_x^2 p_t^g}$

Lemma 2a

$$|\partial_x p_t^g(x,y)| \leq C \frac{(1+|x|^m)}{(1+t^k)}$$

[Follows from Pro 1 & Chapman-Kolmogorov]

Lemma 2.b

$$\|\partial_x^2 p_t^g(x,y)\|_{L_p(B_{x_0,1}\times[t-1,t])}$$

$$\leq C\times(1+|x_0|^\varepsilon)\times\frac{(1+|x_0|^m)}{(1+t^k)}.$$

$$\boxed{\partial_x q_t^g}$$

- $\partial_x q_t^g(x,y) = \int_0^t \partial_x E_x \frac{\partial L}{\partial y}(X_s^y, y)\, p_{t-s}(X_s^y, g, y)\, ds$

Lemma 3

$$|\partial_x E_x g(X_t^y)|$$

$$= |\partial_x E_x g(X_t^y) - \partial_x \langle g, \mu_\infty^y \rangle|$$

$$\leq C \frac{(1+|x|^m)}{(1+t^k)}.$$

[By Pro 1 & Chapman-Kolmogorov]

Lemma 4

$$|\partial_x q_t^g(x,y)| \leq C \frac{(1+|x|^m)}{(1+t^k)}$$

[By Le 3 & Le 1]

Second derivative, $\partial_y^2 p_t^g$

- Representation:

$$
\begin{aligned}
& p_t^{(2)}(x,g,y) \\
&:= \int_0^t ds \quad p_s\big(x, L_{0,2}(\cdot)\, p_{t-s}(\cdot, g)\big) \\
&\quad + \int_0^t ds \sum_{|i|=1} \partial_y\, p_s\big(x, L_{i,2}(\cdot)\, p_{t-s}(\cdot, g)\big),
\end{aligned}
$$

where

$$
L_{i,2} := C_2^i \, \frac{\partial^{2-i} L}{\partial y^{2-i}} .
$$

Equivalently, for using Krylov's est's,

$$
\begin{aligned}
p_t^{(2)}(x,g,y) &:= \int_0^t ds \; E_x\big(L_{0,2}(X_s^y)\, p_{t-s}(X_s^y, g, y)\big) \\
&\quad + \int_0^t ds \sum_{|i|=1} \partial_y^{i} \, E_x\big(L_{i,2}(X_s^y)\, p_{t-s}(X_s^y, g, y)\big).
\end{aligned}
$$

$\boxed{u_{yx}}$

- is analyzed by $\partial_x p_t$ and $\partial_x q_t$