CONSISTENT FAMILIES OF BROWNIAN MOTIONS.

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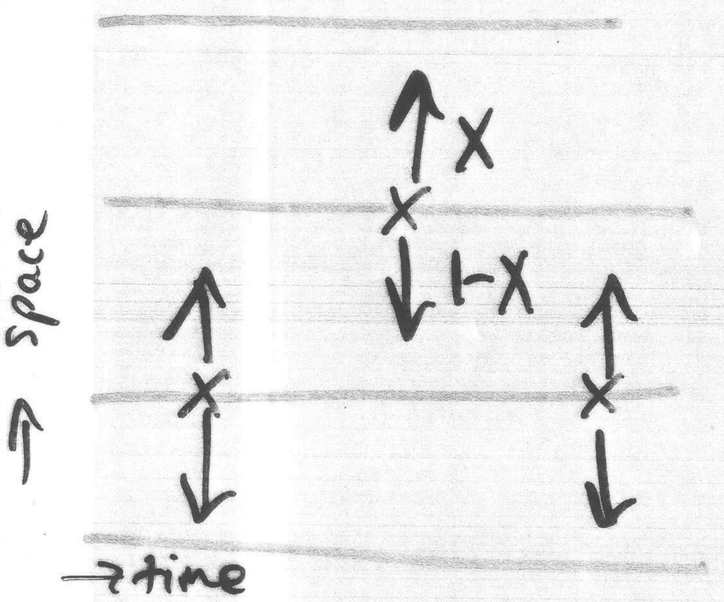
Based on joint work with Chris Howitt.

Flow of kernels: Model for random evolution of distribution of mass

time is $\mathbb{R}$
space is $\mathbb{Z}$

parameter $\mu \in \mathcal{M}_1([0,1])$



- Poisson Point process Λ unit rate on $\mathbb{R} \times \mathbb{Z}$
- Attached to each pt an independent copy of X having distribution μ

Dynamics Mass at a site x at instant t with $(t,x) \in \Lambda$ is divided in ratio $X : 1-X$, with parts moving to neighbouring sites $x+1$ and $x-1$.

Defn of flow

$K_{s,t}(x, A)$ = amount of mass in A at time t of unit mass introduced at x at time s. $(s \leq t)$

(1) $K_{s,t}$ is a <u>random kernel</u> (from $\mathbb{Z}$ to $\mathbb{Z}$)

(2) $K_{s,t} K_{t,u} = K_{s,u}$ $\quad s<t<u$

(3) $K_{t_1,t_2} \perp\!\!\!\perp K_{t_2,t_3} \cdots \perp\!\!\!\perp K_{t_{n-1},t_n}$ $\quad t_1 < t_2 \cdots < t_n$

(4) $K_{s+h,t+h} \stackrel{dist}{=} K_{s,t}$

$(K_{s,t}\,; s \leq t)$ is a <u>stochastic flow of kernels</u> (on $\mathbb{Z}$)

<u>Alternative Interpretation</u>: Marked PPP (Λ, X) is a <u>random environment</u> governing motion of a particle.

- jumps at meeting a space/time pt in Λ
  - up with probability X
  - down with probability $1-X$.

then

$K_{s,t}(x, A)$ = Conditional probability given environment that the particle is in A at time t if it is at x at time s.

<u>N point motions</u> : N particles moving <u>conditionally independently</u> given enviroment.

<u>Averaging</u> over enviroments gives a Markov process on $\mathbb{Z}^N$ with semigroup

$$P^{(N)}_{t-s}(\underline{x}, \underline{A}) = \mathbb{E}\left[K_{s,t}(x_1, A_1) \times \qquad \times K_{s,t}(x_N, A_N)\right]$$

$$\underline{x} = (x_1, \dots x_N)$$
$$\underline{A} = A_1 \times A_2 \times \dots \times A_N.$$

- clump of $(k+l)$ particles splits into k moving up, l moving down at rate $\frac{(k+l)!}{k!\,l!} p(k:l)$ where $p(k:l) = \mathbb{E}\left[X^k (1-X)^l\right]$.

So $(p^{(N)}; N \geq 1)$ determines moments of X determines law of the flow K.

__Theorem__ (Le Jan / Raimond 2004)

- A stochastic flow of kernels (on any space) is characterized by its N pt motions
- Any __consistent__ family of (Feller) Markov semigroups $(P^{(N)}; N \geq 1)$ determines a flow of kernels.

- Consistent means ... any M cpts of N-dim process in family are distributed as M dimensional member of family.

Scaling limits

Do Brownian scaling: lattice is $\frac{1}{\sqrt{n}}\mathbb{Z}$

Poisson point process intensity n

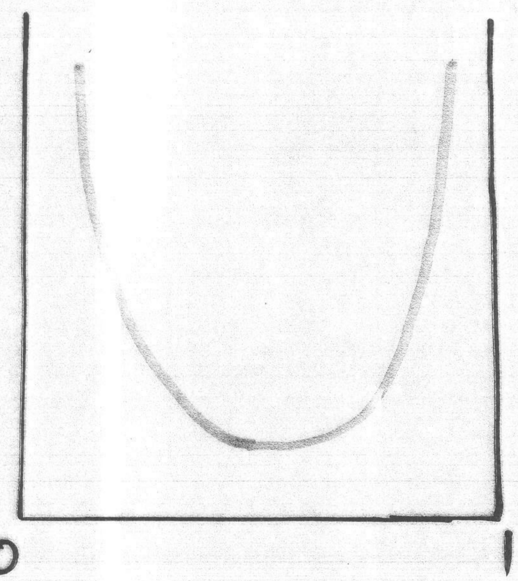
$\mu = \mu_n$ depends on n.

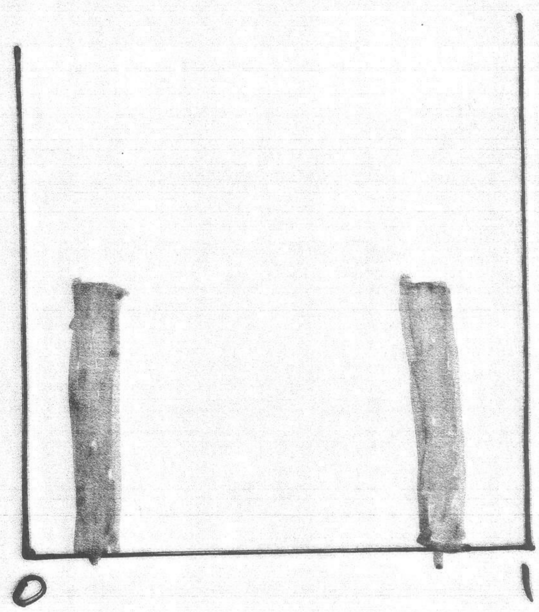
Two Examples

(A) "beta" flow

Le Jan / Raimond / Lemaire

(B) "erosion" flow





$$\mu_n(dx) = C_n\, x^{1/\sqrt{n}-1}(1-x)^{1/\sqrt{n}-1}\,dx$$

$$\mu_n = \tfrac{1}{2}\delta_{1/\sqrt{n}} + \tfrac{1}{2}\delta_{1-1/\sqrt{n}}$$

N pt motions reversible

→ Dirichlet forms.

N point motions not reversible

→ Martingale problem techniques.

Denote

$$P_n(k:l) = \int_0^1 x^k (1-x)^l \, \mu_n(dx)$$

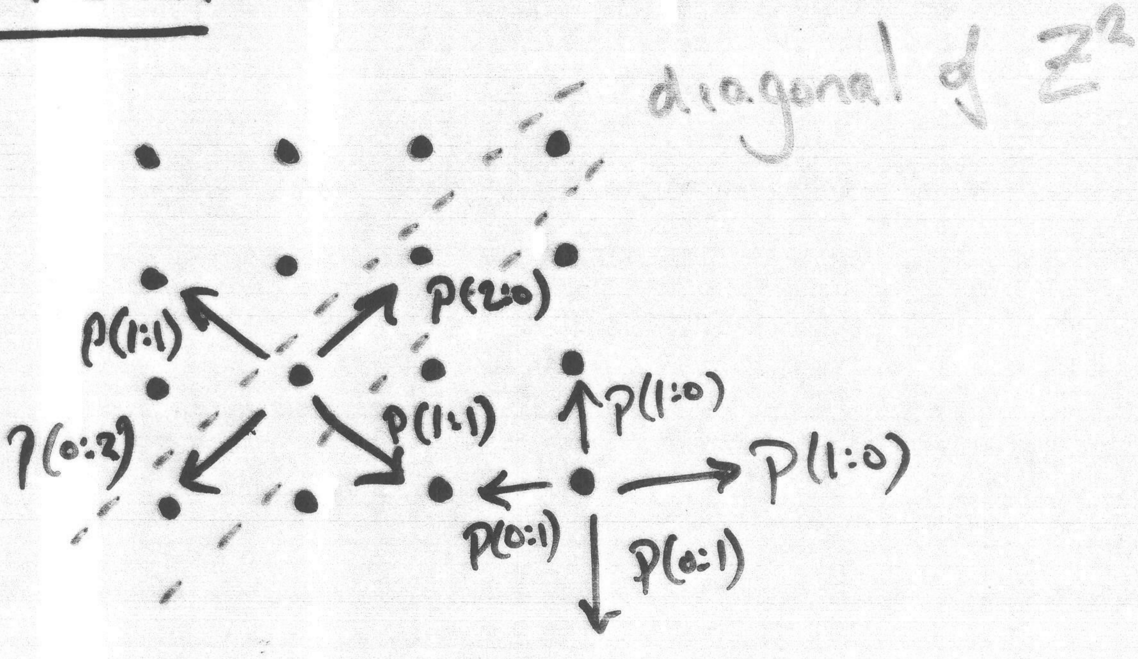
Assume μ_n symmetric about $1/2$ $\Rightarrow$

$P_n(0:1) = P_n(1:0) = 1/2$

One pt Motion

Simple symmetric random walk $\underset{n\to\infty}{\Longrightarrow}$ BM

Two pt Motion



Scaling limit if $\sqrt{n}\, P_n(1:1) \to \theta \in (0,\infty)$ as $n\to\infty$

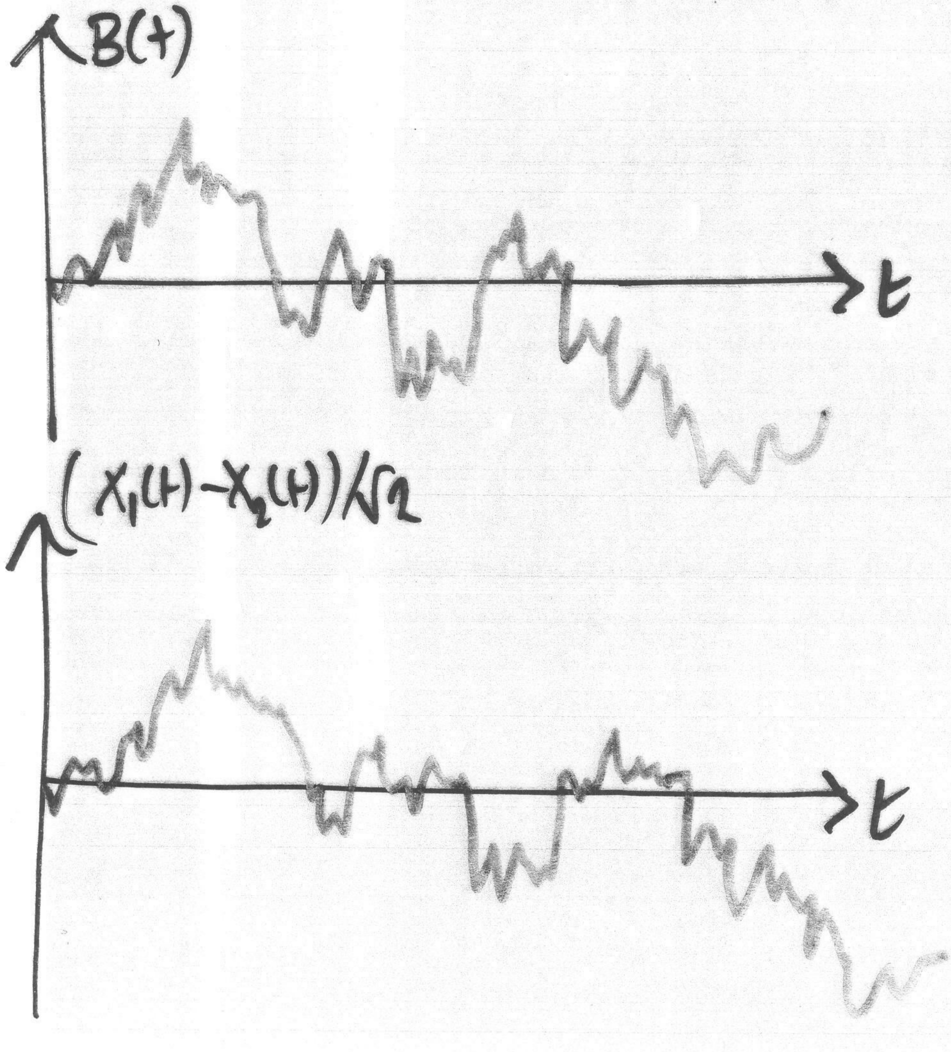
Scaling limit is a diffusion (X_1, X_2) on $\mathbb{R}^2$ with

$$\left.\begin{matrix} X_1 \\ X_2 \end{matrix}\right\} \text{ each BMs}$$

$$\langle X_1, X_2 \rangle(t) = \int_0^t 1_{(X_1(s) = X_2(s))}\, ds$$

$$L^0(|X_1 - X_2|)(t) = 4\theta \int_0^t 1_{(X_1(s) = X_2(s))}\, ds$$

" θ coupled BMs "



- SAME EXCURSIONS
- $\{t : X_1(t) = X_2(t)\}$ is Cantor set with +ve lebesgue measure.

- Action of generator of (X_1, X_2) on C^2-functions <u>doesn't characterize</u>

- By Tanaka

$$|X_1(t) - X_2(t)| - 4\theta \int_0^t 1_{(X_1(s) = X_2(s))} \, ds$$

is a martingale.

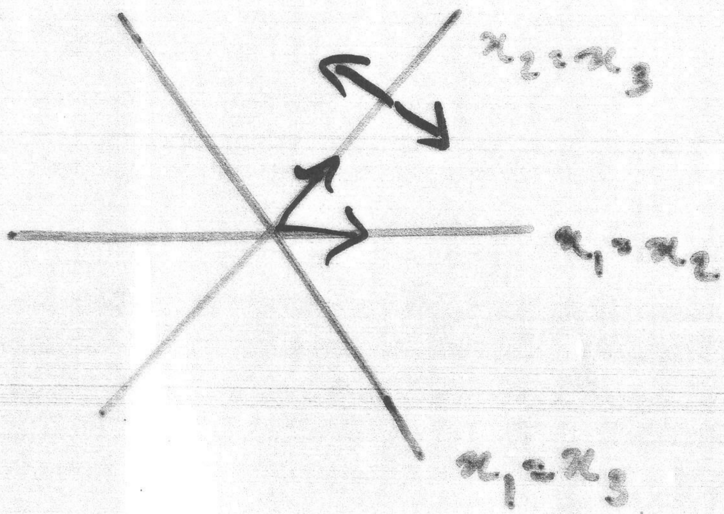
Need to apply generator to piecewise linear function $f(x_1, x_2) = |x_1 - x_2|$ to characterize.

- Need some multidimensional generalization to characterize N pt motions.

$x \in \mathbb{R}^N$ belongs to cell

$$\{ y \in \mathbb{R}^N : y_i \le y_j \text{ iff } x_i \le x_j \}$$

$\boxed{N=3}$ $x_1 + x_2 + x_3 = 0$



13 cells

For each $x \in \mathbb{R}^N$ $V(x)$ <u>set of vectors</u>

"<u>pointing out</u>" to neighbouring cells with form

$$V = V_{IJ} \quad \text{with cpts} \quad V(i) = \begin{cases} 0 & \text{if } i \notin I \cup J \\ +1 & \text{if } i \in I \\ -1 & \text{if } i \in J \end{cases}$$

GIVEN a family of parameters

$(\theta(k:l);\ k,l \ge 1)$ satisfying CONSISTENTCY

$$\theta(k:l) = \theta(k+1:l) + \theta(k:l+1)$$

Put $\theta(V_{IJ}) = \theta(|I|, |J|)$

+ SYMMETRY

- Say $f \in \mathcal{L}^N$ if $f: \mathbb{R}^N \to \mathbb{R}$ cts, linear on each cell.

- Set

$$\mathcal{A}_N^\theta f(x) = \sum_{v \in V(x)} \theta(v) \nabla_v f(x)$$

<u>Defn</u> An $\mathbb{R}^N$-valued process $(X(t); t \geq 0)$ solves the <u>$\mathcal{A}_N^\theta$-martingale problem</u> if

— Each cpt X_i is a BM

— $\langle X_i, X_j \rangle (t) = \int_0^t \mathbb{1}(X_i(s) = X_j(s))\, ds$

— $f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s))\, ds$ is a martingale for each $f \in \mathcal{L}^N$.

<u>THEOREM</u> The $\mathcal{A}_N^\theta$-martingale problem is well-posed.

<u>Moreover</u>:

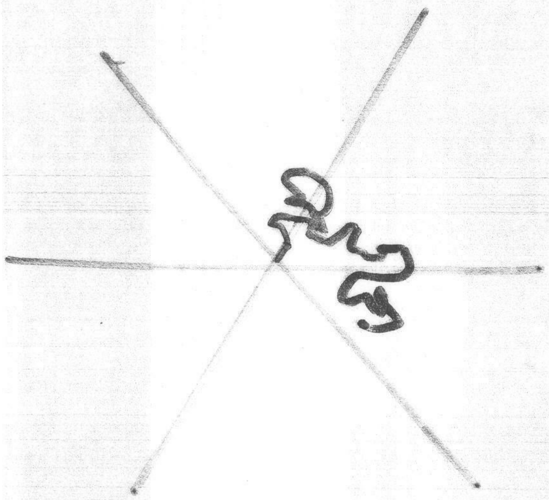
Consistent θ $\Rightarrow$ Consistent diffusions as N varies.

Existence By scaling limits with

$$\sqrt{n}\, P_n(k:l) \longrightarrow \theta(k:l)$$

Uniqueness Martingale problem describes how clump of $(k+l)$ particles splits into clumps of size k and l.

$N=3$ [Ikeda-Watanabe]



Excursion from diagonal D "leaves" in a direction given by some $v \in V(D)$

$$E(v) = \{ x + \alpha v : x \in D, \alpha > 0 \}$$

Proposition – X solves A_N^θ-mart problem starting from diagonal D

– $T_\varepsilon = \inf\{ t \geq 0 : |X_i(t) - X_j(t)| = \varepsilon \text{ some } i,j \}$

Then

$$\lim_{\varepsilon \to 0} \mathbb{P}(X(T_\varepsilon) \in E(v)) = \frac{\theta(v)}{\sum_{u \in V(D)} \theta(u)}.$$