SLE and α-S

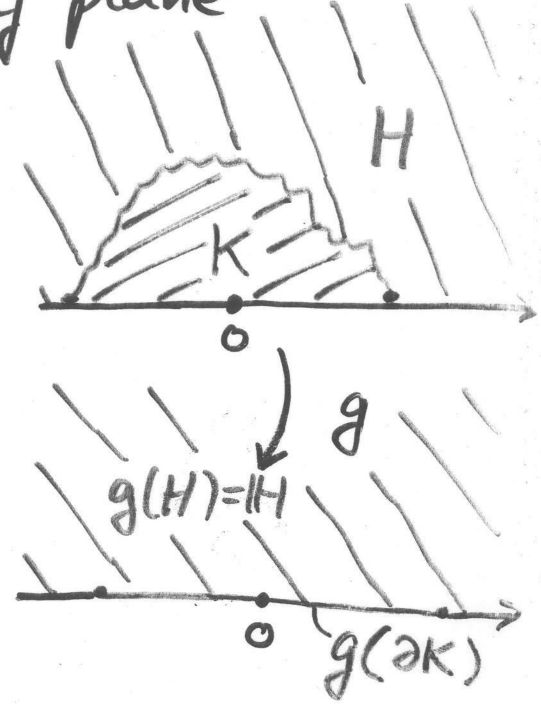
driven by Lévy pro

joint work

1. Growing clusters in $\mathbb{H}$

$\overline{\mathbb{H}} := \{ z \in \mathbb{C} : \operatorname{Im}(z) \geq 0 \}$ complex upper half-plane

We call *cluster* any compact $K \subseteq \overline{\mathbb{H}}$ such that $H := \mathbb{H} \setminus K$ is simply connected.

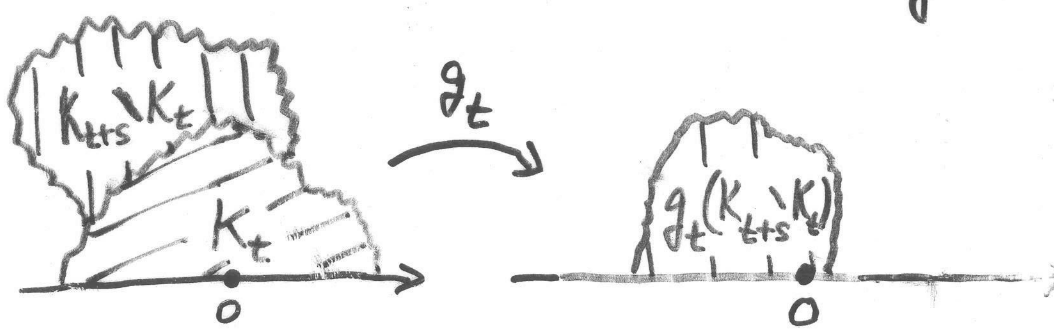


Riemann Mapping Theorem $\Rightarrow \exists!\ g : H \longrightarrow \mathbb{H}$ conformal

such that $g(z) = z + \frac{b}{z} + O\left(\frac{1}{|z|^2}\right)$ as $z \to \infty$,

for some $b \geq 0$ called half-plane *capacity* $\operatorname{hcap}(K)$, a "measure" of the size of K

Consider an *increasing cluster* $(K_t)_{t \geq 0}$ such that $\operatorname{hcap}(K_t) = 2t$



2. Local growth and Loewner Evolutions

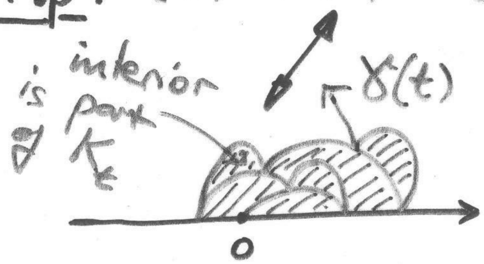
Special cases of increasing clusters: restrict growth to one point. How?

(1) Local growth: $\bigcap_{\varepsilon>0} \overline{\{g_t(z) : z \in K_{t+\varepsilon} \setminus K_t\}} = \{\text{single point}\} =: \{\underset{\mathbb{R}}{\underset{\cap}{U}}(t)\}$

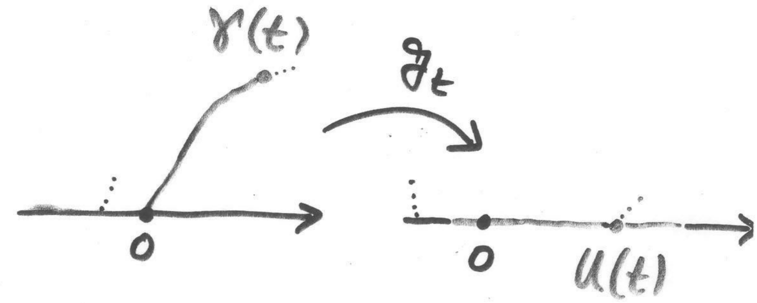
(2) K_t generated by cadlag curve: $\bigcap_{\varepsilon>0} \overline{K_{t+\varepsilon} \setminus K_t} = \{\text{single point}\} =: \{\underset{\overline{\mathbb{H}}}{\underset{\cap}{\gamma}}(t)\}$

(3) K_t is a cadlag curve: $K_t = \overline{\{\gamma(s), 0 \leq s \leq t\}}$, in particular $\bigcap_{\varepsilon>0} (\mathbb{H} \cap K_t \setminus K_{t-\varepsilon}) = \{\gamma(t-)\}$

Prop: (3) $\Rightarrow$ (2) $\Rightarrow$ (1), but (1) $\not\Rightarrow$ (2) $\not\Rightarrow$ (3)



"driving process"



Prop: If (1) holds for cadlag $(U(t))_{t\geq 0}$, then $g_t : \mathbb{H} \setminus K_t \longrightarrow \mathbb{H}$ satisfies

Loewner's equation: $\partial_t g_t(z) = \dfrac{2}{g_t(z) - U(t)}$, $g_0(z) = z$, well-defined on $[0, \zeta(z))$, $z \in \overline{\mathbb{H}}$

right derivative

Conversely: $K_t = \{z \in \overline{\mathbb{H}} : \zeta(z) \leq t\}$

3. Independent growth increments

$U(0)=0$

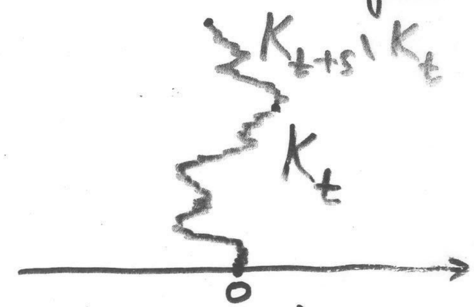
Let $(g_t)_{t\geq 0}$ be a Loewner evolution driven by a cadlag function $(U(t))_{t\geq 0}$

Recall $g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right)$ as $z\to\infty$ $\left[\begin{array}{l}\text{and } g_t(\gamma(t))=U(t)\\ \text{under condition (2)}\end{array}\right]$

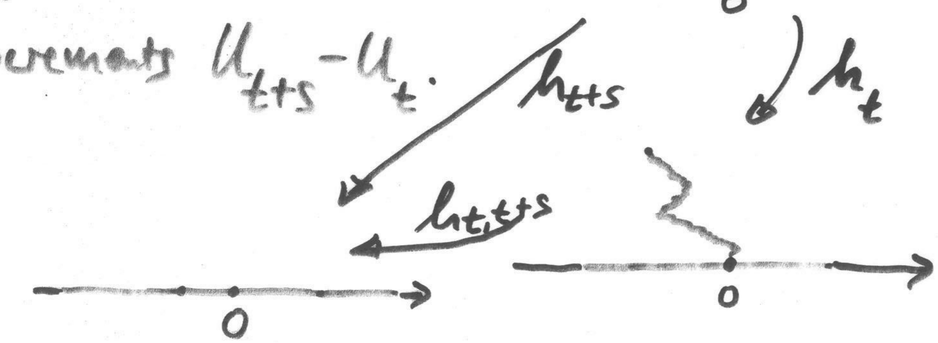
We could also choose different conformal mappings

$$h_t(z) = z - U(t) + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right) = g_t(z) - U(t) \quad \left[\begin{array}{l}\text{and } h_t(\gamma(t))=0\\ \text{under condition (2)}\end{array}\right]$$

Then Loewner's equation implies: $dh_t(z) = \frac{2}{h_t(z)}dt - dU(t)$ "Bessel flow"



Prop: $(h_t)_{t\geq 0}$ has stationary independent "increments" $h_{t,t+s} := h_{t+s}\circ h_t^{-1}$ if and only if $(U(t))_{t\geq 0}$ has stationary independent increments $U_{t+s}-U_t$.

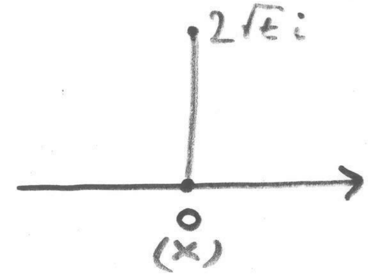


Remark: $a^{\frac{1}{\alpha}}K_{t/a} \overset{d}{=} K_t$ if and only if $\alpha=2$ and $(U(t))_{t\geq 0}$ is a multiple $(\sqrt{\kappa}B_t)_{t\geq 0}$ of Brownian motion

4. Jumps of $(U(t))_{t\geq 0}$ and tree structure of K_t

Examples: 1) $U(t) \equiv 0$, $g_t(z) = \sqrt{z^2+4t}$, $K_t = [0, 2\sqrt{t}\, i]$

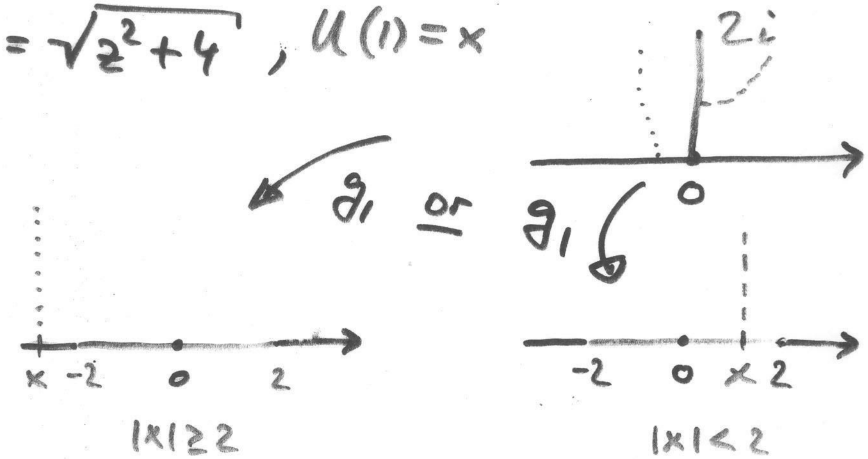
2) $U(t) \equiv x$, $K_t = [x, x+2\sqrt{t}\, i]$, translation of 1) by x.



3) $U(t) = \begin{cases} 0 & \text{for } 0 \leq t < 1 \\ x & \phantom{\text{for }} t \geq 1 \end{cases}$, $g_1(z) = \sqrt{z^2+4}$, $U(1) = x$

Recall $\partial_t g_t(z) = \dfrac{2}{g_t(z) \underbrace{- U(t)}_{=0}}$ $\quad 0 \leq t < 1$

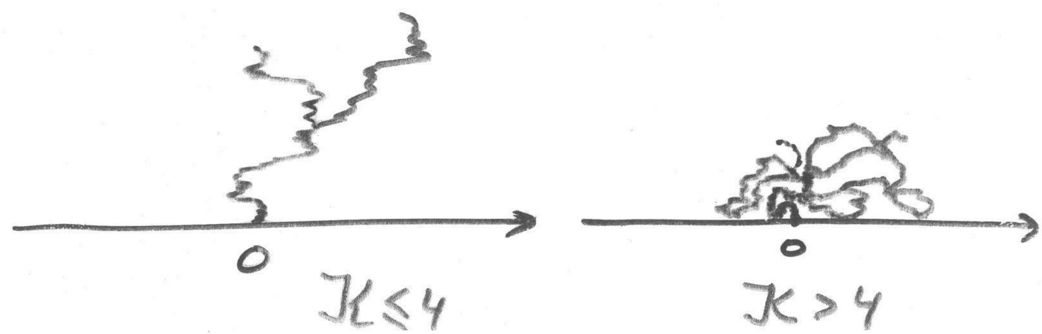
and $\partial_t g_{1,1+t}(z) = \dfrac{2}{g_{1,1+t}(z) - \underbrace{(U(1+t) - U(1))}_{=x}}$ $\quad t \geq 0$



4) $U(t) = \begin{cases} \sqrt{\kappa} B_t & \text{for } 0 \leq t < 1 \\ x + \sqrt{\kappa} B_t & \phantom{\text{for }} t \geq 1 \end{cases}$

where $(B_t)_{t \geq 0}$ Brownian motion

and x "small"



5. Phase transitions $\mathcal{K}=4$, $\alpha=1$

Consider a Lévy process $U(t)=\sqrt{\mathcal{K}}\,B_t+\theta^{1/\alpha}S_t$

$\zeta(z)=\inf\{t\geq 0: z\in K_t\}$

Brownian motion $\perp\!\!\!\perp$ symmetric α-stable

$Var(B_t)=t$ $\qquad$ $\mathbb{E}e^{i\lambda S_t}=e^{-t|\lambda|^\alpha}$

<u>Theorem</u>: (I) $0\leq\mathcal{K}\leq 4$ and $U\not\equiv 0$ $\Rightarrow$ $\mathbb{P}(\zeta(z)=\infty)=1$ for all $z\in\overline{\mathbb{H}}\setminus\{0\}$;

(II) $\mathcal{K}>4$ and $1\leq\alpha<2$ $\Rightarrow$ $\mathbb{P}(\zeta(z)<\infty)=1$ for all $z\in\overline{\mathbb{H}}\setminus\{0\}$;

(III) $\mathcal{K}>4$ and $0<\alpha<1$ $\Rightarrow$ $0<\mathbb{P}(\zeta(z)<\infty)<1$ for all $z\in\overline{\mathbb{H}}\setminus\{0\}$.

Rushkin et al predicted a weaker statement and described their simulations as

$\underbrace{\text{(I) trees} \quad \text{(II)–(III) bushes}}_{\substack{\mathcal{K}=4 \text{ phase transition} \\ \text{for } \theta=0 \text{ known (Schramm)}}}$, $\quad \underbrace{\text{(II) isolated} \quad \text{(III) forests of bushes}}_{\alpha=1 \text{ phase transition}}$

<u>Corollary</u>: (I) $Leb(\bigcup_{t\geq 0}K_t)=0$ $\quad$ (II) $Leb(\mathbb{H}\setminus\bigcup_{t\geq 0}K_t)=0$ $\quad$ (III) $Leb(K_t)>0$

<u>Conjecture</u>: (I) K_t is a tree : $\forall x,y\in K_t$ $\exists!$ $\{\rho(s), 0\leq s\leq 1\}$ simple path in $K_t\cup\mathbb{R}$ $\rho(0)=x, \rho(1)=y$

(II)–(III) K_t is generated by a left-continuous function.

6. α-SLE driven by α-stable processes

Recall $a^{\frac{1}{\alpha}} K_{t/a} \overset{d}{\neq} K_t$ even for α-stable $(U(t))_{t\geq 0}$; SLE

Brownian scaling is intrinsic to Loewner's equation $\partial_t g_t(z) = \frac{2}{g_t(z) - U(t)}$

Consider instead

$$\partial_t g_t(z) = \frac{2\,|g_t(z) - \theta^{1/\alpha} \overbrace{S_t}^{\alpha\text{-stable}}|^{2-\alpha}}{g_t(z) - \theta^{1/\alpha} S_t}, \quad g_0(z) = z, \quad t \in [0, \zeta(z))$$

$$K_t = \{ z \in \overline{\mathbb{H}} : \zeta(z) \leq t \}$$

Theorem: $\exists\, \theta_0(\alpha) > 0$ s. th.

(i) $0 \leq \theta < \theta_0(\alpha) \Rightarrow \mathbb{P}(\zeta(z) = \infty) = 1$ for all $z \in \overline{\mathbb{H}} \setminus \{0\}$; $m\left(\bigcup_{t\geq 0} K_t\right) = 0$;

(ii) $\theta > \theta_0(\alpha) \Rightarrow \mathbb{P}(\zeta(z) < \infty) = 1$ for all $z \in \overline{\mathbb{H}} \setminus \{0\}$; $m\left(\mathbb{H} \setminus \bigcup_{t\geq 0} K_t\right) = 0$

Remarks: 1) $a^{\frac{1}{\alpha}} K_{t/a} \overset{d}{=} K_t$ is scale-invariant

2) g_t is NOT conformal