Gluon scattering amplitudes at strong coupling

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Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills.

- It can give non trivial information about more realistic theories but is more tractable. In the last years, many tools become available.
- The strong coupling regime can be study, by means of the gauge/string duality, through a weakly coupled string sigma model.
- Higher loop (MHV) amplitudes appear to have a remarkable iterative structure, leading to a proposal for all loops n-point amplitudes.

Aim of this talk

Prescription for computing gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills at strong coupling by using the AdS/CFT correspondence.

Gauge theory results

Bern, Dixon, Smirnov, (Anastasiou, Carrasco, Johansson, Kosover, Roiban,...)

$$A_n^{L,Full} \sim \sum_\rho Tr(T^{a_{\rho(1)}}...T^{a_{\rho(n)}})A_n^{(L)}(\rho(1),...,\rho(2))$$

- Leading N_c color ordered $n-$points amplitude at L loops: $A_n^{(L)}$
- The amplitudes are divergent so we need to introduce a regulator.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon)$
- Scale out the tree amplitude $M_n^{(L)}(\epsilon) = A_n^{(L)}/A_n^{(0)}$. Up to few loops, $M_n^{(L)}(\epsilon)$ can be written in terms of lower order amplitudes!

$$M_4^{(2)}(\epsilon) = \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)M_4^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)$$

Motivated by the infrared behavior (plus additional studies) of multi-loop amplitudes...

MHV amplitudes: all loops proposal!

$$\mathcal{M}_n \equiv 1 + \sum_{L=1} \alpha^L M_n^{(L)}(\epsilon) = exp\left[\sum_{\ell=1}^{\infty} \alpha^\ell \left(f^\ell(\epsilon) M_n^{(1)}(\ell\epsilon) + C^{(\ell)} + \mathcal{O}(\epsilon)\right)\right]$$

$$\alpha \sim \frac{\lambda\mu^{2\epsilon}}{8\pi^2}, \qquad f^\ell(\epsilon) = f_0^\ell + \epsilon f_1^\ell + \epsilon^2 f_2^\ell$$

We will perform explicit computations for $n = 4$.

4 point amplitude

$$\mathcal{A} = \mathcal{A}_{tree} \left(\mathcal{A}_{div,s}\right)^2 \left(\mathcal{A}_{div,t}\right)^2 \exp\left\{\frac{f(\lambda)}{8}(\ln s/t)^2 + const\right\}$$

$$\mathcal{A}_{div,s} = \exp\left\{-\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right\}$$

- Divergent piece plus dynamical part of finite term are characterized by two functions.
- $\left(\lambda\frac{d}{d\lambda}\right)^2 f^{(-2)}(\lambda) = f(\lambda)$: Cusp anomalous dimension, controls leading divergence.
- $\lambda\frac{d}{d\lambda} g^{(-1)}(\lambda) = g(\lambda)$: Subleading divergence.

AdS/CFT duality

Consider a stack of $D3$-branes in type IIB string theory. Two equivalent descriptions of the same system.

- Low energy theory (quantum field theory) for the degrees of freedom of the branes.
- String theory on a curved background, induced by the matter density of the branes.

AdS/CFT duality (Maldacena)

Four dimensional maximally SUSY Yang-Mills $\Leftrightarrow$ Type IIB string theory on $AdS_5 \times S^5$.

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'}$$

$$\Downarrow$$

We can study a strongly coupled gauge theory by means of a weakly coupled sigma model

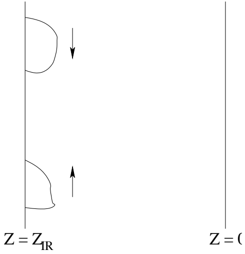
We will study scattering amplitudes at strong coupling by using the AdS/CFT duality.

- Set up the computation: Use a $D - brane$ as IR cut-off.
- Actual computations: Dimensional regularization.

String theory set up

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$$

- Place a D-brane extended along x_{3+1} and located at some large z_{IR}.



- The asymptotic states are open strings ending on the D-brane.
- Consider the scattering of these open strings.
- The proper momentum of these strings, $k_{pr} = k\frac{z_{IR}}{R}$ is very large, so we are interested in the regime of fixed angle and very high momentum.

This regime was considered in flat space (Gross and Mende)

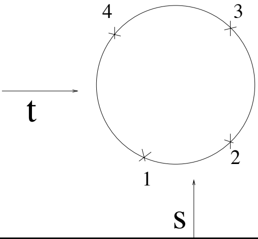
Key feature

The amplitude is dominated by a saddle point of the classical action.

$\Downarrow$

We need to consider a classical string on AdS

World-sheet with the topology of a disk with vertex operator insertions (corresponding to external states)



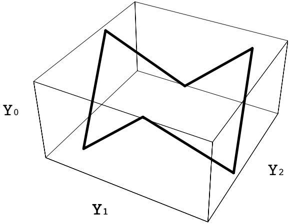
- Near each vertex operator, the momentum of the external state fixes the form of the solution.
- In the boundary of the world-sheet $z = z_{IR}$

- "T-duality": $ds^2 = w^2(z)dx_\mu dx^\mu \to \partial_\alpha y^\mu = iw^2(z)\epsilon_{\alpha\beta}\partial_\beta x^\mu$
- Boundary conditions: x^μ carries momentum $k^\mu \to y^\mu$ has winding $\Delta y^\mu = 2\pi k^\mu$.
- After a change of coordinates $r = R^2/z$ we end up again with AdS_5

$$ds^2 = R^2 \frac{dy_{3+1}^2 + dr^2}{r^2}$$

World-sheet whose boundary is located at $r = R^2/z_{IR}$ and is a particular line constructed as follows...

- For each particle with momentum k^μ draw a segment joining two points separated by $\Delta y^\mu = 2\pi k^\mu$



- Concatenate the segments according to the ordering of the insertions on the disk (particular color ordering)
- Momentum conservation: Closed diagram.

- As $z_{IR} \to \infty$ the boundary of the world-sheet moves to $r = 0$.
- Exactly same computation as when computing the expectation value of a Wilson-Loop given by a sequence of light-like segments!

Prescription

- $\mathcal{A}_n$: Leading exponential behavior of the $n-$point scattering amplitude.
- $A_{min}(k_1^\mu, k_2^\mu, ..., k_n^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

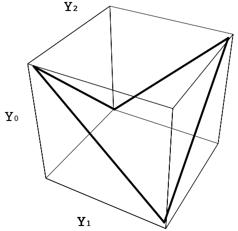
$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- Prefactors are subleading in $1/\sqrt{\lambda}$, and we don't compute them.
- In particular our computation is blind to helicity, etc.

Four point amplitude at strong coupling

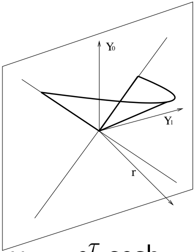
Consider $k_1 + k_3 \to k_2 + k_4$

$$s = -(k_1 + k_2)^2 = -4k^2 \sin^2 \phi/2$$
$$t = -(k_1 + k_4)^2 = -4k^2 \cos^2 \phi/2$$



Need to find the minimal surface ending on such sequence of light-like segments.

- Warm up: Try to find the solution near one of the cusps.



The surface can be embedded in AdS_3

$$ds^2 = \frac{-dy_0^2 + dy_1^2 + dr^2}{r^2}$$

$$y_0 = e^\tau \cosh\sigma, \quad y_1 = e^\tau \sinh\sigma, \quad r = e^\tau w(\tau)$$

$$\Downarrow$$

$$S_{NG} \sim \int d\sigma \int d\tau \frac{\sqrt{1-(w(\tau)+w'(\tau))^2}}{w(\tau)^2}$$

Solution for the cusp (Kruczenski)

$$w = \sqrt{2} \rightarrow r = \sqrt{2}\sqrt{y_0^2 - y_1^2} = \sqrt{2y^+y^-}, \quad y^\pm = y^0 \pm y^1$$

Embedding coordinates

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$$

$$Y^\mu = \frac{y^\mu}{r}, \quad \mu = 0, ..., 3$$

$$Y_{-1} + Y_4 = \frac{1}{r}, \quad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r}$$

Surface of the cusp in embedding coordinates

$$Y_0^2 - Y_{-1}^2 = Y_1^2 - Y_4^2, \quad Y_2 = Y_3 = 0$$

Back to the four segments...

- S_{NG}: Poincare coordinates (r, y_0, y_1, y_2) and parametrize the surface by its projection to (y_1, y_2) plane.
- Action for two fields $r(y_1, y_2), y_0(y_1, y_2)$. *E.g.* if $s = t$ the fields live on a square parametrized by y_1, y_2.

$$S_{NG} = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2}$$

- By scale invariance, edges of the square at $y_1, y_2 = \pm 1$

Boundary conditions

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \quad y_0(\pm 1, y_2) = \pm y_2, \quad y_0(y_1, \pm 1) = \pm y_1$$

- We know the solution near the cusps. We can make some guess

$$y_0(y_1, y_2) = y_1 y_2, \quad r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)}$$

- Easily seen to satisfy all the conditions and actually solves the eoms!

Solution in conformal gauge?

$$ds_{WS}^2 = \frac{dy_1^2}{(1 - y_1^2)^2} + \frac{dy_2^2}{(1 - y_2^2)^2} = du_1^2 + du_2^2, \qquad y_i = \tanh u_i$$

- In terms of u coordinates

$$y_1 = \tanh u_1 , \qquad y_2 = \tanh u_2$$
$$r = \frac{1}{\cosh u_1 \cosh u_2} , \qquad y_0 = \tanh u_1 \tanh u_2$$

Conformal gauge action

$$iS = -\frac{R^2}{2\pi} \int du_1 du_2 \frac{1}{2} \frac{\partial r \partial r + \partial y_\mu \partial y^\mu}{r^2}$$

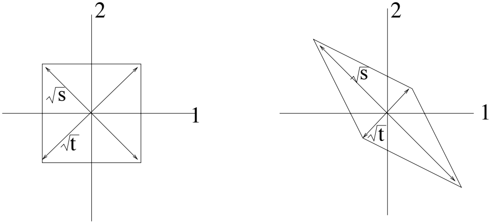
- In embedding coordinates

$$Y_0 = \sinh u_1 \sinh u_2 , \quad Y_1 = \sinh u_1 \cosh u_2 , \quad Y_2 = \cosh u_1 \sinh u_2$$
$$Y_{-1} = \cosh u_1 \cosh u_2 , \qquad Y_4 = Y_3 = 0$$

Embedding coordinates surface

$$Y_0 Y_{-1} = Y_1 Y_2 \qquad Y_3 = Y_4 = 0$$

- We would like to capture the kinematical dependence of the amplitude. We need to consider $s \neq t$.
- The square will be deformed to a rhombus



(a) (b)

- Start with $Y_0 Y_{-1} = Y_1 Y_2, \;\; Y_4 = 0$ and perform a boost in the 04 direction!

After the boost

$$Y_4 = 0, \quad Y_0 Y_{-1} = Y_1 Y_2 \rightarrow Y_4 - vY_0 = 0, \quad \sqrt{1-v^2}\, Y_0 Y_{-1} = Y_1 Y_2$$

$$\Downarrow$$

Solution for the rhombus

$$r = \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2},$$
$$y_0 = r\sqrt{1+b^2} \sinh u_1 \sinh u_2$$
$$y_1 = r \sinh u_1 \cosh u_2, \qquad y_2 = r \cosh u_1 \sinh u_2$$

- The parameters a and b encode the dynamical information.

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \qquad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}$$

Let's compute the area...

- Small problem: The area diverges!
- Dimensional reduction scheme: Theory in $D = 4 - 2\epsilon$ dimensions but with 16 supercharges.
- For integer D this is exactly the low energy theory living on $Dp-$branes $(p = D - 1)$

Gravity dual

$$ds^2 = h^{-1/2} dx_D^2 + h^{1/2} \left(dr^2 + r^2 d\Omega_{9-D}^2 \right), \qquad h = \frac{c_D \lambda_D}{r^{8-D}}$$

$$\lambda_D = \frac{\lambda \mu^{2\epsilon}}{(4\pi e^{-\gamma})^\epsilon} \qquad c_D = 2^{4\epsilon} \pi^{3\epsilon} \Gamma(2+\epsilon)$$

T-dual coordinates

$$ds^2 = \sqrt{\lambda_D c_D}\left(\frac{dy_D^2 + dr^2}{r^{2+\epsilon}}\right) \rightarrow S_\epsilon = \frac{\sqrt{\lambda_D c_D}}{2\pi}\int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}$$

- Presence of ϵ will make the integrals convergent.
- Problem: The equations of motion will depend on ϵ.
- We couldn't find the general solution but for the cusp we can:

$$r = \sqrt{2+\epsilon}\sqrt{y^+y^-} \rightarrow -iS_\epsilon = \frac{4}{\epsilon^2}\frac{A_\epsilon}{(2y_c^+y_c^-)^{\epsilon/2}}$$

- Note: In general, the $1/\epsilon^2$ poles will come from the region near the cusps.

- When computing the action, we will be interested in terms up to ϵ^0 order.
- We need to know the ϵ-corrected solution only near the cusps, and we actually do!
- Plugging everything into the action...

$$iS = -\frac{\sqrt{\lambda_D c_D}}{2\pi a^{\epsilon}}\left(\frac{\pi\Gamma\left[-\frac{\epsilon}{2}\right]^2}{\Gamma\left[\frac{1-\epsilon}{2}\right]}\,{}_2F_1\left(\frac{1}{2},-\frac{\epsilon}{2},\frac{1-\epsilon}{2};b^2\right)+1\right)+\mathcal{O}(\epsilon)$$

- Just expand in powers of ϵ...

Final answer

$$\mathcal{A} = e^{iS} = \exp\left[iS_{div} + \frac{\sqrt{\lambda}}{8\pi}\left(\log\frac{s}{t}\right)^2 + \tilde{C}\right]$$

$$S_{div} = 2S_{div,s} + 2S_{div,t}$$

$$S_{div,s} = -\frac{1}{\epsilon^2}\frac{1}{2\pi}\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon}\frac{1}{4\pi}(1-\log 2)\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}}$$

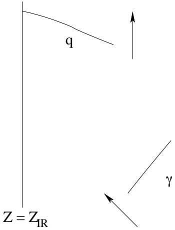
- Should be compared to the field theory answer

$$\mathcal{A} \sim (\mathcal{A}_{div,s})^2(\mathcal{A}_{div,t})^2 \exp\left\{\frac{f(\lambda)}{8}(\ln s/t)^2 + const\right\}$$

$$\mathcal{A}_{div,s} = \exp\left\{-\frac{1}{8\epsilon^2}f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon}g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right\}$$

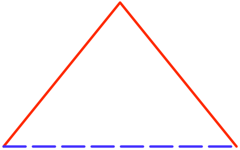
- The general structure is in perfect agreement with the field theory expectations (BDS conjecture).
- We can extract the strong coupling value for the cusp anomalous dimension, $f(\lambda) = \frac{\sqrt{\lambda}}{\pi}$ in agreement with the well known result.
- Strong coupling value for the function $g(\lambda)$ characterizing subleading divergences: $g = \frac{\sqrt{\lambda}}{4\pi} 2(1 - \log 2)$.
- The detailed dependence on the kinematics of the finite piece agrees with the expected value.

Other processes (something like $\gamma \longrightarrow q + \bar{q}$)



- Brane I extending along (x^0, x^1) at $Z = Z_{IR}$.
- Brane II extending along (x^0, x^1, z)
- Photon $\rightarrow (II, II)$, quarks $\rightarrow (I, II)$

- $(0, k, ik) \rightarrow (k/2, k/2, *) + (-k/2, k/2, *)$
- After T-duality, a triangle in the (y^0, y^1) plane with boundary conditions for r
- $r = 0$ in the red lines (quarks), $r = \infty$ in the blue line (photon).



What have we done?

- A prescription for computing planar gluon scattering amplitudes on $\mathcal{N} = 4$ SYM at strong coupling by using the AdS/CFT duality.
- We have done detailed computations for $n = 4$ but the prescription is valid for any number of gluons.
- Our results agree in all detail with the conjecture of Bern, Dixon and Smirnov.
    - A comparison allows to extract the strong coupling behavior of the function controling subleading divergences.
- A small step towards understanding the iterative structures for gluon amplitudes from the string theory point of view.

What things need to be done?

- Try to make explicit computations for $n > 4$.
  - Test the conjectured iterative structure for arbitrary n.
  - One could study some particular limits...
- We haven't assume/use at all the machinery of integrability. (see Jevivki, Kalousios, Spradlin, and Volovich; Mironov, Morozov and Tomaras)
- Subleading corrections in $1/\sqrt{\lambda}$? Information about helicity of the particles, etc. (see Abel, Forste, Khose; Kruczenski, Roiban, Tirziu, Tseytlin)
- Gross and Mende computed higher genus amplitudes (in flat space) using similar ideas, can we do the same?
- Can we repeat the computation in other backgrounds?
- Deeper relation among Wilson loops and scattering amplitudes? (Drummond, Korchemsky and Sokatchev; Brandhuber, Heslop and Travaglini)
- Further developmens:
  - IR behavior at strong coupling (Buchbinder)
  - Universality of scattering amplitudes (Komargodsky and Razamat)