Integrability in $\mathcal{N} = 4$ SYM: an overview

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SYM with 16 supercharges – conformal field theory in $d = 4$

Textbook description:
- anomalous dimensions of gauge invariant operators
- 3-point functions
- sewing relations

• spectral problem of $\mathcal{N} = 4$ SYM: find all anomalous dimensions

- $\mathcal{O}_n$ "good" operators – diagonal RG flow $\langle \mathcal{O}_n(0)\mathcal{O}_m(x)\rangle = \dfrac{\delta_{mn}}{|x|^{2\Delta_n}}$

  available tricks for special "good" operators

- O_n "natural" operators $\Rightarrow$ operator mixing – $\langle O_n(0)O_m(x)\rangle \neq \delta_{mn}$

  "good" operators = combinations of "natural" operators

◇ Clean formulation in terms of dilatation generator ($\mathcal{D} \in PSU(2,2|4)$)

$$\mathcal{D}O_n = \sum_m \gamma_{nm}\, O_m \quad \longrightarrow \quad \mathcal{D}\mathcal{O}_n = \Delta_n\, \mathcal{O}_n$$

- Further simplification: restrict to single-trace O_n in 't Hooft limit

- Known explicit higher loop anomalous dimensions in $\mathcal{N} = 4$ SYM:
  - BMN operators (2-loops) — Gross, Mikhailov, RR
  - single-magnon dispersion relation (all loops) — Gross, Mikhailov, RR; Santambrogio, Zanon
  - 2- and 3-loop cusp anomalous dimension — Kotikov, Lipatov, Onishchenko, Velizhanin; Bern, Dixon, Smirnov
  - 4-loop cusp anom. dim. tour de force — Bern, Czakon, Dixon, Kosower, Smirnov;
    - further improvements — Cachazo, Spradlin, Volovich

- Strong coupling – from AdS/CFT
  - Operators with large quantum numbers — Frolov, Tseytlin; Kruczensky, Ryzhov, Tseytlin;
  - Cusp/folded string (LO, NLO, NNLO) — Gubser, Klebanov, Polyakov; Frolov, Tseytlin; Kruczensky; Frolov, Tirziu, Tseytlin; RR, Tirziu, Tseytlin

Outline

- What is integrability and why do we believe in it?
- Hamiltonian vs. (2-dimensional) S-matrix: 1,2,3,4. . . ∞?
- The idea of the Bethe ansatz
- BES vs direct calculations
- Is this the end?
- Outlook and open problems

What is integrability . . .

`Wikipedia:` “In Hamiltonian mechanics, an integrable system refers to a Hamiltonian system that has constants of motion other than the energy itself. A completely integrable system is a system that has n degrees of freedom, n constants of motion, and whose constants of motion are in involution: that is, the Poisson bracket between each pair of constants of motion vanishes.”

- spectrum – degeneracies unexplained by lowest conservation laws
- no algorithm to identify integrable Hamiltonians

Sasha Migdal: "... but why do you believe in integrability for $\mathcal{N} = 4$ SYM?"

Matthias Staudacher: "I guess unlimited optimism :)"

12$^{\text{th}}$ Itzykson meeting, Paris 2007

From weak coupling end:

- low order evolution operators define integrable H — Lipatov; Minahan, Zarembo; Beisert, Staudacher
- tree-level Yangian consistent with 1-loop dilatation operator — Dolan, Nappi, Witten
- higher-loop Yangian — Agarwal, Rajeev; Zwiebel
- degenerate pairs and higher multiplets — Beisert, Kristjansen, Staudacher; Beisert, Spill; Torrielli; Matsumoto; Moriyama,...

From strong coupling end:

- classical AdS string – infinitely many integrals of motion — Bena, Polchinski, RR
- seem to survive quantization — Berkovits
- structures characteristic to integrable systems in quantum strings
  - classical spectrum on cylinder — Kazakov, Marshakov, Minahan, Zarembo; Beisert, Kazakov, Sakai, Zarembo

Hamiltonian vs. (2-dimensional) S-matrix

$$\mathcal{D} O_n = \sum_m \gamma_{nm} O_m \quad \longrightarrow \quad \mathcal{D} \mathcal{O}_n = \Delta_n \mathcal{O}_n$$

- Interpret as a Schrödinger equation with $\mathcal{D}$ as Hamiltonian; γ is its matrix representation in the basis $\{O_n\}$

- Need vacuum; choose scalar 1/2−BPS operator Tr $[Z^J]$ $J \to \infty$

Excitations: – replace vacuum field Z by one of remaining fields

– enforce gauge invariance

Berenstein, Maldacena, Nastase

$\mapsto$ 4 real scalars

4 fermions

arbitrary covariant derivatives of vacuum fields

Hamiltonian vs. (2-dimensional) S-matrix

$$\mathcal{D}O_n = \sum_m \gamma_{nm}\, O_m \qquad \longrightarrow \qquad \mathcal{D}\mathcal{O}_n = \Delta_n\, \mathcal{O}_n$$

- Interpret as a Schrödinger equation with $\mathcal{D}$ as Hamiltonian; γ is its matrix representation in the basis $\{O_n\}$

- Need vacuum; choose scalar 1/2−BPS operator

- γ has block-diagonal structure due to charge conservation

	ϕ^1	ϕ^2	Z	A_μ	ψ^1	ψ^2	ψ^3	ψ^4	Q^1	Q^2	Q^3	Q^4
J_{12}	1	0	0	0	1/2	-1/2	-1/2	1/2	-1/2	1/2	1/2	-1/2
J_{34}	0	1	0	0	-1/2	1/2	-1/2	1/2	1/2	-1/2	1/2	-1/2
J_{56}	0	0	1	0	-1/2	-1/2	1/2	1/2	1/2	1/2	-1/2	-1/2

$SU(2)$ sector		$Z,\ \phi$
$SU(1\|1)$ sector		$Z,\ \psi_1$
$SU(n\|m)$ sector	$1 \le n \le 2$ $1 \le m \le 3$	$Z,\ \phi_1,\ \phi_2,\ \psi_\alpha$
$SL(2)$		$D^n Z,\ n = 0, \dots, \infty$

Beisert – complete classification

- vacuum – Tr $[Z^J]$, $J \to \infty$

- 't Hooft limit → fields are ordered → lattice

How does one find γ?

V1: from renormalization factors $\gamma = \lim_{\epsilon \to 0} \epsilon \mathbf{Z}^{-1} \frac{d\mathbf{Z}}{d \ln g_{YM}}$

V2: from 2-point function of basis elements

V3: twist-2 operators – from scattering amplitudes/AP kernel

V4: algebraically

- Constraints on the L-loop Hamiltonian
  - global symmetries and structure of Feynman diagrams
  - vacuum energy and the energy of other BPS states
  - results of explicit calculations
  - allow/introduce terms not affecting the eigenvalues
    - similarity transformations: $\mathcal{H}' = U^{-1}\mathcal{H}U$ & $U^\dagger \neq U^{-1}$

- Existing results:
  - 5-loop $SU(2)$ assuming integrability, smooth continuum limit Beisert, Dippel, Staudacher
  - 2-loop $SL(2)$ (w/o); larger sectors (w/)
  - 4-loop $SU(2)$; no assumptions Beisert, McLoughlin, RR

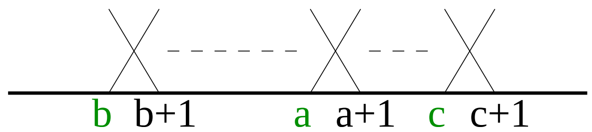
$$\mathcal{H}_0 = +\{\}$$

$$\mathcal{H}_1 = +2\{\} - 2\{1\}$$

$$\mathcal{H}_2 = -8\{\} + 12\{1\} - 2\big(\{1,2\} + \{2,1\}\big)$$

$$\mathcal{H}_3 = +60\{\} - 104\{1\} + 4\{1,3\} + 24\big(\{1,2\} + \{2,1\}\big) - 4i\epsilon_2\{1,3,2\} + 4i\epsilon_2\{2,1,3\} - 4\big(\{1,2,3\} + \{3,2,1\}\big)$$

$$\begin{aligned}\mathcal{H}_4 = &+\big(-560 - 4\beta_{2,3}\big)\{\} \\ &+ \big(+1072 + 12\beta_{2,3} + 8\epsilon_{3a}\big)\{1\} \\ &+ \big(-84 - 6\beta_{2,3} - 4\epsilon_{3a}\big)\{1,3\} \\ &- 4\{1,4\} \\ &+ \big(-302 - 4\beta_{2,3} - 8\epsilon_{3a}\big)\big(\{1,2\} + \{2,1\}\big) \\ &+ \big(+4\beta_{2,3} + 4\epsilon_{3a} + 2i\epsilon_{3c} - 4i\epsilon_{3d}\big)\{1,3,2\} \\ &+ \big(+4\beta_{2,3} + 4\epsilon_{3a} - 2i\epsilon_{3c} + 4i\epsilon_{3d}\big)\{2,1,3\} \\ &+ \big(4 - 2i\epsilon_{3c}\big)\big(\{1,2,4\} + \{1,4,3\}\big) \\ &+ \big(4 + 2i\epsilon_{3c}\big)\big(\{1,3,4\} + \{2,1,4\}\big) \\ &+ \big(+96 + 4\epsilon_{3a}\big)\big(\{1,2,3\} + \{3,2,1\}\big) \\ &+ \big(-12 - 2\beta_{2,3} - 4\epsilon_{3a}\big)\{2,1,3,2\} \\ &+ \big(+18 + 4\epsilon_{3a}\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big) \\ &+ \big(-8 - 2\epsilon_{3a} - 2i\epsilon_{3b}\big)\big(\{1,2,4,3\} + \{1,4,3,2\}\big) \\ &+ \big(-8 - 2\epsilon_{3a} + 2i\epsilon_{3b}\big)\big(\{2,1,3,4\} + \{3,2,1,4\}\big) \\ &- 10\big(\{1,2,3,4\} + \{4,3,2,1\}\big)\end{aligned}$$



- $\{...bac\} = ...P_{b,b+1}P_{a,a+1}P_{c,c+1}$
- β=undetermined; directly computable
- ϵ=similarity parameters

$$\mathcal{H} \to U(\epsilon)^{-1}\mathcal{H}U(\epsilon)$$

Higher-loop $\mathcal{H}$ is computable in principle; Direct calculation of the all-loop $\mathcal{H}$ appears impractical; partial calculations contain nontrivial information

Eigenvectors and eigenvalues: illustrate on $SU(2)$ sector

- 2-particle states: $\mathcal{H}|\Psi\rangle = E|\Psi\rangle$

$$|\Psi\rangle = \sum_{1\le l_1<l_2\le J} \Psi(l_1,l_2)|\ldots Z\underset{\underset{l_1}{\downarrow}}{\phi}Z\ldots Z\underset{\underset{l_2}{\downarrow}}{\phi}Z\ldots\rangle$$

  - If $l_1 - l_2 \ge L+1$ then $\Psi(l_1,l_2) = e^{i(l_1p_1+l_2p_2)}$
  - If $l_1 - l_2 \le L$ details of $\mathcal{H}$ matter;

- Bethe ansatz – asymptotic plane waves   Bethe

$$\Psi(l_1,l_2) = e^{i(l_1p_1+l_2p_2)} + S(p_1,p_2,\lambda)e^{i(l_1p_2+l_2p_1)}$$

- We may think of $S(p_1,p_2,\lambda)$ as a $(1+1)$-dimensional S-matrix

- Spectrum from periodicity of wave function

$$e^{ip_1J} = S(p_1,p_2,\lambda) \qquad e^{ip_2J} = S(p_2,p_1,\lambda)$$

Eigenvectors and eigenvalues: illustrate on $SU(2)$ sector

- 2-particle states: $\mathcal{H}|\Psi\rangle = E|\Psi\rangle$

$$|\Psi\rangle = \sum_{1\le l_2 < l_2 \le J} \Psi(l_1,l_2)|\ldots Z\underset{\underset{l_1}{\downarrow}}{\phi} Z\ldots Z\underset{\underset{l_2}{\downarrow}}{\phi} Z\ldots\rangle$$

  - If $l_1 - l_2 \ge L+1$ then $\Psi(l_1,l_2) = e^{i(l_1p_1+l_2p_2)}$
  - If $l_1 - l_2 \le L$ details of $\mathcal{H}$ matter;

e.g. 1-loop $SU(2)$ sector $\mathcal{H}_1 = \mathbb{1} - \mathcal{P}$

$$\Psi(l_1,l_2) = e^{i(l_1p_1+l_2p_2)} + S(p_1,p_2,\lambda)e^{i(l_1p_2+l_2p_1)}$$

$$E\Psi(l_1,l_2) = 2\Psi(l_1,l_2) - \Psi(l_1-1,l_2) - \Psi(l_1,l_2+1)$$

$$S(p_1,p_2) = -\frac{e^{i(p_1+p_2)} - 2e^{ip_1} + 1}{e^{i(p_1+p_2)} - 2e^{ip_2} + 1}$$

$$E = E(p_1) + E(p_2)$$
$$E(p) = 4\sin^2\tfrac{p}{2}$$

Integrability=factorization $\leftrightarrow$ Yang-Baxter equation

$$S(p_1,p_2)S(p_1,p_3)S(p_2,p_3) = S(p_2,p_3)S(p_1,p_3)S(p_1,p_2)$$

- multi-particle S-matrix is a sequence of 2-particle S-matrices
- particle momenta are separately conserved ($\exists$ higher IoM)

Wave function periodicity yields spectrum (Bethe equations)

$$e^{ip_k J} = \prod_{j\neq k=1}^{M} S(p_k, p_j, \lambda) \qquad E = \sum_{k=1}^{M} E(p_k, \lambda)$$

Larger sectors

- Nested Bethe ansatz:
  - spectrum from periodicity
  - kind of row reduction
  - repeat 1d Bethe ansatz with a different choice of vacuum at each stage

$\diamond$ The message: it suffices to know $S(p_1,p_2,\lambda)_{ij}^{kl}$ with $i,j,k,l \in (4|4)$

Constraints on the S-matrix from $\mathcal{N}=4$ SYM

- symmetries preserved by the choice of vacuum ($PSU(2|2)^2$)

  ! symmetry algebra: extended and w/ quantum corrections

$$[\mathfrak{R}^a{}_b,\, \mathfrak{J}^c] = \delta^c_b \mathfrak{J}^a - \frac{1}{2}\delta^a_b \mathfrak{J}^c,$$
$$[\mathfrak{L}^\alpha{}_\beta,\, \mathfrak{J}^\gamma] = \delta^\gamma_\beta \mathfrak{J}^\alpha - \frac{1}{2}\delta^\alpha_\beta \mathfrak{J}^\gamma,$$
$$\{\mathfrak{Q}^\alpha{}_a,\, \mathfrak{S}^b{}_\beta\} = \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \delta^b_a \delta^\alpha_\beta \mathfrak{C},$$
$$\{\mathfrak{Q}^\alpha{}_a,\, \mathfrak{Q}^\beta{}_b\} = \varepsilon^{\alpha\beta}\varepsilon_{ab}\mathfrak{P},$$
$$\{\mathfrak{S}^a{}_\alpha,\, \mathfrak{S}^b{}_\beta\} = \varepsilon^{ab}\varepsilon_{\alpha\beta}\mathfrak{K}$$

$$Q^\alpha{}_a|\phi^b\rangle = a(p,\lambda)\,\delta^b_a|\psi^\alpha\rangle,$$
$$Q^\alpha{}_a|\psi^\beta\rangle = b(p,\lambda)\,\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^b Z^+\rangle,$$
$$S^a{}_\alpha|\phi^b\rangle = c(p,\lambda)\,\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^\beta Z^-\rangle,$$
$$S^a{}_\alpha|\psi^\beta\rangle = d(p,\lambda)\,\delta^\beta_\alpha|\phi^a\rangle$$

$$\mathcal{G}\mathcal{S}|\Psi\Psi\rangle = \mathcal{S}\mathcal{G}|\Psi\Psi\rangle$$

  ! symmetry may change the vacuum charge

- unusual feature: scattering may change the vacuum charge
- factorization (à la Yang-Baxter)
- "algebraic crossing"

  ! only indirect string theory justification

  ! enhancement of $psu(2|2)$ to a Hopf algebra

$$\mathcal{S}_{12}|\phi_1^a\phi_2^b\rangle = A_{12}|\phi_2^{\{a}\phi_1^{b\}}\rangle + B_{12}|\phi_2^{[a}\phi_1^{b]}\rangle + \frac{1}{2}C_{12}\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi_2^\alpha\psi_1^\beta\mathcal{Z}^-\rangle,$$

$$\mathcal{S}_{12}|\psi_1^\alpha\psi_2^\beta\rangle = D_{12}|\psi_2^{\{\alpha}\psi_1^{\beta\}}\rangle + E_{12}|\psi_2^{[\alpha}\psi_1^{\beta]}\rangle + \frac{1}{2}F_{12}\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi_2^a\phi_1^b\mathcal{Z}^+\rangle,$$

$$\mathcal{S}_{12}|\phi_1^a\psi_2^\beta\rangle = G_{12}|\psi_2^\beta\phi_1^a\rangle + H_{12}|\phi_2^a\psi_1^\beta\rangle,$$

$$\mathcal{S}_{12}|\psi_1^\alpha\phi_2^b\rangle = K_{12}|\psi_2^\alpha\phi_1^b\rangle + L_{12}|\phi_2^b\psi_1^\alpha\rangle.$$

$$A_{12} = S_{12}^0\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+},$$

$$B_{12} = S_{12}^0\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\left(1 - 2\,\frac{1 - g^2/2x_2^- x_1^+}{1 - g^2/2x_2^- x_1^-}\,\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-}\right),$$

$$C_{12} = S_{12}^0\,\frac{g^2\gamma_2\gamma_1}{\alpha x_2^- x_1^-}\,\frac{1}{1 - g^2/2x_2^- x_1^-}\,\frac{x_2^+ - x_1^+}{x_2^- - x_1^+},$$

$$D_{12} = -\,S_{12}^0,$$

$$E_{12} = -\,S_{12}^0\left(1 - 2\,\frac{1 - g^2/2x_2^+ x_1^-}{1 - g^2/2x_2^+ x_1^+}\,\frac{x_2^- - x_1^-}{x_2^- - x_1^+}\right),$$

$$F_{12} = -\,S_{12}^0\,\frac{2\alpha(x_2^+ - x_2^-)(x_1^+ - x_1^-)}{\gamma_2\gamma_1 x_2^+ x_1^+}\,\frac{1}{1 - g^2/2x_2^+ x_1^+}\,\frac{x_2^- - x_1^-}{x_2^- - x_1^+},$$

$$G_{12} = S_{12}^0\,\frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \qquad H_{12} = S_{12}^0\,\frac{\gamma_1}{\gamma_2}\,\frac{x_2^+ - x_2^-}{x_2^- - x_1^+},$$

$$K_{12} = S_{12}^0\,\frac{\gamma_2}{\gamma_1}\,\frac{x_1^+ - x_1^-}{x_2^- - x_1^+} \qquad L_{12} = S_{12}^0\,\frac{x_2^- - x_1^-}{x_2^- - x_1^+}.$$

$$S_0(x_l, x_k) = \frac{1 - g^2/2x_k^+ x_l^-}{1 - g^2/2x_k^- x_l^+}\,\sigma_{kl}^2$$

$\sigma_{kl} = e^{i\theta_{kl}}$ – universal

$\neq 0$ at $\lambda \to \infty$

$= \mathcal{O}(\lambda^4)$ at $\lambda \to 0$

$$x^+ + \frac{g^2}{2x^+} - x^- - \frac{g^2}{2x^-} = i$$

Asymptotic all-loop Bethe equations

Beisert, Staudacher

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}},$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L \prod_{\substack{j=1\\j\neq k}}^{K_4} \left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma^2(x_{4,k},x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}},$$

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}},$$

--- -- ------------------------------------

$$E(g) = 2\sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\right), \qquad \Delta = \Delta_0 + g^2\,E(g)\,.$$

$$1 = \prod_{j=1}^{K_4}\left(\frac{x^+_{4,j}}{x^-_{4,j}}\right), \qquad u_k = x_k + \frac{g^2}{x_k}, \qquad u_k \pm \tfrac{i}{2} = x^\pm_k + \frac{g^2}{x^\pm_k}\,.$$

The phase:

$\diamond$ Universal to all sectors; general structure: Arutyunov, Frolov, Staudacher; Beisert, Klose; Beisert, Tseytlin

$$\theta(u_1,u_2) = \sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g)\Big[q_r(u_1)q_{r+1+2\nu}(u_2) - q_r(u_2)q_{r+1+2\nu}(u_1)\Big]$$

$$q_{r,k} = \frac{1}{r-1}\left(\frac{i}{(x_k^+)^{r-1}} - \frac{i}{(x_k^-)^{r-1}}\right) \qquad\qquad g = \frac{\sqrt{\lambda}}{\pi}$$

- various suggestions for the origin of the phase:
  - relativistic 2d sigma model — Mann, Polchinski; Gromov, Kazakov, Viera, Sakai
  - fill to physical vacuum — Sakai, Satoh; Rej, Staudacher, Zieme
- constrained by crossing at strong coupling — Janik
  - morally similar to $\quad S^{p,q}(i\pi - \theta) = \mathcal{C}^p S^{\bar{p},q}(\theta)\mathcal{C}^p$
  - for AdS: $\sigma_{12}\sigma_{\bar{1}2} = f_{12}$ but $f_{12} \neq f_{\bar{1}2}$

    solution: compatibility of algebra and $\mathcal{C}$: $1 \mapsto \bar{1} \mapsto \bar{\bar{1}} \neq 1$

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$$\theta(u_1,u_2) = \sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g)\Big[q_r(u_1)q_{r+1+2\nu}(u_2) - q_r(u_2)q_{r+1+2\nu}(u_1)\Big]$$

- strong coupling: $c_{r,s}(g) = g^{2-r-s}\beta_{r,s}(g) = \sum_n c_{r,s}^{(n)} g^{1-n}$

$$c_{r,s}^{(n)} = \frac{(1-(-)^{r+s})(r-1)(s-1)}{2(-2\pi)^n \Gamma(n-1)} \frac{\Gamma(\frac{1}{2}(s+r+n-3))\Gamma(\frac{1}{2}(s-r+n-1))}{\Gamma(\frac{1}{2}(s+r-n+1))\Gamma(\frac{1}{2}(s-r-n+3))}$$

! not unique — Beisert, Hernandez, Lopez

- weak coupling: $\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu}\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$

• analytic continuation from strong coupling — Beisert, Eden, Staudacher

An inspired guess?

$$f(g) = -\frac{1/g}{1-1/g} = \frac{1}{1-g}$$

- Series expansion:

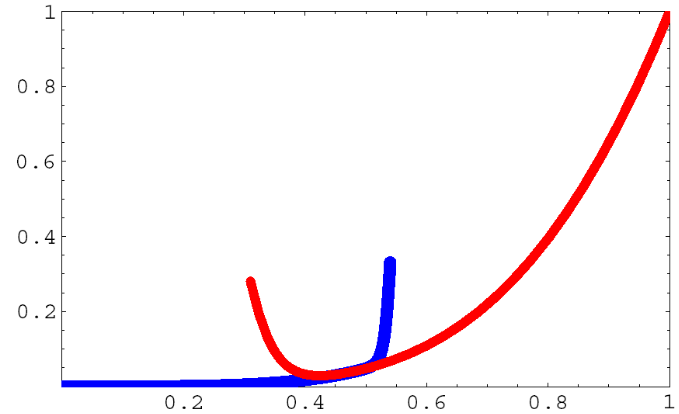
$g \to \infty$	$g \to 0$
$f(g) = \sum b_n g^{-n} \quad (b_n = -1)$	$f(g) = \sum a_n g^{+n} \quad (a_n = +1)$

◇ Analytic continuation "rule": $a_n = -b_{-n}$ Beisert, Eden, Staudacher

$$c_{rs}(g) = \sum_n c_{r,s}^{(n)} g^{1-n} \mapsto -\sum_n c_{r,s}^{(-n)} g^{1+n}$$

Various appealing features:

- only integer powers of λ
- first nonzero contribution at 4 loops
- Lipatov's transcedentality
- expected radius of convergence



$c_{23}/(c_{23}+1)$ vs. $\sqrt{\lambda}/(\sqrt{\lambda}+\pi)$

The phase:

◇ Universal to all sectors; general structure: Arutyunov, Frolov, Staudacher; Beisert, Klose; Beisert, Tseytlin

$$\theta(u_1,u_2) = \sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g)\Big[q_r(u_1)q_{r+1+2\nu}(u_2) - q_r(u_2)q_{r+1+2\nu}(u_1)\Big]$$

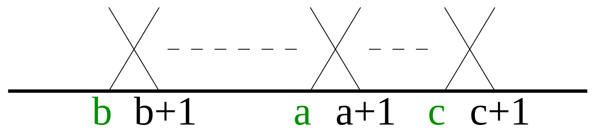
- weak coupling: $\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu}\beta^{(r+\nu+\mu)}_{r,r+1+2\nu}$

• analytic continuation from strong coupling – several possibilities:
Beisert, Eden, Staudacher

L	no $\zeta(2n+1)$	with $\zeta(2n+1)$
4	$\beta^{(3)}_{2,3} = 2\zeta(3)$	$\beta^{(3)}_{2,3} = 4\zeta(3)$
5	$\beta^{(4)}_{2,3} = -20\zeta(5)$	$\beta^{(4)}_{2,3} = -40\zeta(5)$
6	$\beta^{(5)}_{2,3} = 210\zeta(7)$,	$\beta^{(5)}_{2,3} = 420\zeta(7)$,
	$\beta^{(5)}_{3,4} = 12\zeta(5)$, $\beta^{(5)}_{2,5} = -4\zeta(5)$	$\beta^{(5)}_{3,4} = 24\zeta(5)$, $\beta^{(5)}_{2,5} = -8\zeta(5)$

- phase coefficients may be identified at the level of the Hamiltonian

$$
\begin{aligned}
\mathcal{H}_4 = &+\big(-560 - 4\beta_{2,3}\big)\{\} \\
&+ \big(+1072 + 12\beta_{2,3} + 8\epsilon_{3a}\big)\{1\} \\
&+ \big(-84 - 6\beta_{2,3} - 4\epsilon_{3a}\big)\{1,3\} \\
&- 4\{1,4\} \\
&+ \big(-302 - 4\beta_{2,3} - 8\epsilon_{3a}\big)\big(\{1,2\} + \{2,1\}\big) \\
&+ \big(+4\beta_{2,3} + 4\epsilon_{3a} + 2i\epsilon_{3c} - 4i\epsilon_{3d}\big)\{1,3,2\} \\
&+ \big(+4\beta_{2,3} + 4\epsilon_{3a} - 2i\epsilon_{3c} + 4i\epsilon_{3d}\big)\{2,1,3\} \\
&+ \big(4 - 2i\epsilon_{3c}\big)\big(\{1,2,4\} + \{1,4,3\}\big) \\
&+ \big(4 + 2i\epsilon_{3c}\big)\big(\{1,3,4\} + \{2,1,4\}\big) \\
&+ \big(+96 + 4\epsilon_{3a}\big)\big(\{1,2,3\} + \{3,2,1\}\big) \\
&+ \big(-12 - 2\beta_{2,3} - 4\epsilon_{3a}\big)\{2,1,3,2\} \\
&+ \big(+18 + 4\epsilon_{3a}\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big) \\
&+ \big(-8 - 2\epsilon_{3a} - 2i\epsilon_{3b}\big)\big(\{1,2,4,3\} + \{1,4,3,2\}\big) \\
&+ \big(-8 - 2\epsilon_{3a} + 2i\epsilon_{3b}\big)\big(\{2,1,3,4\} + \{3,2,1,4\}\big) \\
&- 10\big(\{1,2,3,4\} + \{4,3,2,1\}\big)
\end{aligned}
$$



- $\{\ldots bac\} = \ldots P_{b,b+1} P_{a,a+1} P_{c,c+1}$
- β=undetermined; directly computable
- ϵ=similarity parameters

$$\mathcal{H} \to U(\epsilon)^{-1}\mathcal{H}U(\epsilon)$$

- Direct calculation of $\mathcal{H}_4 \Longrightarrow \beta_{23}^{(3)} = 4\zeta(3)$ Beisert, McLoughlin, RR
- expected ζ-constants at 5, 6, 7, 8-loops McLoughlin, RR (unpublished)

Twist operators; $SL(2)$ sector

$$O_{n_1\ldots n_{\mathcal{L}}} = \mathrm{Tr}\ [D^{n_1}Z\ldots D^{n_{\mathcal{L}}}Z] \qquad \sum_{i=1}^{\mathcal{L}} n_i = S \qquad \mathcal{L} = \Delta_0 - S$$

- Hamiltonian is complicated; unknown beyond 2-loops

Various scaling limits:

1) $S \ll \mathcal{L}$ — $\lambda\frac{S}{\mathcal{L}^2}$
2) $\mathcal{L}\to\infty,\ S\to\infty,\ (\ln S)/\mathcal{L} < 1$ — $\frac{\lambda}{\mathcal{L}}\ln\frac{S}{\mathcal{L}}$
3) $\mathcal{L}\to\infty,\ S\to\infty,\ (\ln S)/\mathcal{L} \gg 1$ — $\sqrt{\lambda}\ln S/\sqrt{\lambda}$

- Contact with perturbative gauge theory data – need small $\mathcal{L}$

- AP kernel – close relation to $\mathcal{L}=2$;

$$\gamma(g,S) \xrightarrow{S\to\infty} f(g)\ln S$$
$$f(g) = \Gamma_{\mathrm{cusp}}$$

However...

- BA defined with $\mathcal{L}\to\infty$ and fixed S; order of limits issue?

- $S \gg \mathcal{L}$ really removes twist dependence?

- small twist is outside asymptotic regime; does it matter?

Twist operators; $SL(2)$ sector; Bethe equations for $(\ln S)/\mathcal{L} \gg 1$

$$\left(\frac{x_k^+}{x_k^-}\right)^{\mathcal{L}} = \prod_{\substack{j=1\\ j\neq k}}^{S} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} e^{2i\theta(u_k,u_j)}$$

$$u \pm \frac{i}{2} = x^{\pm} + \frac{g^2}{2x^{\pm}}$$

$$E(g) = \sum_{j=1}^{S} \left[\frac{i}{x_j^+} - \frac{i}{x_j^-}\right]$$

- take logarithm; sums $\rightarrow$ integrals as $S \gg 1$; $u_i \rightarrow \rho(u)$
- take d/du; perturb around $\rho_0 = \rho\Big|_{g=0}$: $\rho(u) = \rho_0(u) - g^2 \frac{E_0}{S}\sigma(u)$;
  Fourier-transform $\sigma(u) \mapsto \hat{\sigma}(t)$

$$\hat{\sigma}(t) = \frac{t}{e^t - 1}\left[K(2gt,\, 0) - 4g^2 \int_0^{\infty} dt'\, \widehat{K}(2gt,\, 2gt')\, \hat{\sigma}(t')\right]$$

BES

$$\widehat{K}(t,\, t') = \frac{J_1(t)J_0(t') - J_1(t')J_0(t)}{t - t'} + \widehat{K}_d(t,\, t')$$

$$\widehat{K}_d(t,\, t') = \frac{4}{tt'} \sum_{\mu>\nu} (-)^{\nu} g^{2\mu+1} \left(\beta^{(2\rho+\nu+\mu)}_{2\rho,2\rho+1+2\nu} J_{2\rho+2\nu}(t) J_{2\rho-1}(t') + \beta^{(2\rho+1+\nu+\mu)}_{2\rho+1,2\rho+2+2\nu} J_{2\rho}(t) J_{2\rho+1+2\nu}(t')\right)$$

$$f(g) = E(g)/\ln S = 16 g^2 \hat{\sigma}(0)$$

- Weak coupling: Beisert, Eden, Staudacher

$$\pi^2 f(g) = \frac{\lambda}{2} - \frac{\lambda^2}{96} + \frac{11\,\lambda^3}{23040} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\lambda^4 \ldots$$

  - agrees within numerical accuracy with direct calculation of $\Gamma_{\rm cusp}$
    Bern, Czakon, Dixon, Kosower, Smirnov; Cachazo, Spradlin, Volovich
  - λ^5 agrees with Padé extrapolation of 4-loop $\Gamma_{\rm cusp}$
    Bern, Czakon, Dixon, Kosower, Smirnov

◇ Bethe ansatz works unexpectedly well despite potential issues

- Strong coupling limit:
  - Direct extrapolation of BES Benna, Benvenutti, Klebanov, Scardicchio
    ◇ correct LO and NLO; uncertainties w/ procedure at NNLO
    ◇ NNLO: apparent inconsistency with ws Feynman diagramatics
    RR, Tirziu, Tseytlin
  - Use strong coupling phase Casteill, Kristjansen
    ◇ correct LO and NLO; NNLO N/A

Does this imply that asymptotic BA is complete?

- Further tests at finite spin and finite length

  Kotikov, Lipatov, Rej, Staudacher, Velizhanin

  – use BFKL

$$\frac{\omega^2}{-g^2} = \Psi(-g^2 E(g)) + \Psi(1 + g^2 E(g)) - 2\Psi(1)$$

$$\Psi(x) = \frac{d}{dx} \ln \Gamma(x)$$
$$\omega = S + 1$$

Balitsky, Fadin, Kuraev, Lipatov

  - Encodes the t-channel exchange of pomeron resonance

$$\text{Tr}\ [Z D^{-1+\omega} Z]$$

  - Sensitive to complete dependence on λ and S

    – different organization of Feynman diagrams

  - valid for negative spin around $\omega = 0$

- Comparison with asymptotic BA predictions require

  – calculation at finite spin

  – continuation to $S < 0$ and expansion around $S = -1$

The comparison:

- BFKL

$$E(g)^{\mathsf{BFKL}} = \frac{-4g^2}{\omega} - 0\left(\frac{-4g^2}{\omega}\right)^2 + 0\left(\frac{-4g^2}{\omega}\right)^3 - 2\zeta(3)\left(\frac{-4g^2}{\omega}\right)^4 + \ldots$$

- ABA

$$E(g)^{\mathsf{ABA}} = \frac{-4g^2}{\omega} - 0\left(\frac{-4g^2}{\omega}\right)^2 + 0\left(\frac{-4g^2}{\omega}\right)^3 - \frac{(-4g^2)^4}{\omega^7} + \ldots$$

- ABA breaks down at finite spin and finite length at 4-loops

Kotikov, Lipatov, Rej, Staudacher, Velizhanin

The comparison:

- ABA breaks down at finite spin and finite length at 4-loops

Kotikov, Lipatov, Rej, Staudacher, Velizhanin

- Proposed fix: change dressing phase coefficients ($\beta_{23}^{(3)}$)

$$\zeta(3) \mapsto \frac{47}{24}\zeta(3) - \frac{1}{4}S_{-4} + \frac{3}{4}S_{-2}S_1 + \frac{3}{8}S_1S_2 + \frac{3}{4}S_3 + \frac{1}{6}S_{-2,1} - \frac{17}{24}S_{2,1}$$

- makes S-matrix state-dependent (depends on S) – Universality?

- Assuming all previous assumptions hold → conjecture for γ_{Konishi}

$$O = \sum_i \text{Tr}\,[\phi^i \bar{\phi}_i] \quad \gamma(g) = 12g^2 - 48g^4 + 336g^6 - \left(\frac{5307}{2} + 564\zeta(3)\right) g^8 \ldots$$

Kotikov, Lipatov, Rej, Staudacher, Velizhanin

Outlook . . .

- Great progress toward finding spectrum of $\mathcal{N} = 4$ SYM
- Bethe Ansatz works better than expected but incomplete
- Not immediately clear what it is diagonalizing
  - $\lambda \to 0$ complete 1-loop; higher loops in some sectors
  - $\lambda \to \infty$ a continuum leading order Hamiltonian

. . . and some open problems

- ⋆ Prove integrability
- ⋆ What integrable model describes the $\mathcal{N} = 4$ spectral problem? What about the world sheet sigma model?
- ⋆ Is the dressing phase truly correct and who ordered it?
  - – Other solutions? Other analytic continuation?
  - – more reliable computations are needed at strong coupling
- ⋆ Use of integrability for finite quantum numbers?
  - – no direct calculation of a wrapping effect
  - – why some quantities are insensitive to it?
- ⋆ New computational techniques?
- ⋆ are there extra symmetries waiting to be discovered?
- ⋆ Is integrability restricted to the spectral problem?
  - – consequences/extra structure in scattering amplitudes?