Graphical Gaussian Models with Symmetries

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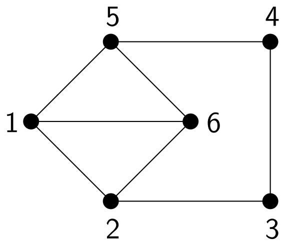
Graphical Models

In the following we will be considering undirected Graphical Models, which are defined by an undirected graph, e.g.



and the (Global) Markov Property

$$A \perp_{\mathcal{G}} B \mid S \Rightarrow A \perp\!\!\!\perp B \mid S$$

where $\perp_{\mathcal{G}}$ denotes graph separation.

Graphical Gaussian Models

We will assume that our random variables are multivariate Gaussian with concentration matrix K.

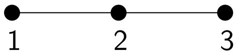
K satisfies the Markov Property w.r.t. a graph $\mathcal{G}$ if

$$i \not\sim j \text{ in } \mathcal{G} \implies k_{ij} = 0$$

For instance

$$\mathrm{K} = \begin{pmatrix} k_{11} & k_{12} & 0 \\ k_{12} & k_{22} & k_{23} \\ 0 & k_{23} & k_{33} \end{pmatrix}$$

satisfies the Markov Property w.r.t.



Graphical Gaussian Models with Symmetries

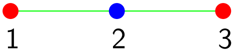
We will now introduce three types of symmetry restrictions on Gaussian models

1. RCON models: Symmetry restrictions on concentrations
2. RCOR models: Symmetry restrictions on partial correlations
3. RCOP models: Permutation symmetry

which we shall represent by graph colourings.

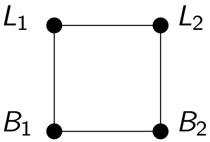
An RCON example:

$$K = \begin{pmatrix} k_{11} & k_{12} & 0 \\ k_{12} & k_{22} & k_{23} \\ 0 & k_{23} & k_{33} \end{pmatrix} = \begin{pmatrix} \alpha & \gamma & 0 \\ \gamma & \beta & \gamma \\ 0 & \gamma & \alpha \end{pmatrix}$$



An RCOP Example: Frets Heads

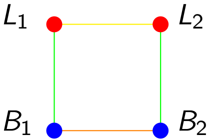
- Study of heredity of head dimensions: Length (L) and breadth (B) of the heads of 25 pairs of first and second sons were measured (Frets, 1921)
- Previous analyses support a model with the following conditional independence relations.



An RCOP Example: Frets Heads

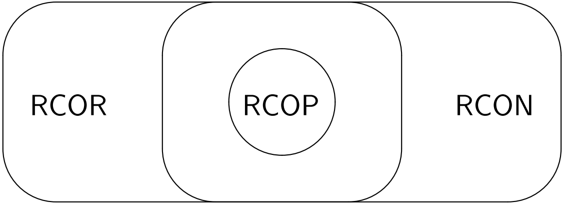
Since there is a symmetry between the two sons, it makes sense to consider a model which remains unaltered if the two sons are interchanged.

This gives the model the following colouring.



Relations between model types

1. RCOR and RCON models repesent the same model restrictions if and only if any pair of edges in the same colour class connect the same vertex colour classes. We shall call this criterion the RCOR-RCON criterion.
2. RCOP models are a subclass of models which are both, RCOR and RCON. (Hence they satisfy the above condition.)



RCOP Models: A Formal Definition

The central idea is that an RCOP model is invariant under the application of particular permutations to its variables.

All such permutations, σ say, must preserve

1. the structure of the graph, i.e. the zero entries of K
2. the graph colouring, i.e. the imposed symmetries of K

We can translate this into a condition to be satisfied by K and σ, for which we will need the following definition.

If σ is a permutation of the elements of $V = \{X_1, \ldots, X_d\}$, i.e. if $\sigma \in S(V)$, let its $d \times d$ permutation matrix $G(\sigma)$ be defined by

$$G(\sigma)_{\alpha\beta} = \begin{cases} 1 & \text{if } \sigma(\beta) = \alpha \\ 0 & \text{otherwise} \end{cases}$$

RCOP Models: A Formal Definition

If $Y \sim \mathcal{N}_{|V|}(0, \Sigma)$ then

$$G(\sigma)Y \sim \mathcal{N}_{|V|}(0, G(\sigma)\Sigma G(\sigma)^T)$$

Let $\Gamma \subseteq S(V)$. Then the distribution of Y is invariant under the action of Γ if and only if

$$G(\sigma)\Sigma G(\sigma)^T = \Sigma \quad \text{for all } \sigma \in \Gamma$$

Since $G(\sigma)$ satisfies $G(\sigma)^T = G(\sigma^{-1}) = G(\sigma)^{-1}$, this is equivalent to

$$G(\sigma)\Sigma G(\sigma)^{-1} = \Sigma \quad \text{for all } \sigma \in \Gamma.$$

Taking inverses of both sides gives

$$G(\sigma)KG(\sigma)^{-1} = K \quad \text{for all } \sigma \in \Gamma.$$

RCOP Models: A Formal Definition

We can show that such a set Γ must in fact be a group. (Properties to check: closure, associativity, identity element, inverse element.)

For Γ to capture all symmetries of the model, we also need it to act transitively on each of the vertex colour classes. (A group G acts transitively on a set Ω if for all $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha = \beta g$.)

This leads to the following definition.

We shall say that a model with imposed symmetry restrictions on K is generated by a group of permutations of its vertices Γ if

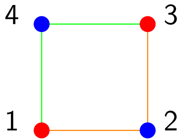
(i) $G(\sigma) K G(\sigma)^{-1} = K$ for all $\sigma \in \Gamma$

(ii) Γ acts transitively on the model's vertex colour classes.

(In the Frets Heads example $\Gamma = \{Id, (12)(34)\}$.)

Example: Not all RCON Models are RCOP Models

Consider the RCON model on four random variables satisfying the following conditional independence statements:



Its concentration matrix K has the form

$$K = \begin{pmatrix} \alpha & \gamma & 0 & \delta \\ \gamma & \beta & \gamma & 0 \\ 0 & \gamma & \alpha & \delta \\ \delta & 0 & \delta & \beta \end{pmatrix}$$

Example: Not all RCON Models are RCOP Models

The permutations which preserve the vertex colour classes are $\sigma = Id, (13), (24)$ and $(13)(24)$. Clearly Id satisfies both of the conditions above, but for $\sigma = (24)$ and $(13)(24)$ condition (i) is violated:

$$G(\sigma)KG(\sigma)^{-1} = \begin{pmatrix} \alpha & \delta & 0 & \gamma \\ \delta & \beta & \delta & 0 \\ 0 & \delta & \alpha & \gamma \\ \gamma & 0 & \gamma & \beta \end{pmatrix} \neq K = \begin{pmatrix} \alpha & \gamma & 0 & \delta \\ \gamma & \beta & \gamma & 0 \\ 0 & \gamma & \alpha & \delta \\ \delta & 0 & \delta & \beta \end{pmatrix}$$

Further, $\sigma = (13)$ does not act transitively on the vertex colour class $\{2, 4\}$, violating condition (ii).

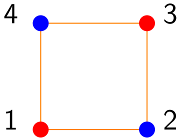
Hence only Id satiesfies both of the conditions (i) and (ii) and the model cannot be RCOP.

Example: Not all RCON Models are RCOP Models

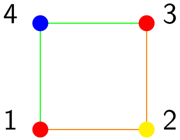
We can turn it into one in two ways:

1. By setting $\gamma = \delta$, forcing condition (i) to hold for $\sigma = (13)(24)$. This model is generated by $\Gamma = \{Id, (13)(24)\}$.

2. By letting 2 and 4 be in two distinct colour classes, forcing condition (ii) to hold for $\sigma = (13)$. This model is generated by $\Gamma = \{Id, (13)\}$.



Model 1



Model 2

Note that model 1 is contained in the original one, which in turn is contained in model 2.

Current work

We would like to develop an algorithm which

- checks whether a given graph colouring represents an RCOP model and finds its generating group if it is
- finds symmetry conditions to be released/imposed to turn a model which has been found not to be RCOP into one that is

We would like to exploit the following result:

A model is RCOP only if inside each vertex colour class, each vertex has the same colouring of incident edges, i.e. only if each vertex has the same number of incident edges inside each edge colour class. For otherwise permuting the vertices inside the colour classes would change the graph colouring. We shall call this criterion the incident edges criterion.

In particular this implies that the subgraphs induced by each vertex colour class must be regular (i.e. all vertices must be of the same degree) in order for the model to be RCOP.

Current work

Note that with the incident edges criterion we would have been able to classify the example given on page 11 as not RCOP without looking at the groups, since the colourings of the edges incident with 2 and 4 are different.

We would like to start our algorithm for checking whether a given model is RCOP with:

1) Identify the model's vertex and edge colour classes.
2) Check whether the model is both RCOR and RCON (RCOR-RCON criterion). If not, the model cannot be RCOP.
3) Check whether the incident edges criterion is satisfied. If not, the model cannot be RCOP.

Current work

We suspect that in order for a graph colouring to be group generated *each pair of vertex colour classes must be connected by one edge colour class only*. (Our example on page 11 supports this conjecture.) If this was true, then *together with the RCOR-RCON criterion* this would mean that *the vertex colouring implies the edge colouring for RCOP models* and that if both of the above criteria were satisfied by a graph, the edge colouring would be redundant. Assuming the result holds, in order to finish our algorithm for checking for permutation symmetry, we would need to *find the automorphism groups of each of the (regular) subgraphs induced by the vertex colour classes and check whether they (or any of their transitive subgroups) are compatible*. The direct product of those will then form the generating group of the model.

References

Søren Højsgaard, Steffen L. Lauritzen, *Graphical Gaussian Models with Edge and Vertex Symmetries*, 2007