# Bifurcation Phenomena in the Flow through a Sudden Expansion in a Pipe

Andrew Cliffe[1], Ed Hall[1], Paul Houston[1], Eric Phipps[2] and Andy Salinger[2]

[1]University of Nottingham

[2]Sandia National Laboratories

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Computational Linear Algebra for Partial Differential Equations
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# Acknowledgements

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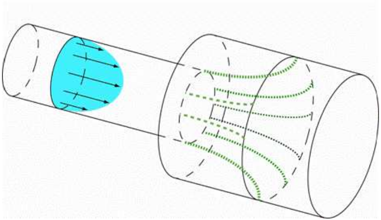

# Overview

- Introduction
- Bifurcation in the presence of $O(2)$ symmetry
- *A posteriori* error estimation
- Numerical results
- Summary and conclusions

# Introduction

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  - $Z_2$ symmetry-breaking Hopf bifurcation.

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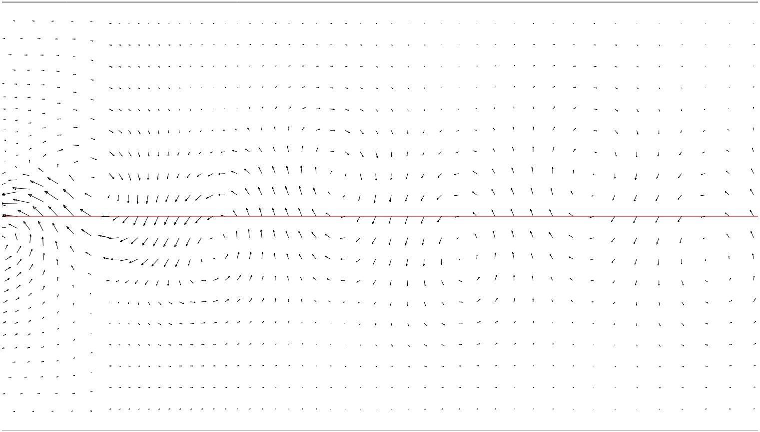

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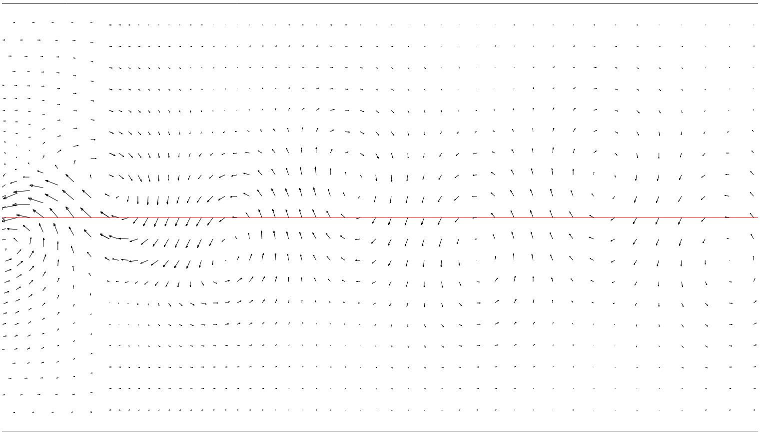

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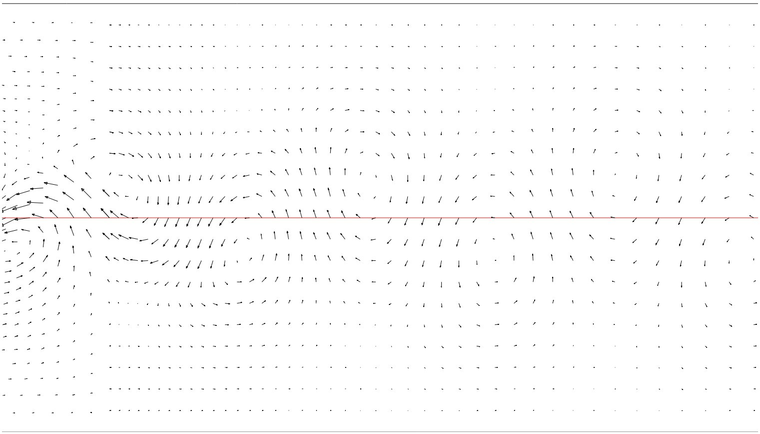

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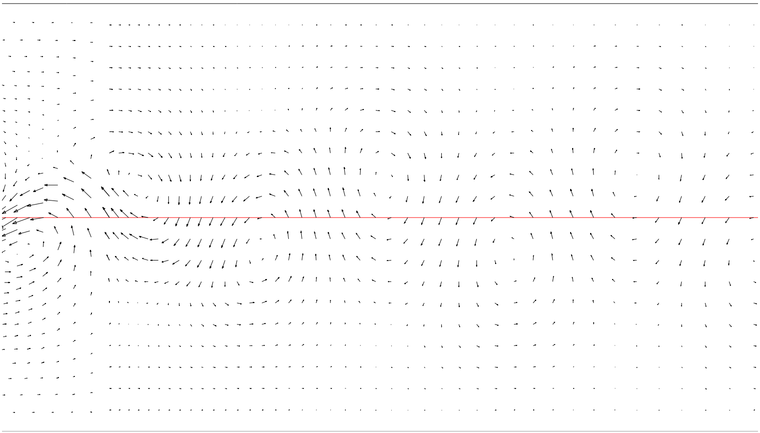

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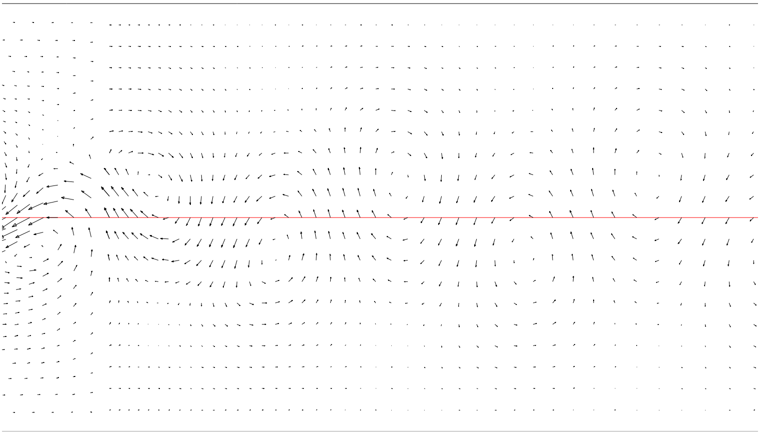

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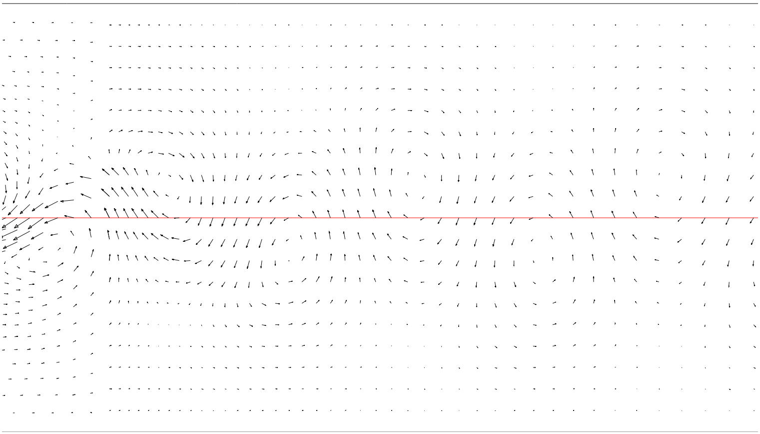

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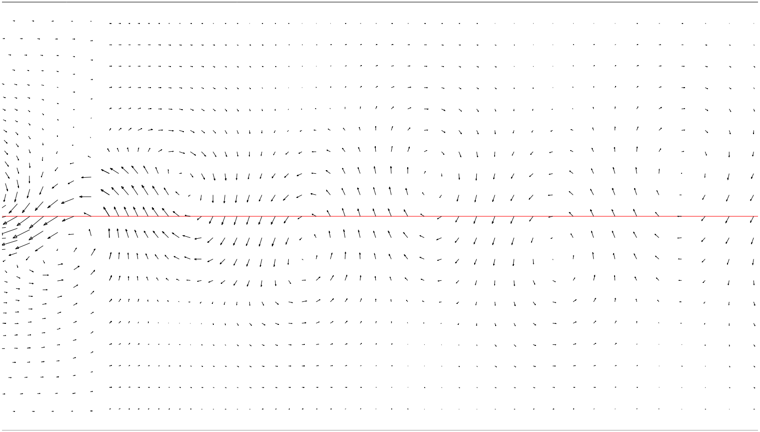

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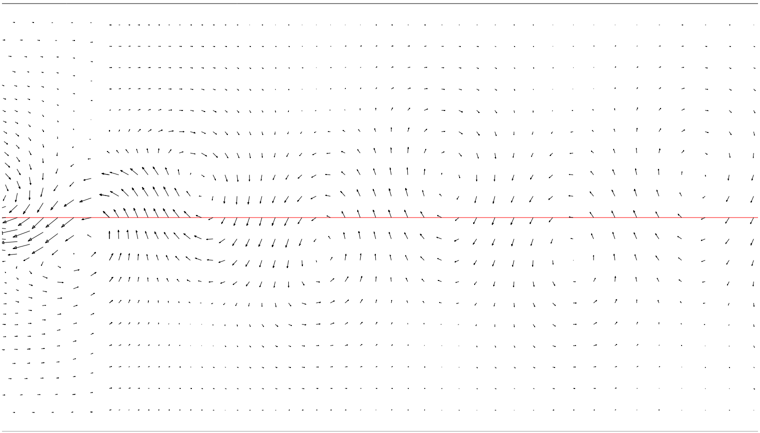

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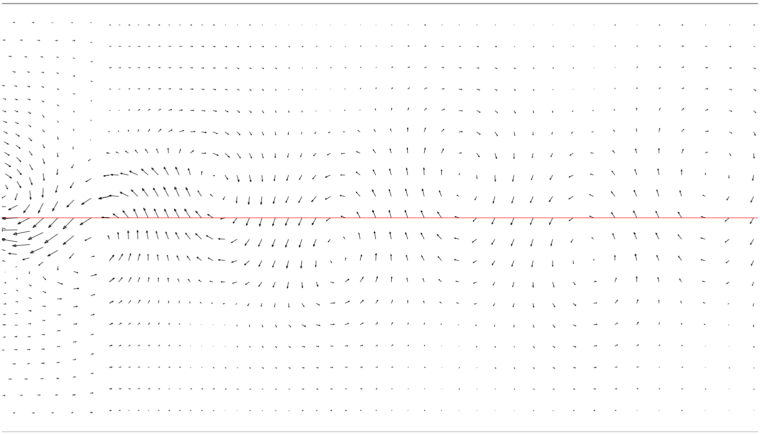

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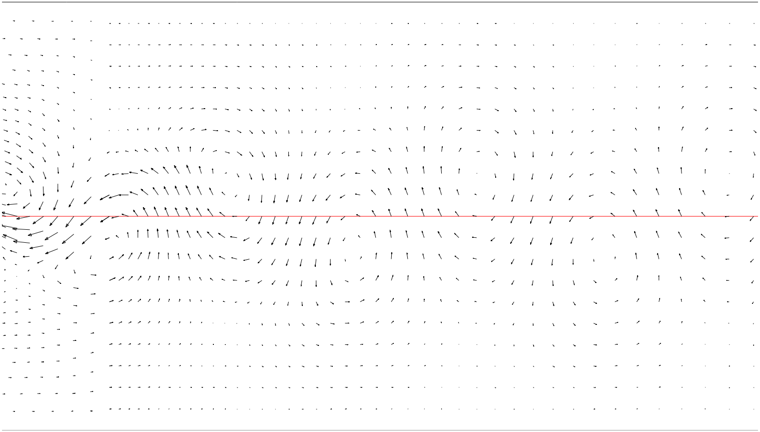

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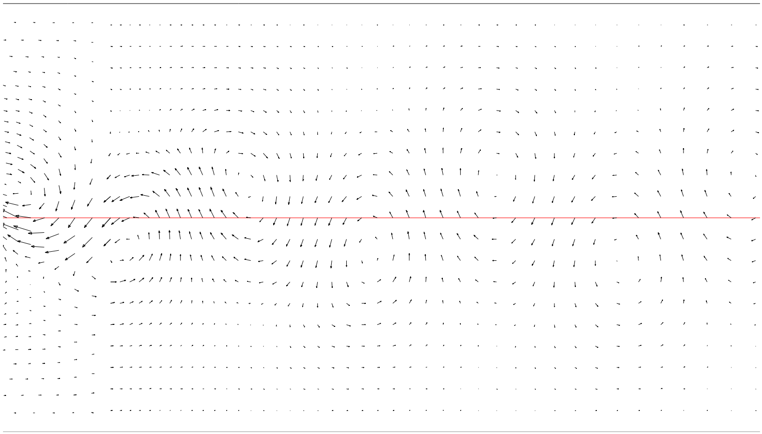

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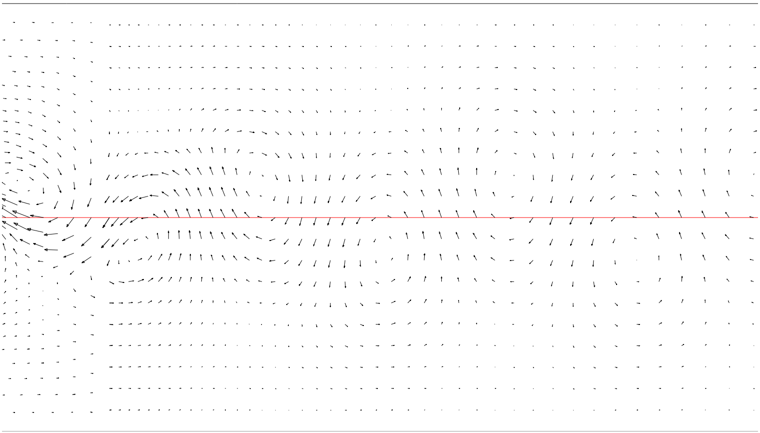

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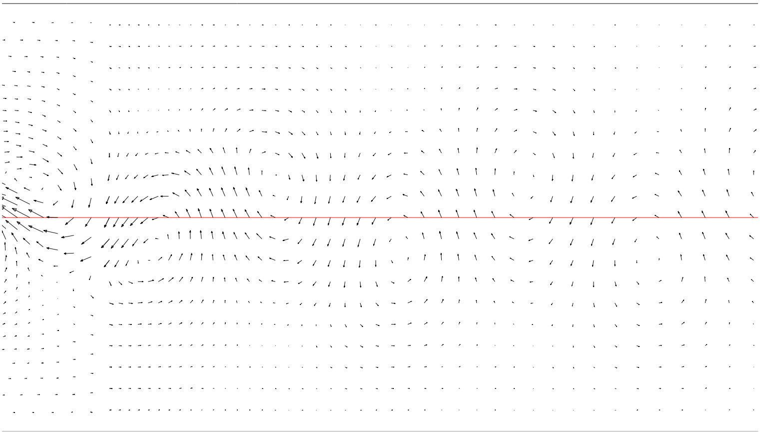

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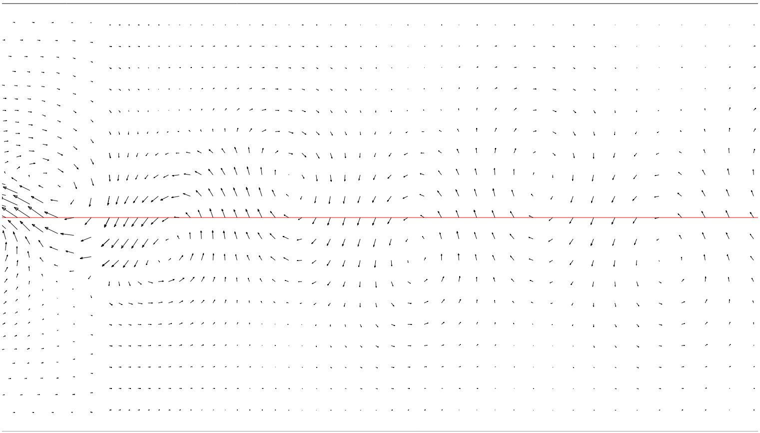

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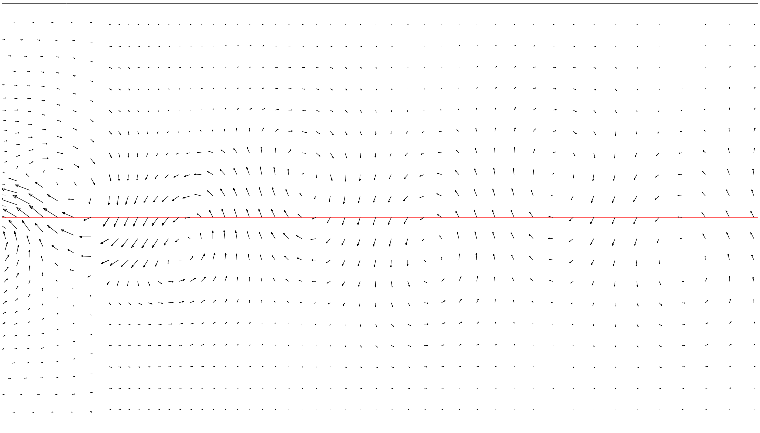

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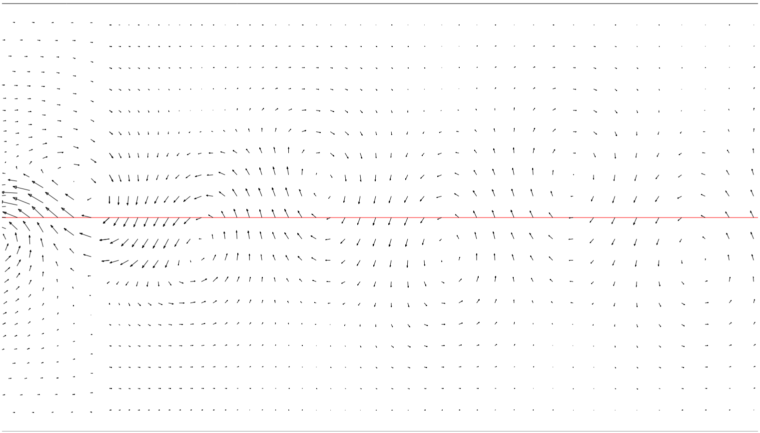

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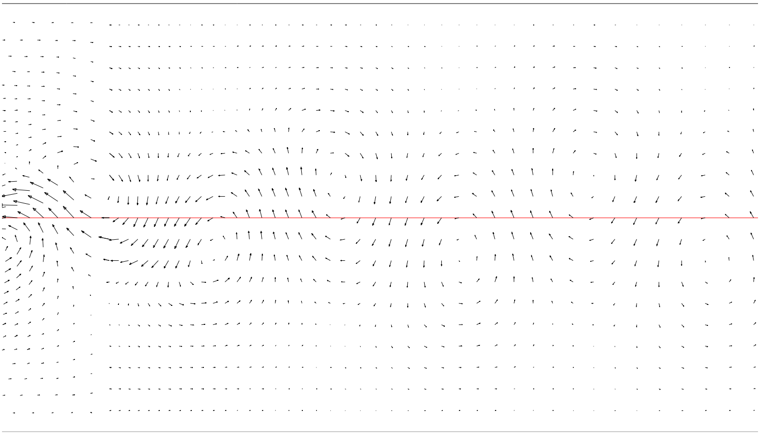

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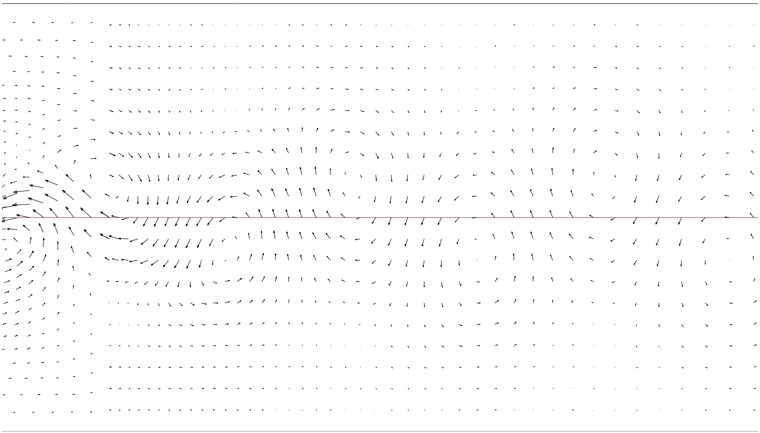

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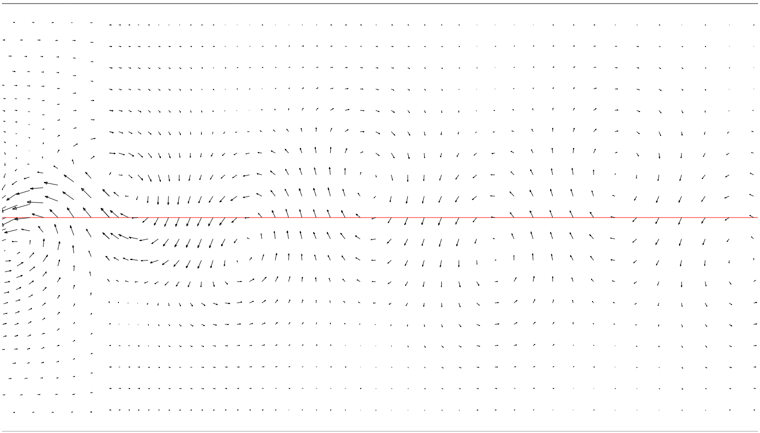

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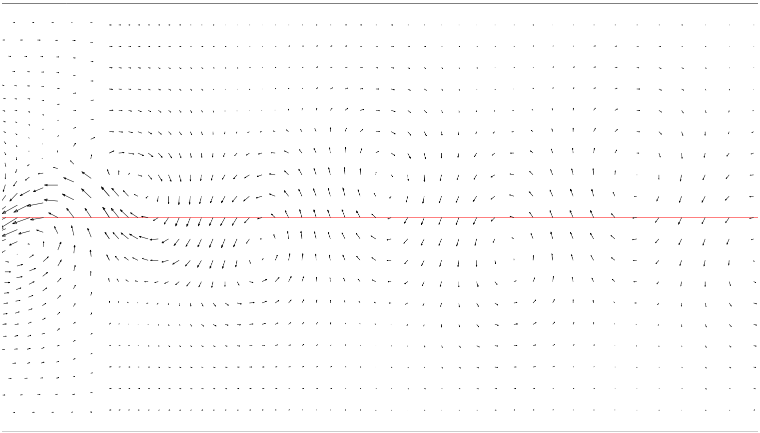

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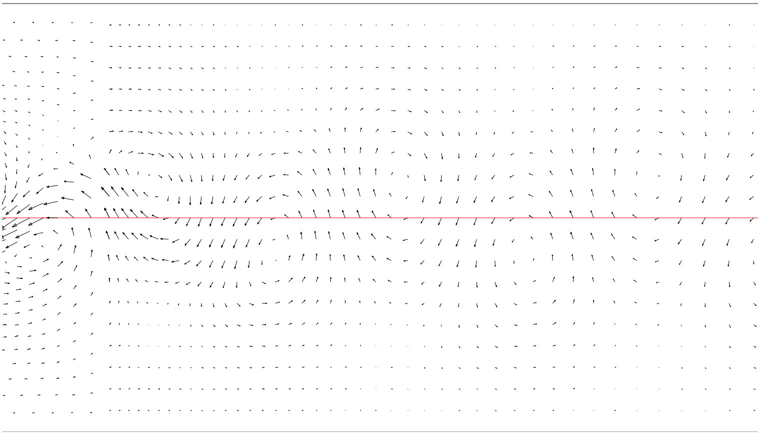

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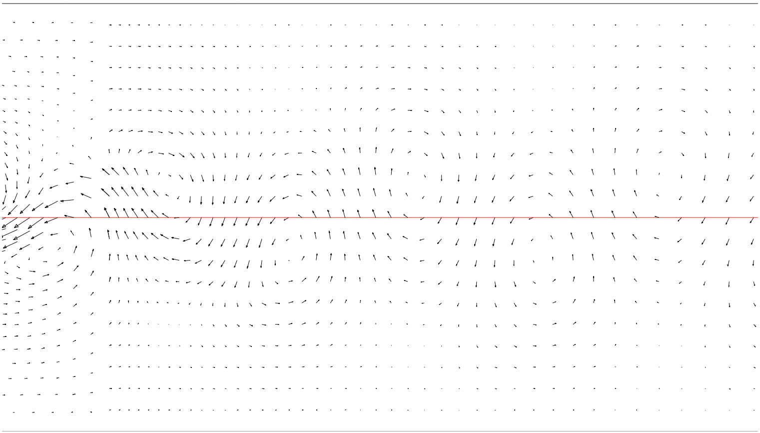

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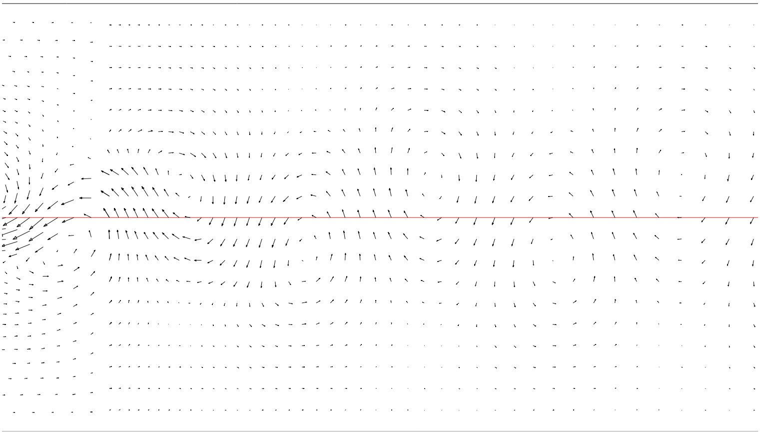

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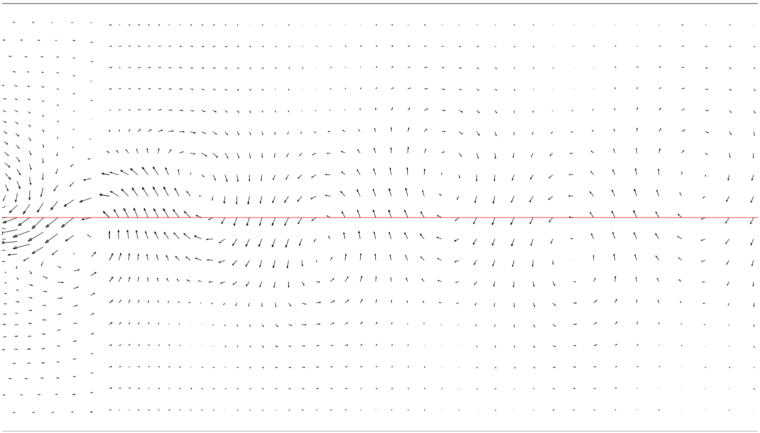

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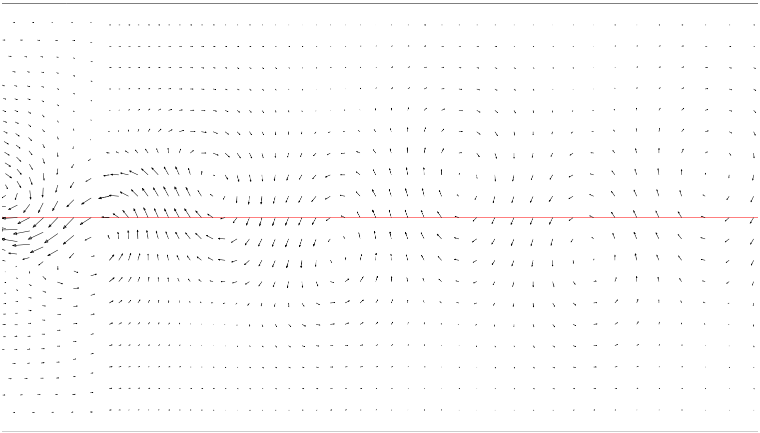

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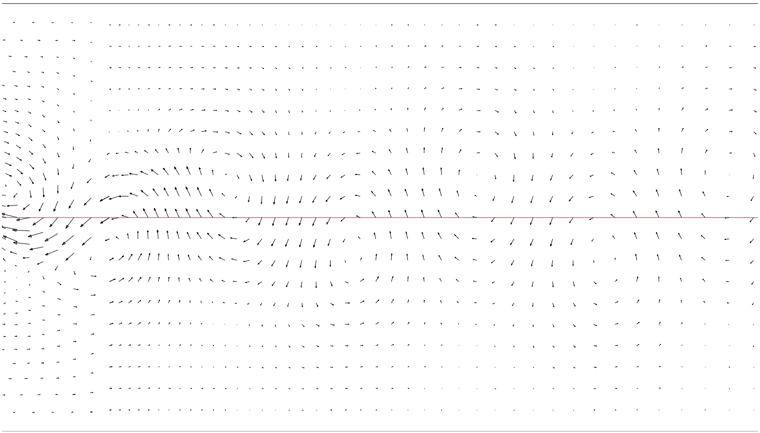

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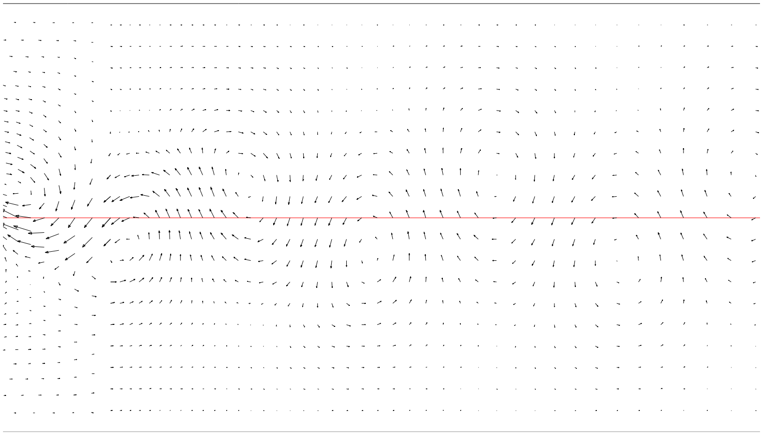

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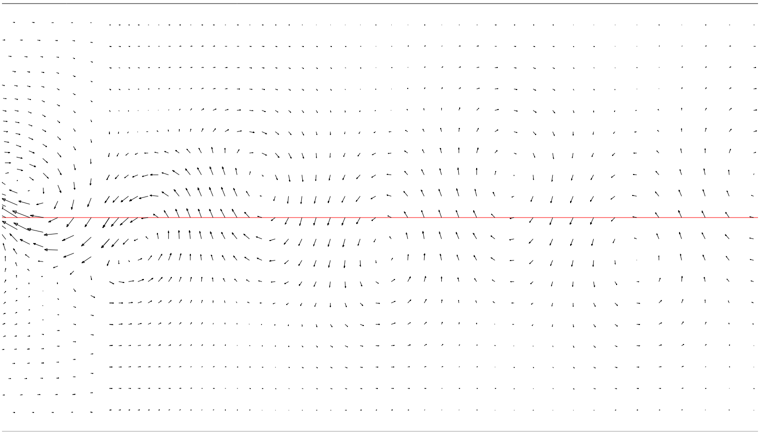

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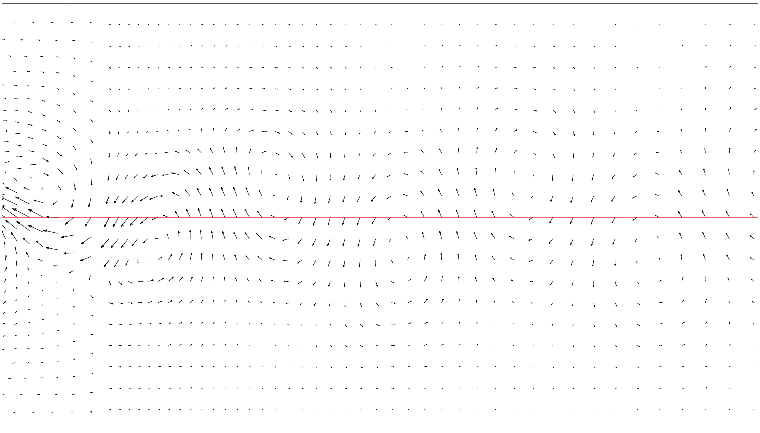

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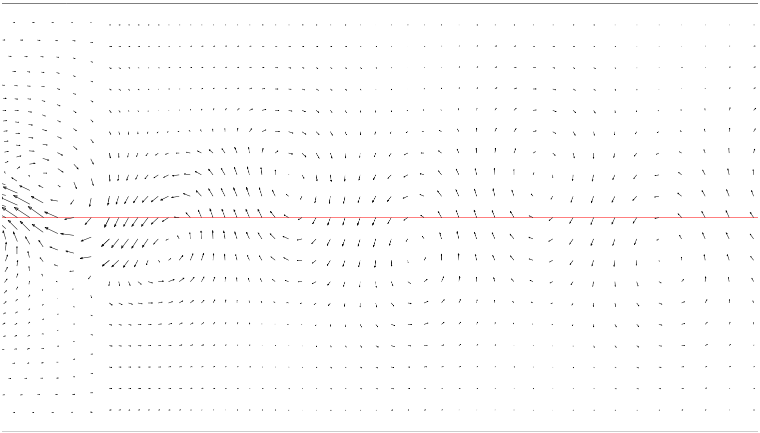

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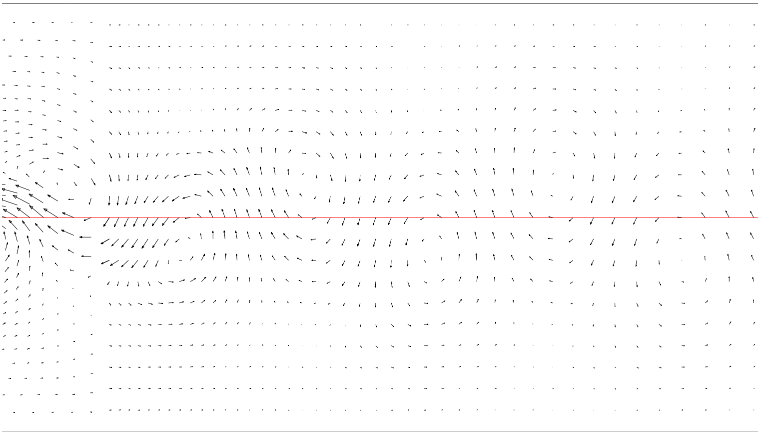

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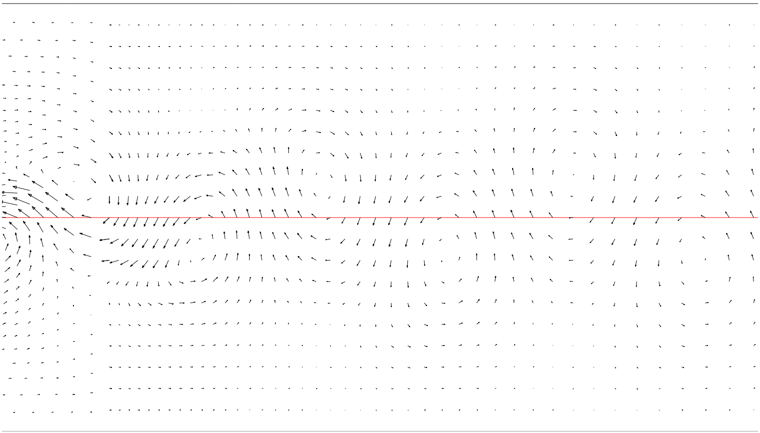

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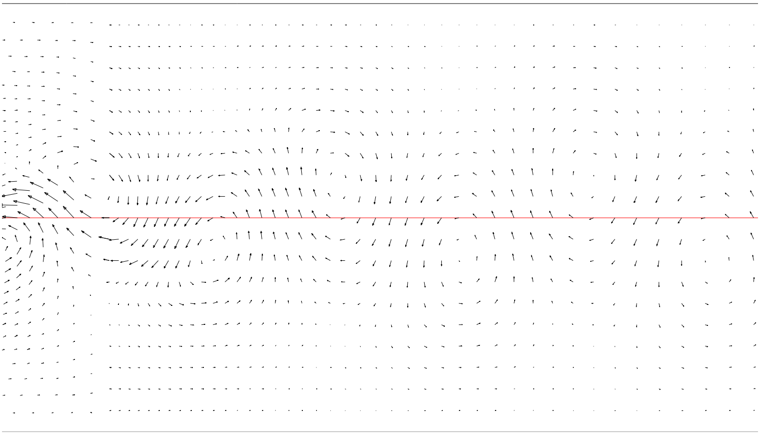

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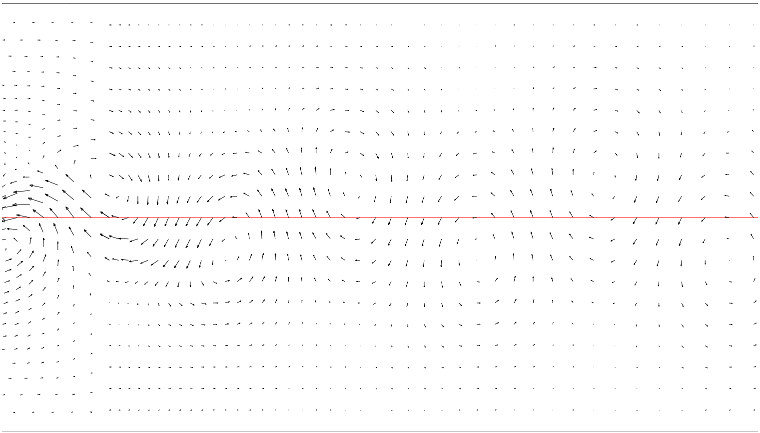

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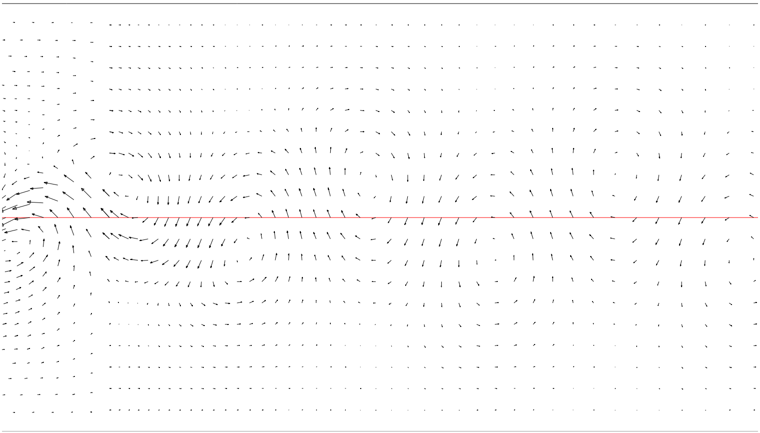

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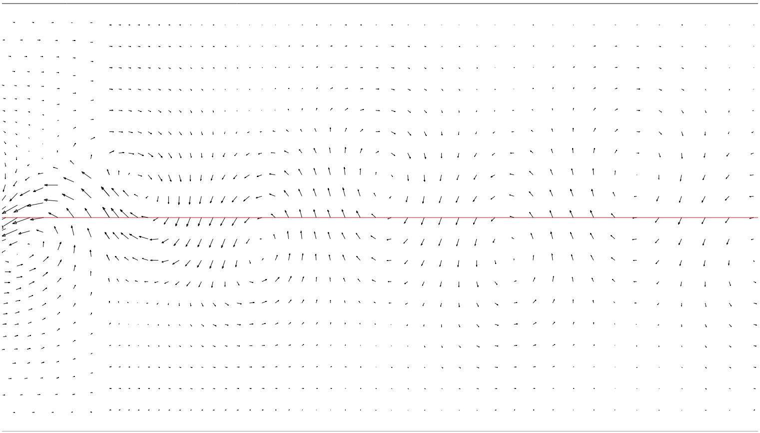

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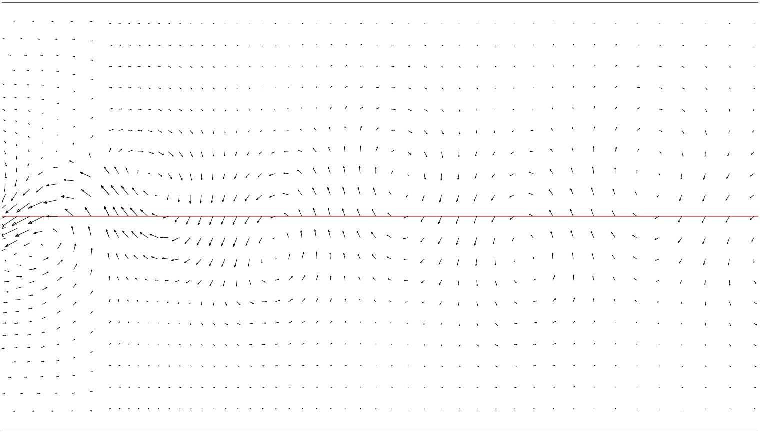

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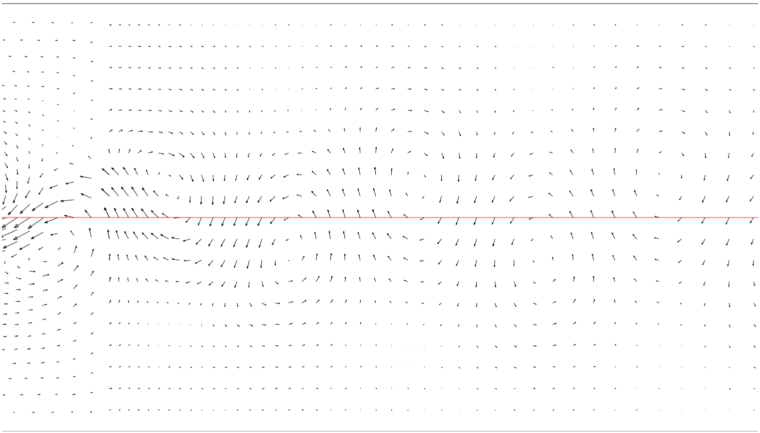

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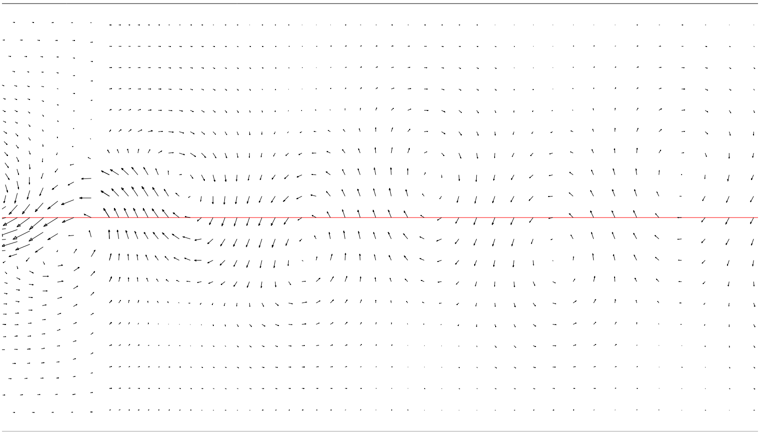

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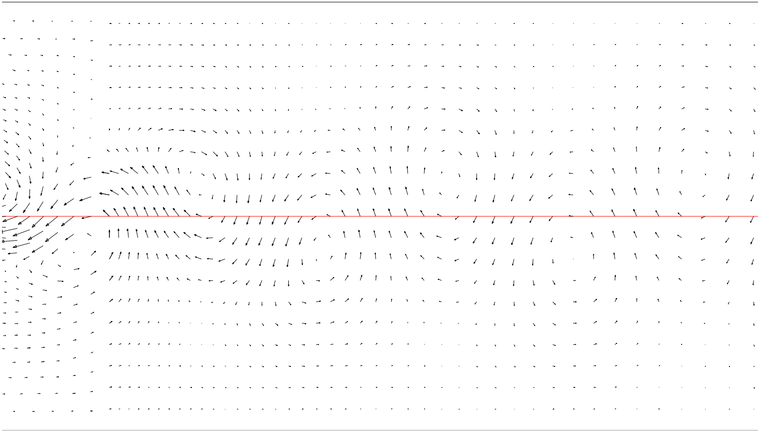

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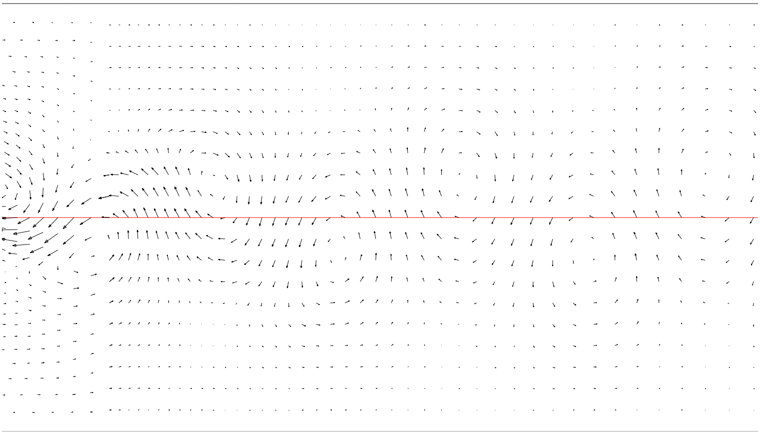

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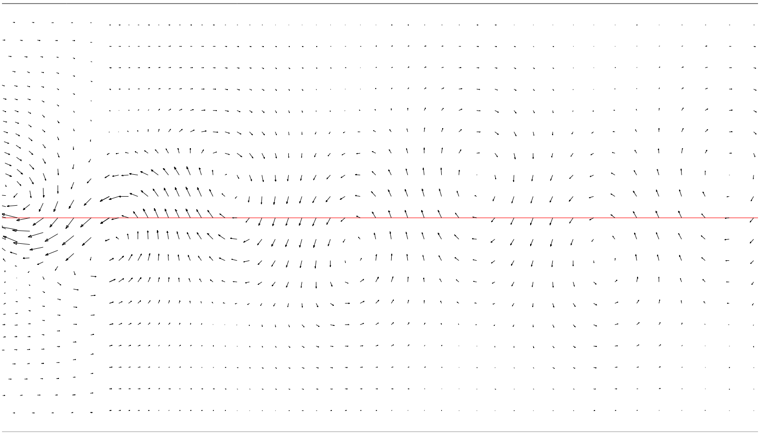

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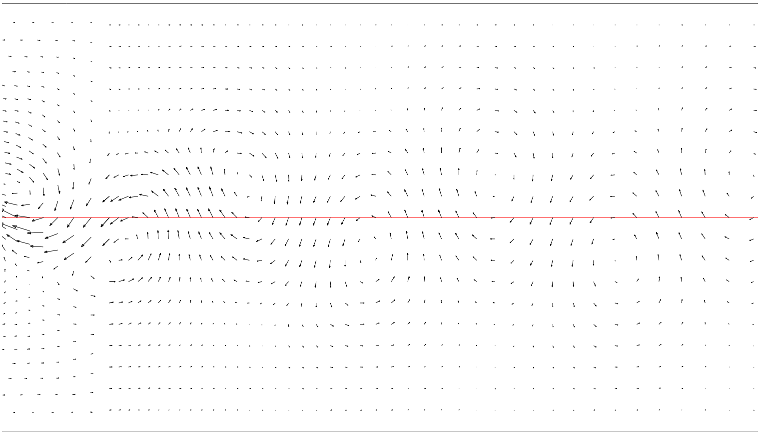

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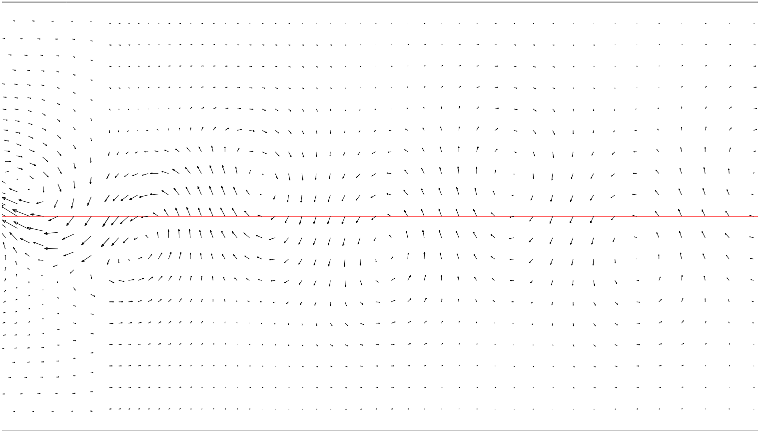

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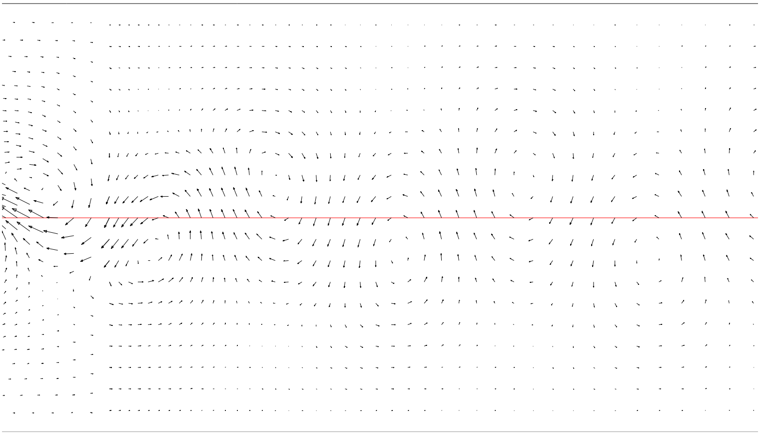

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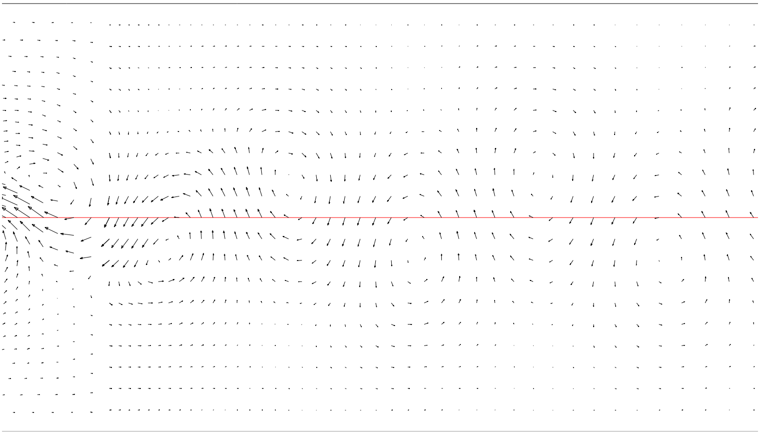

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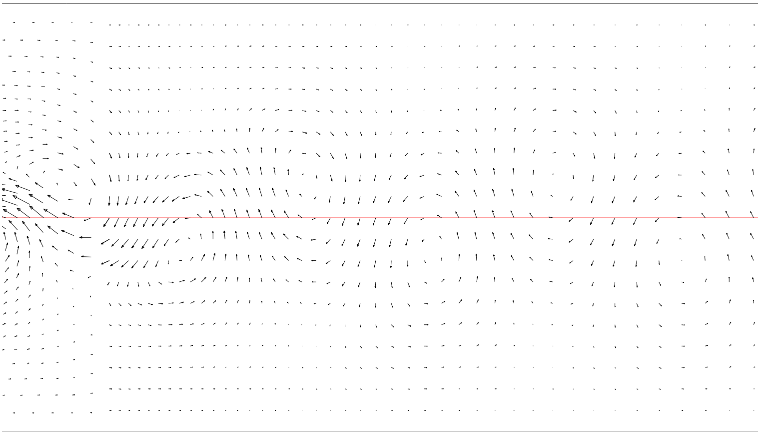

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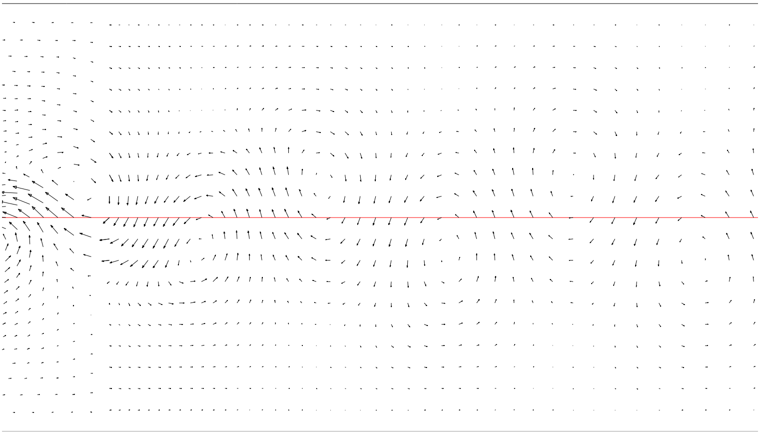

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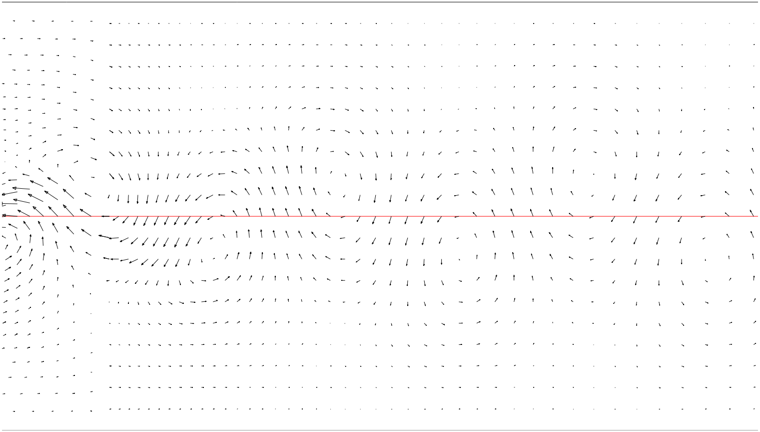

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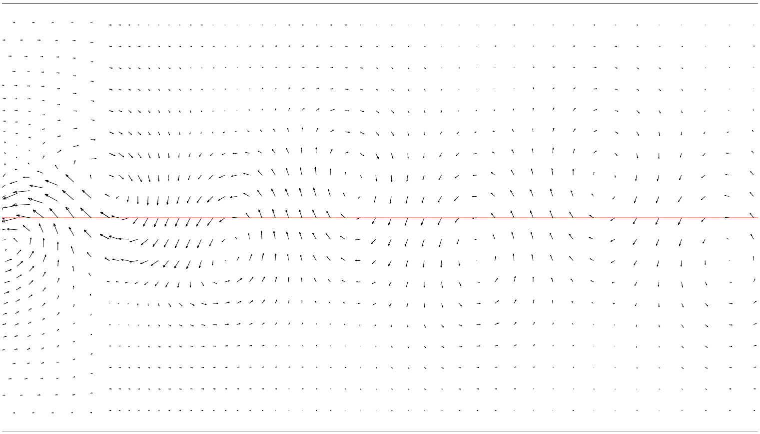

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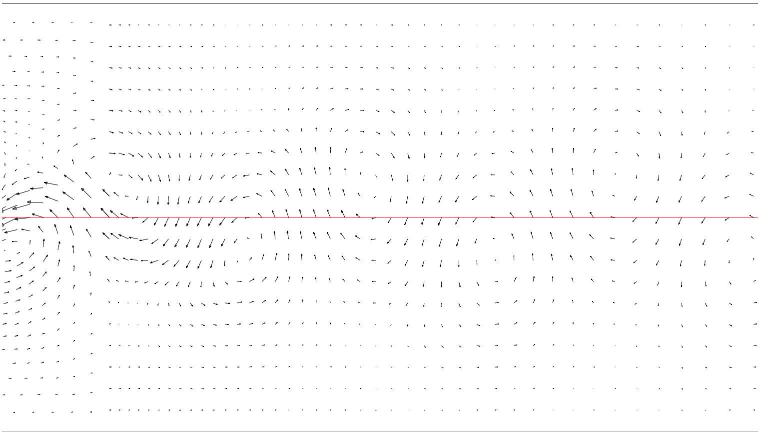

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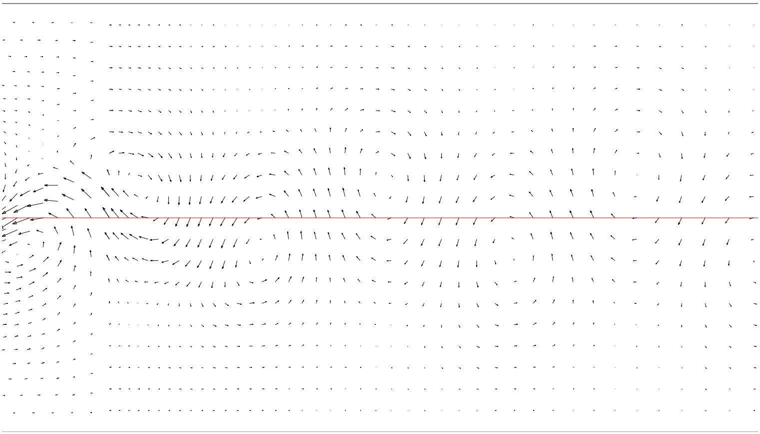

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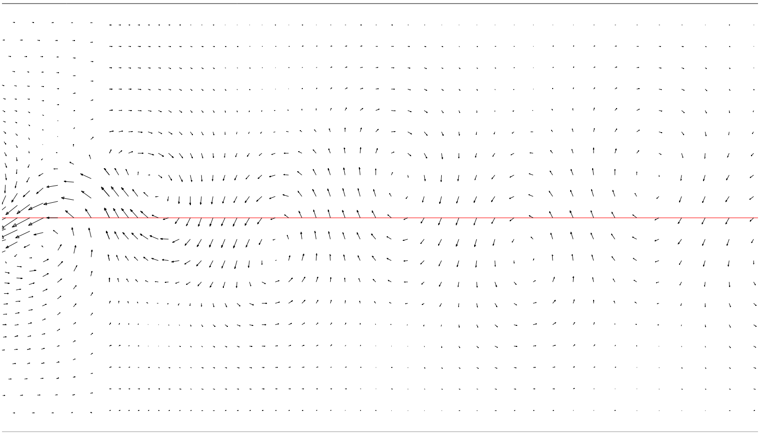

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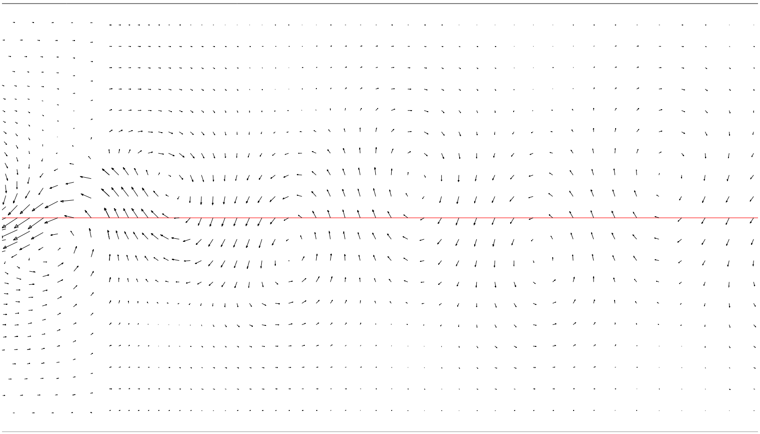

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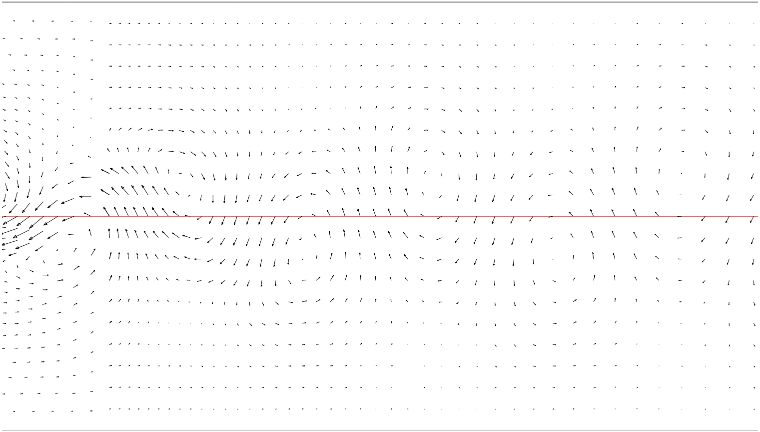

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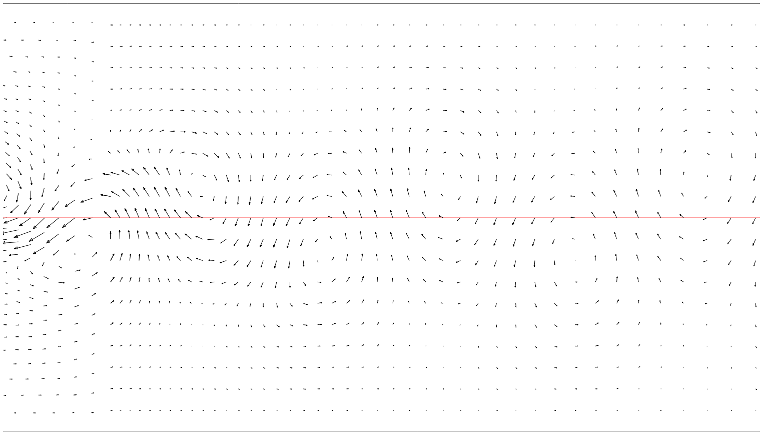

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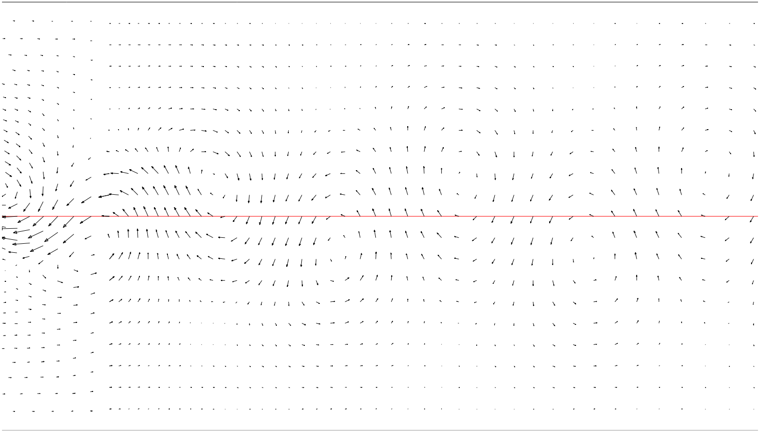

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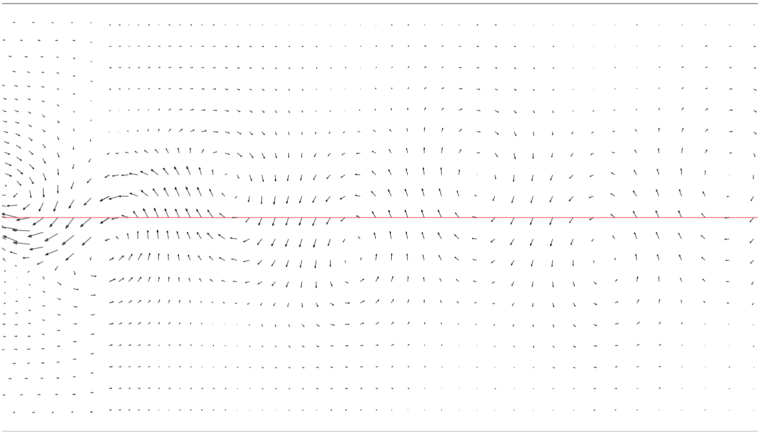

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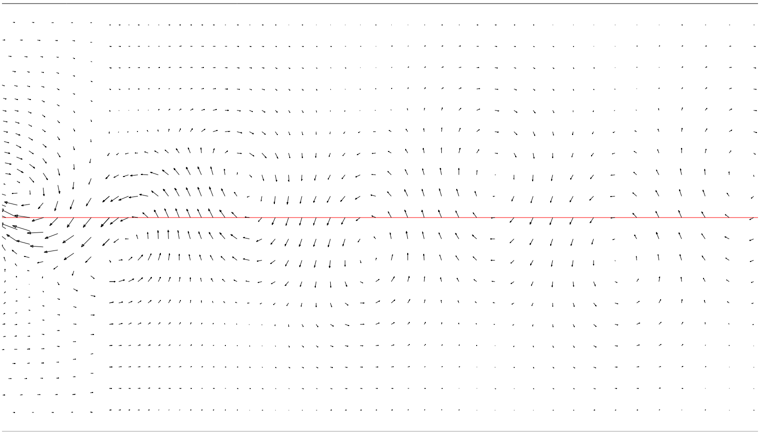

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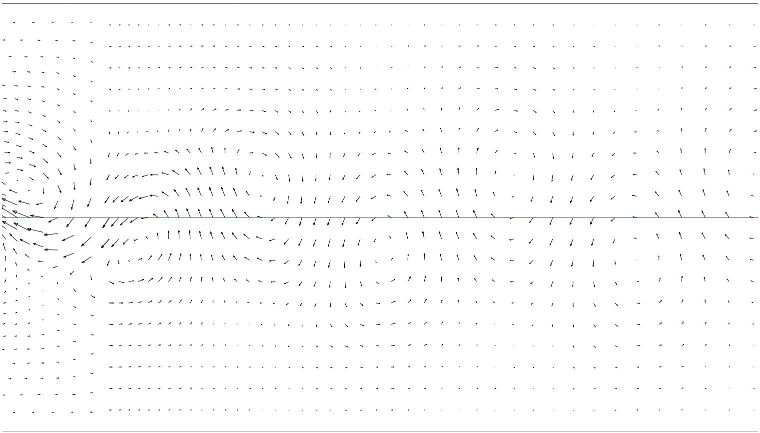

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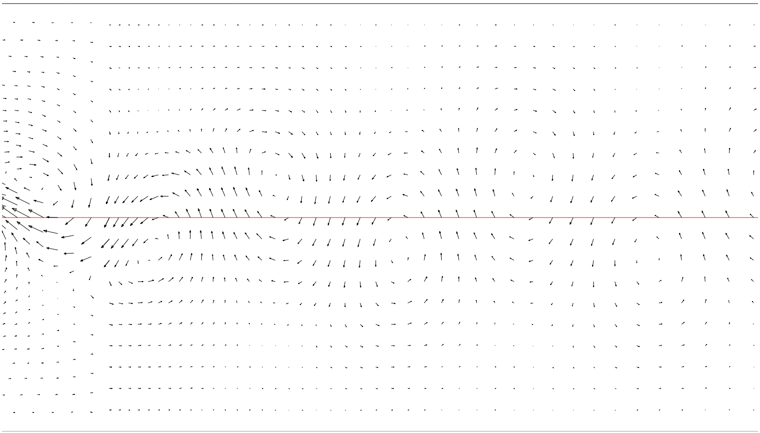

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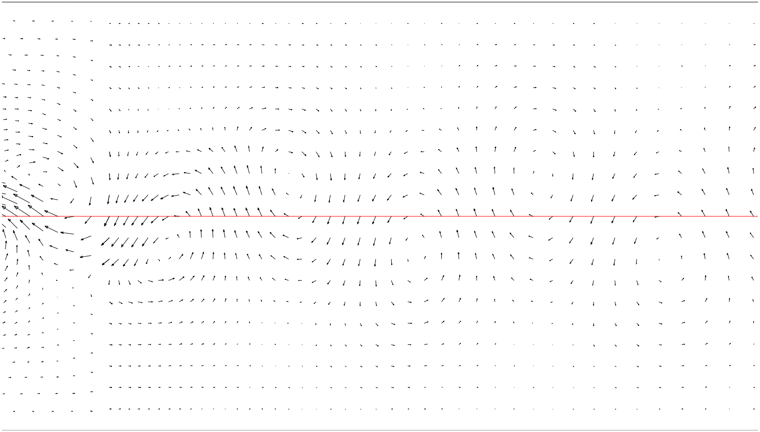

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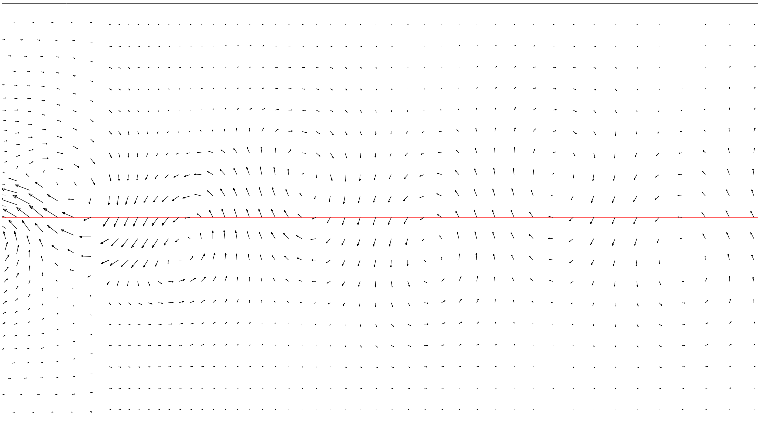

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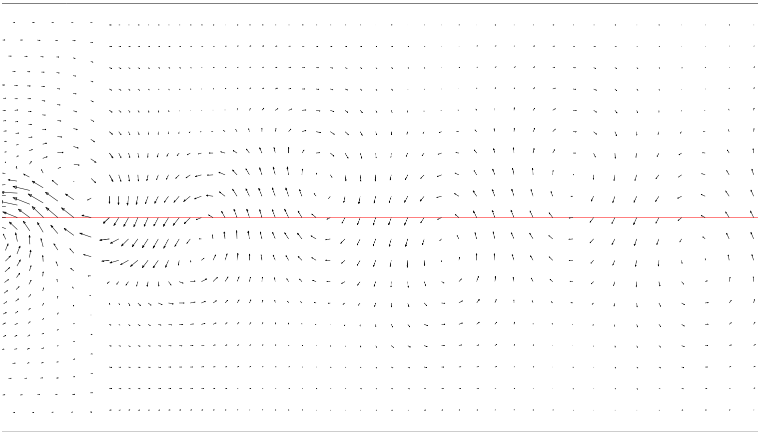

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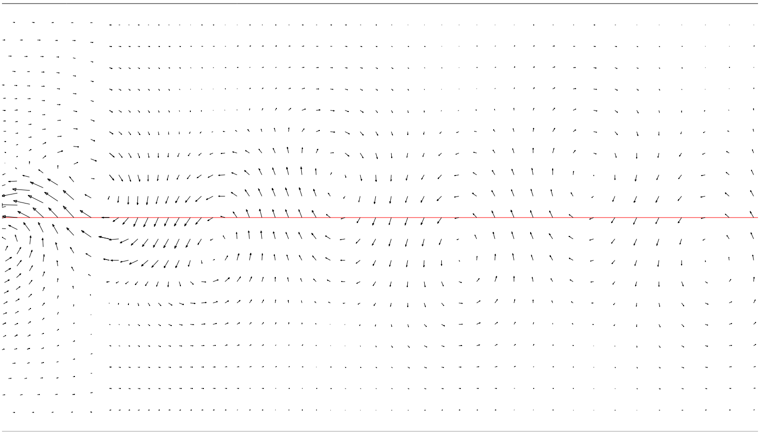

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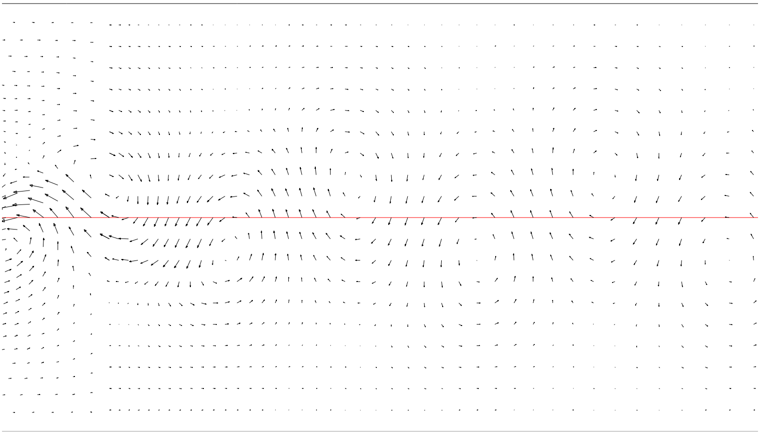

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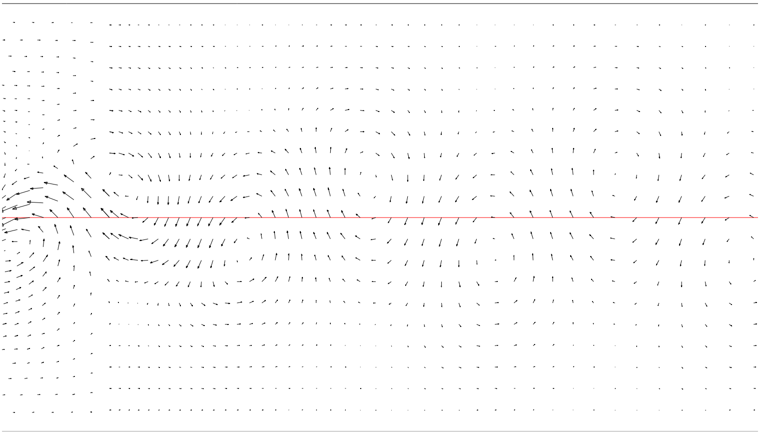

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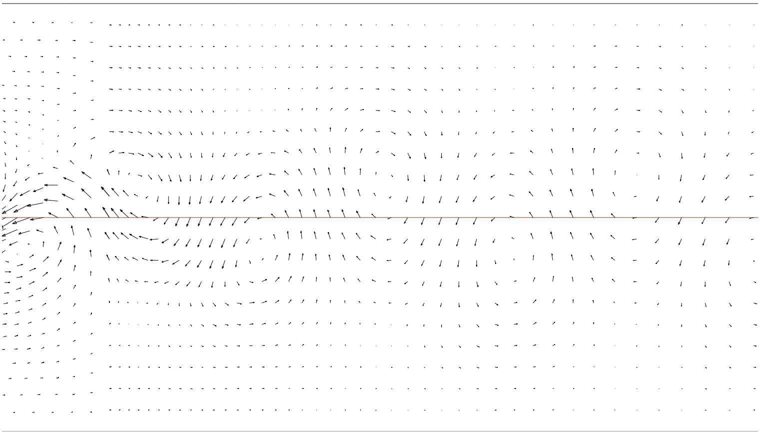

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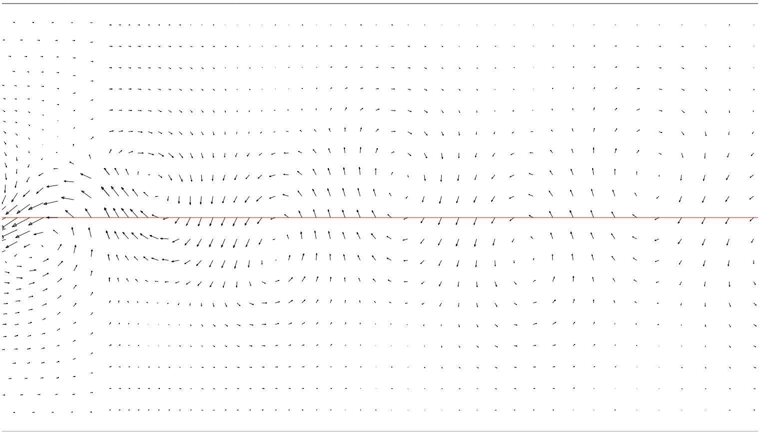

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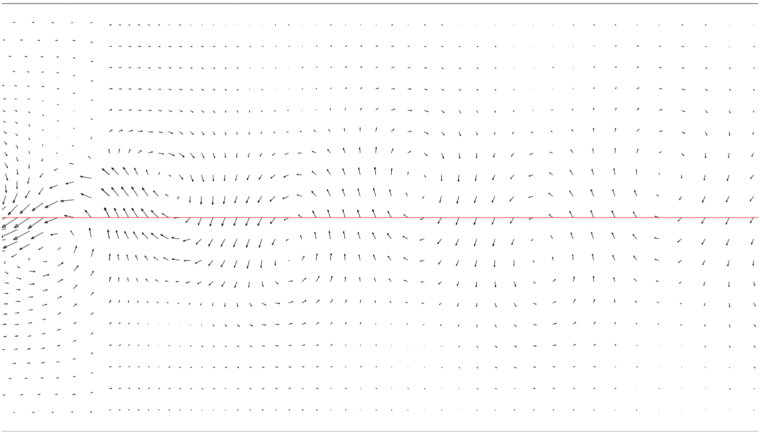

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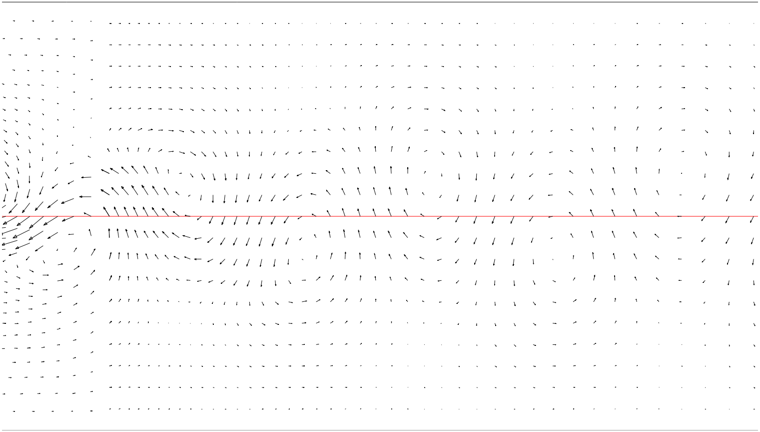

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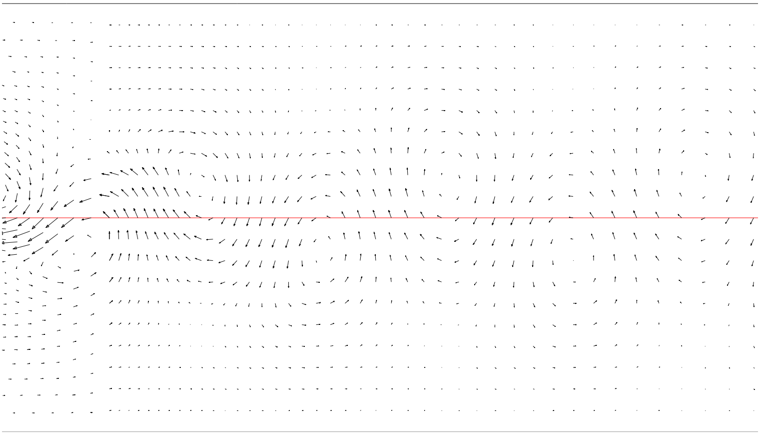

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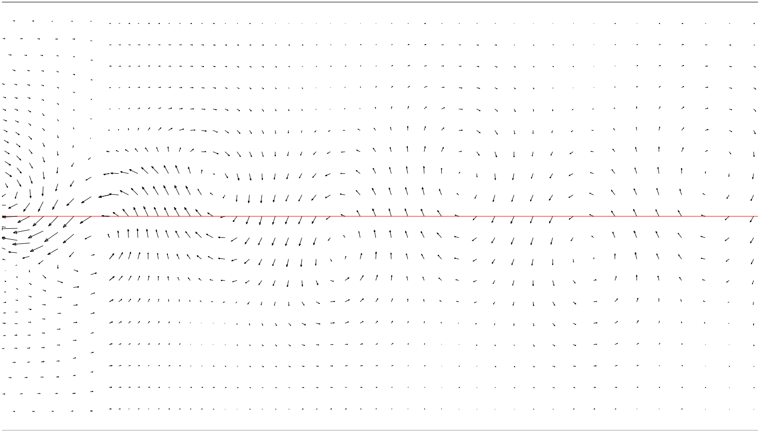

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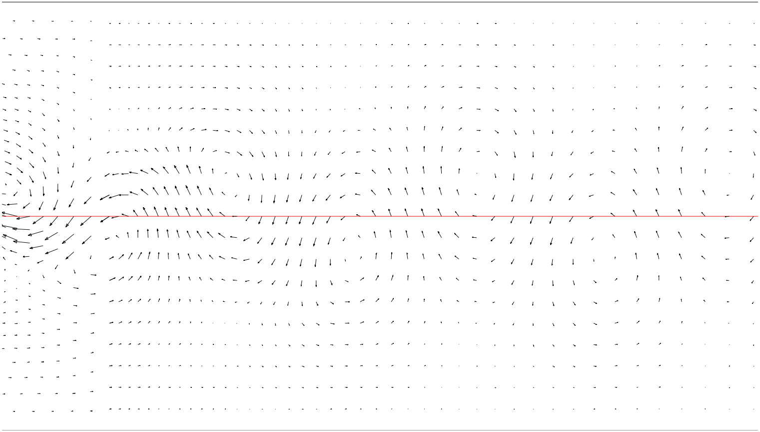

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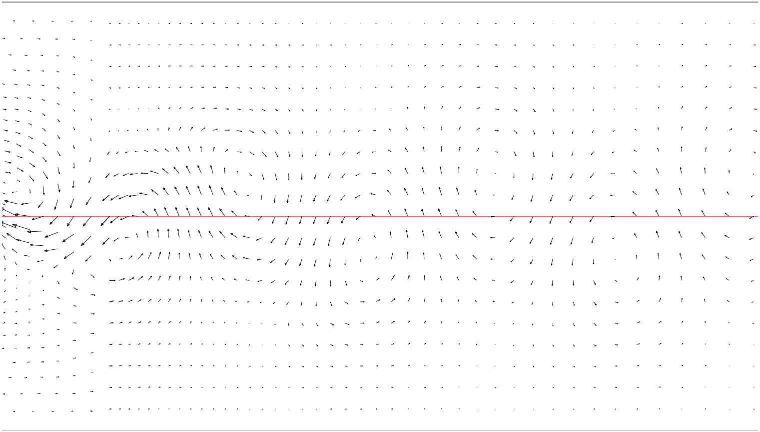

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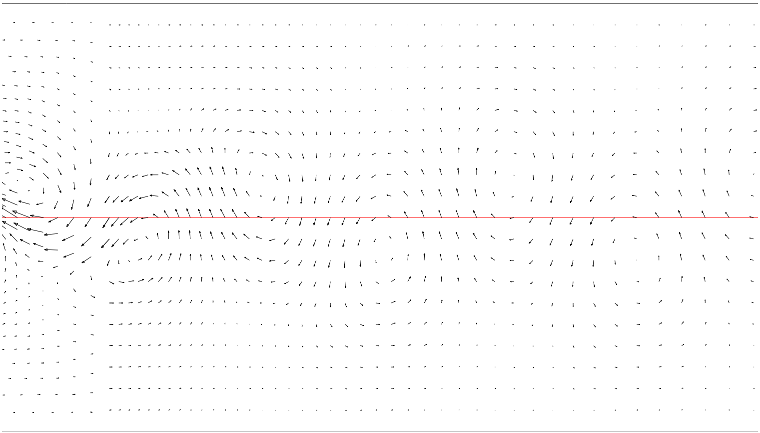

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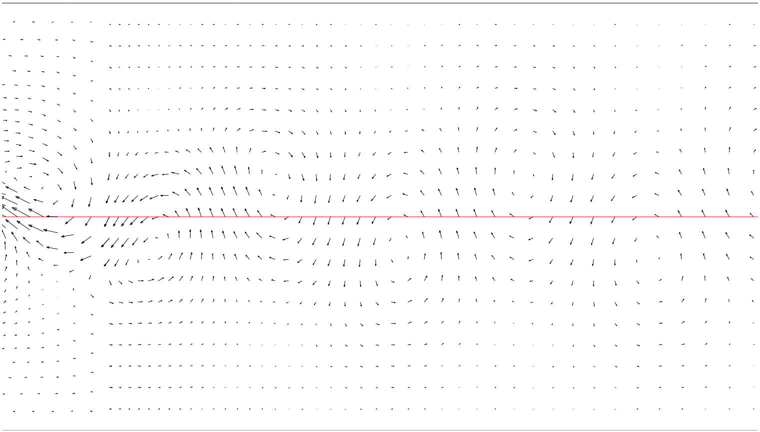

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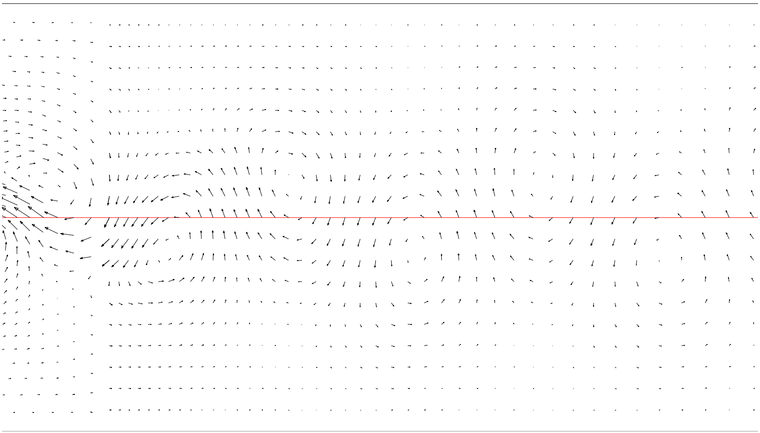

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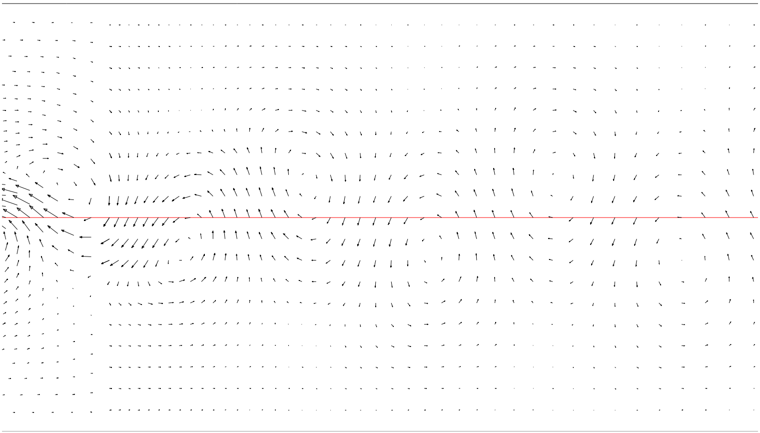

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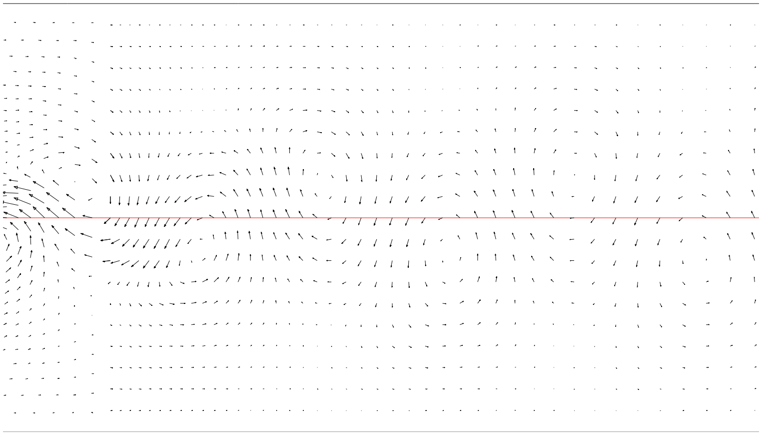

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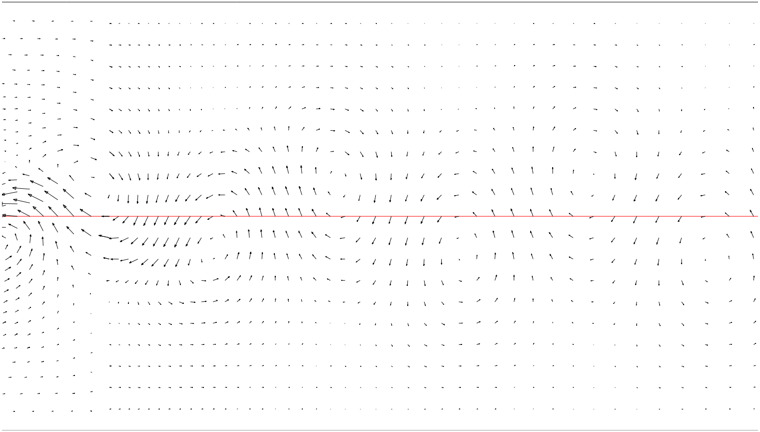

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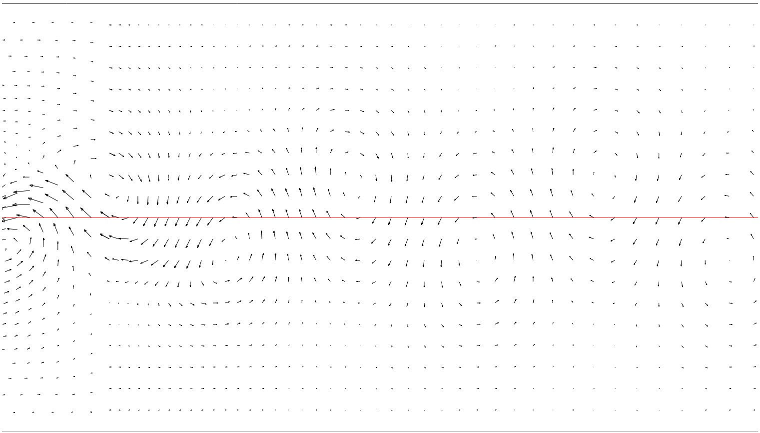

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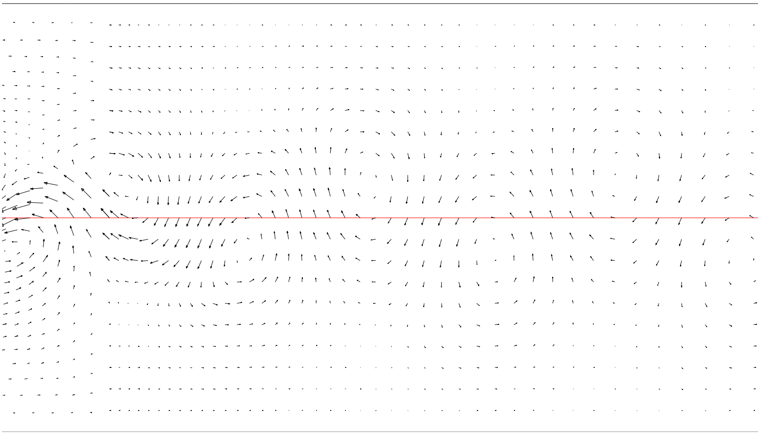

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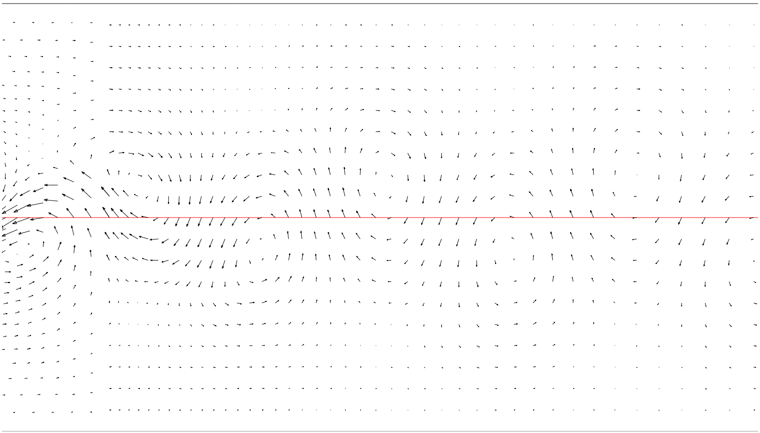

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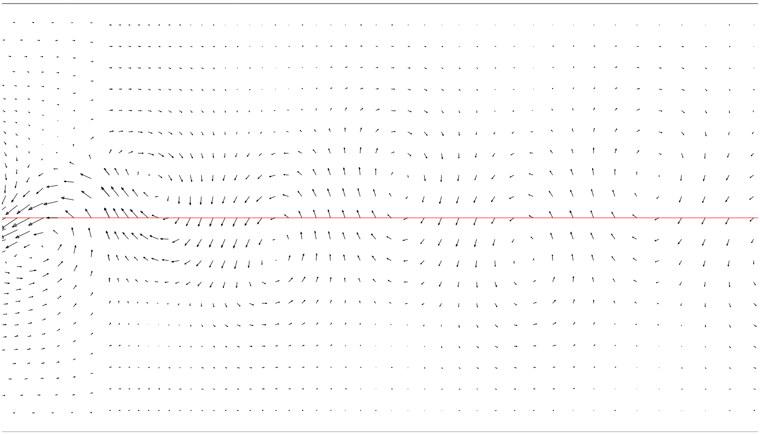

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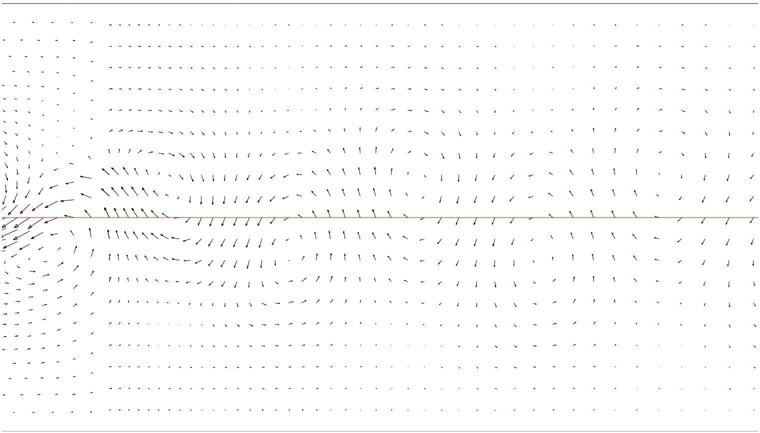

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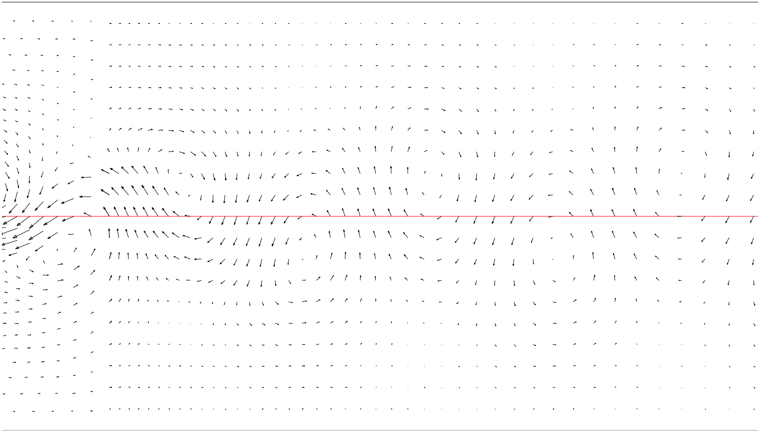

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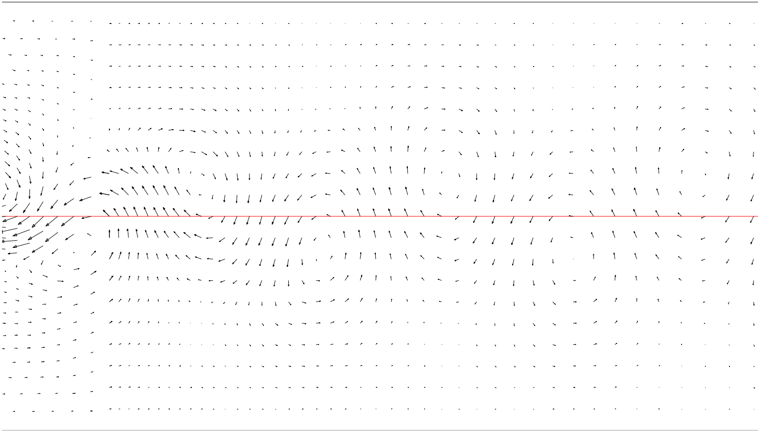

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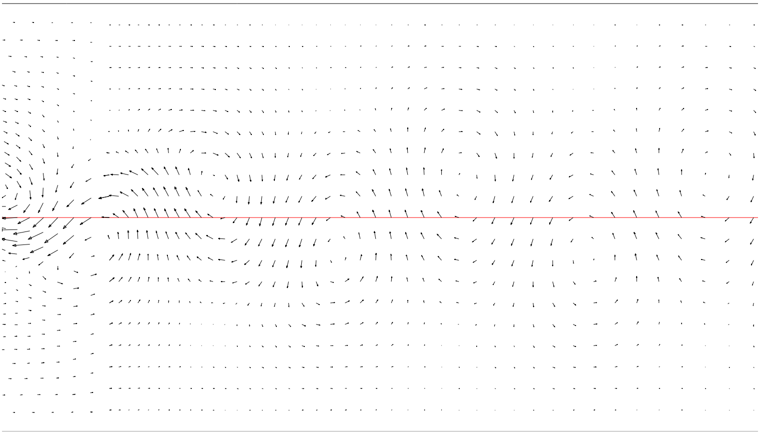

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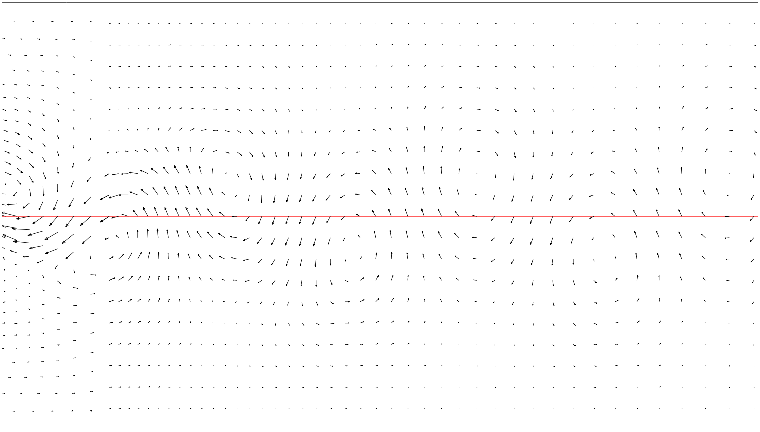

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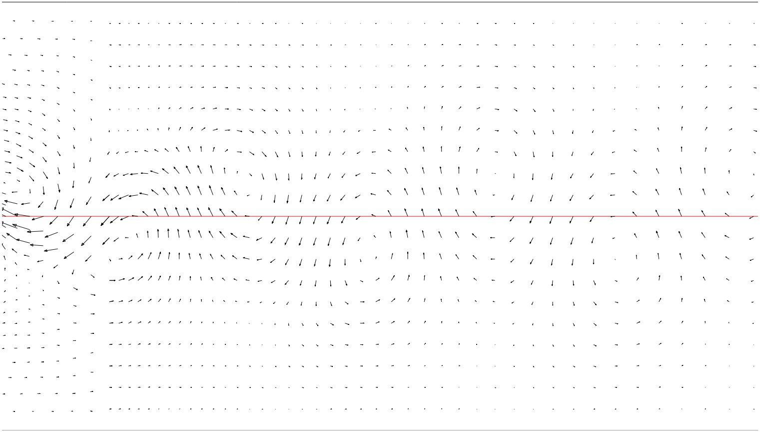

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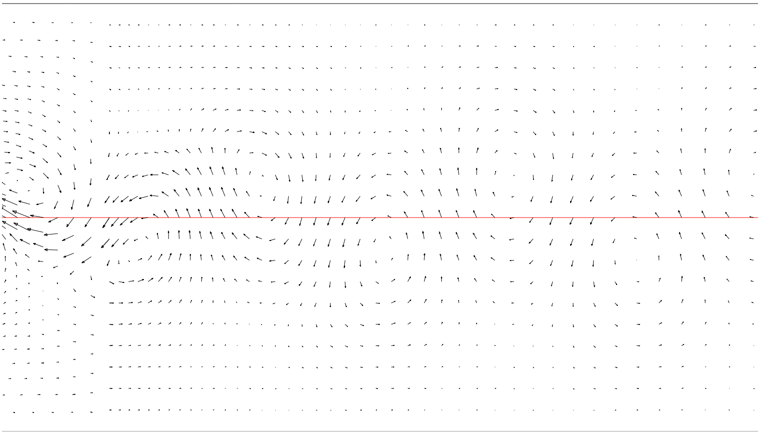

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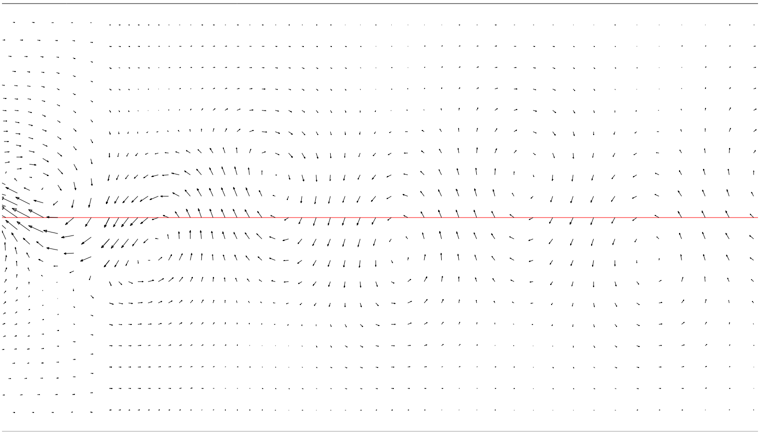

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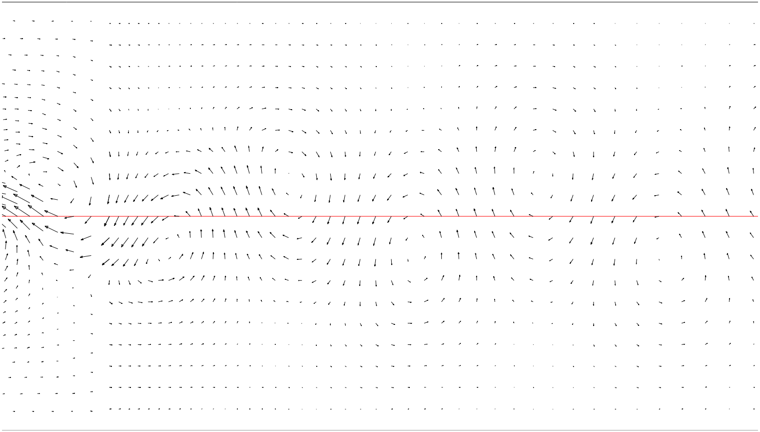

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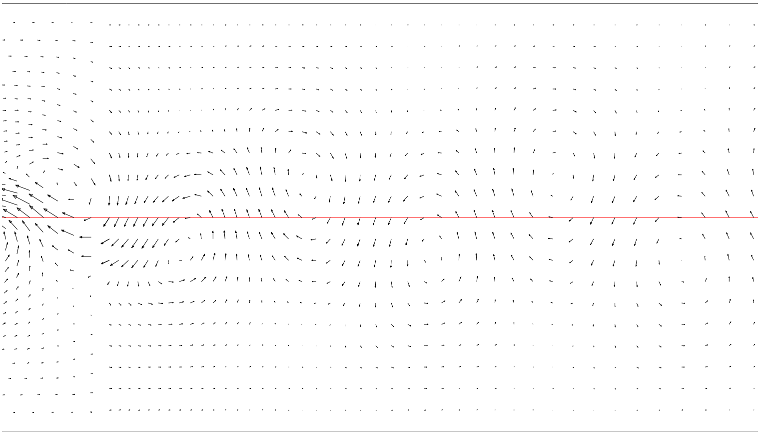

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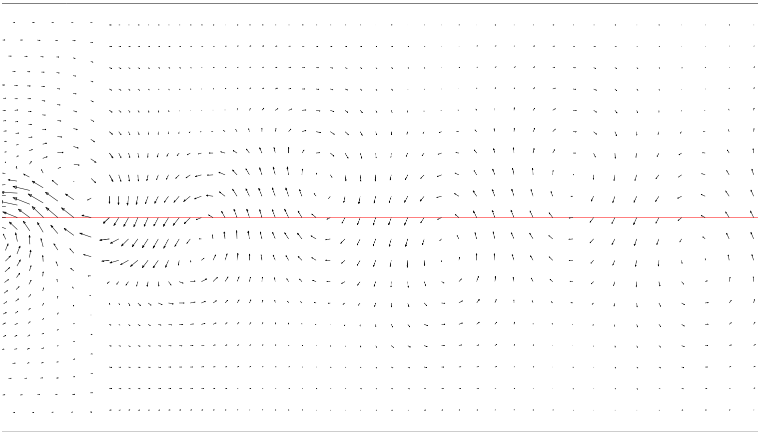

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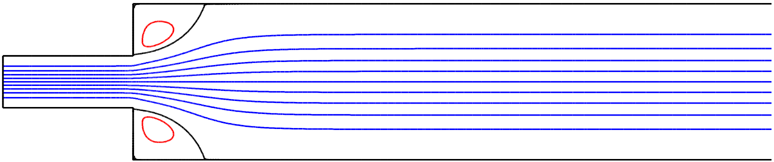
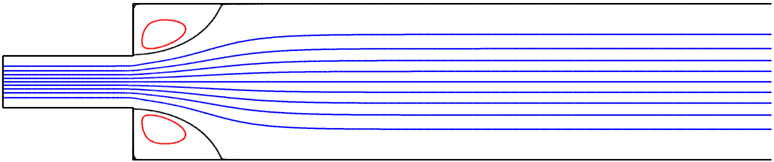
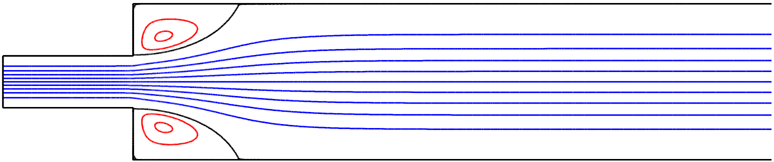
# Channel with a Sudden Expansion - Re $= 20$



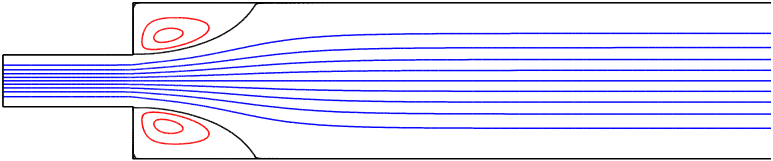
# Channel with a Sudden Expansion - Re = 25



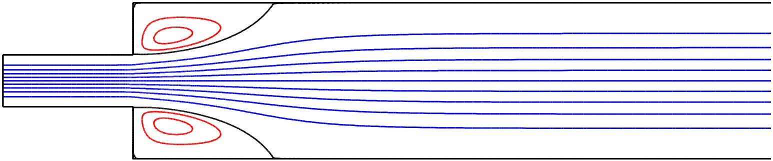
# Channel with a Sudden Expansion - Re = 30



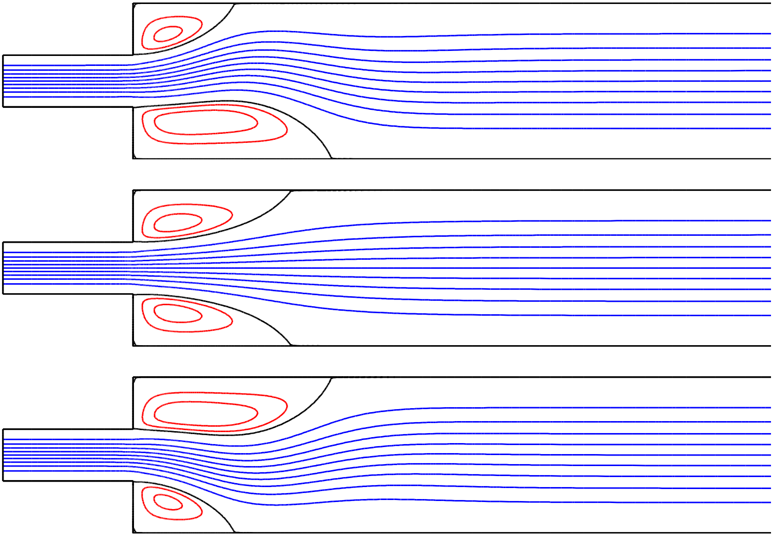
# Channel with a Sudden Expansion - Re = 35



# Channel with a Sudden Expansion - Re = 40



# Channel with a Sudden Expansion - Re = 45



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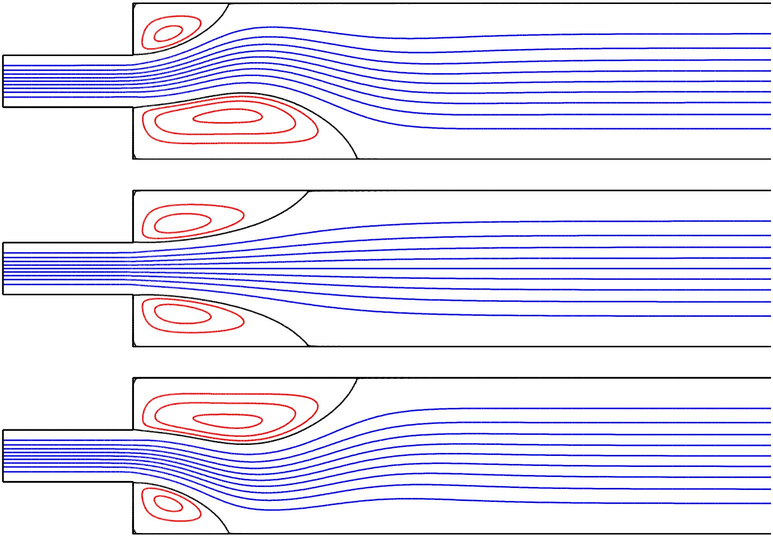

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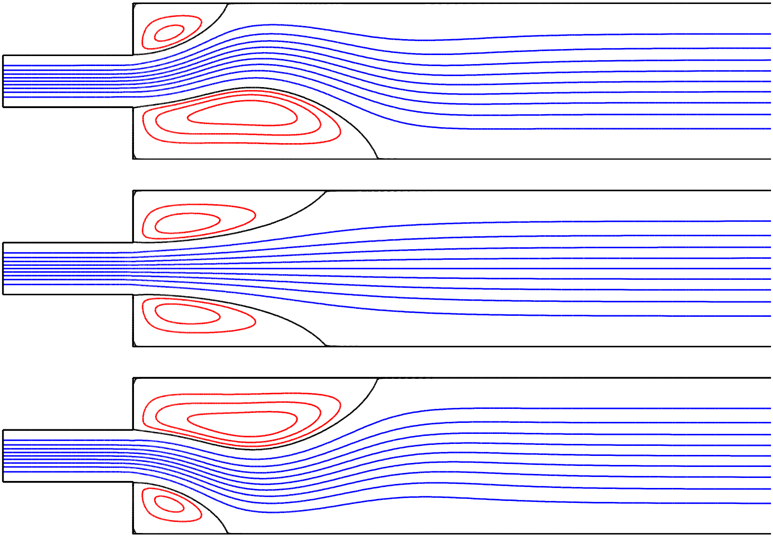

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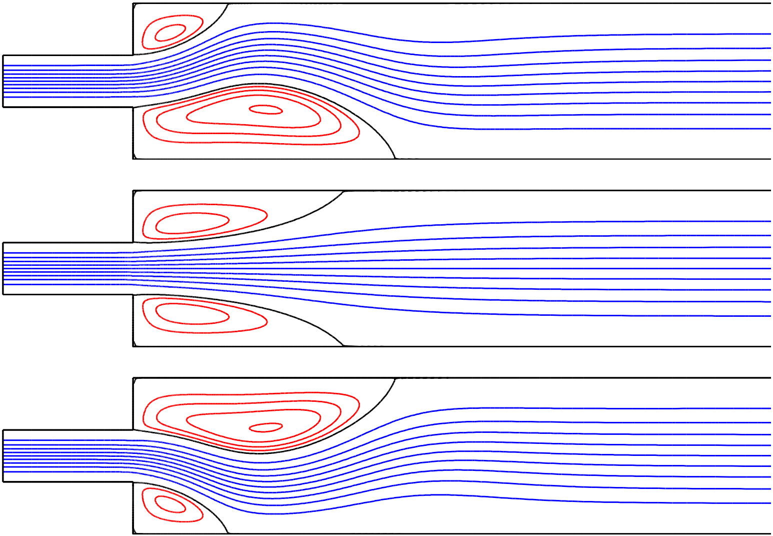

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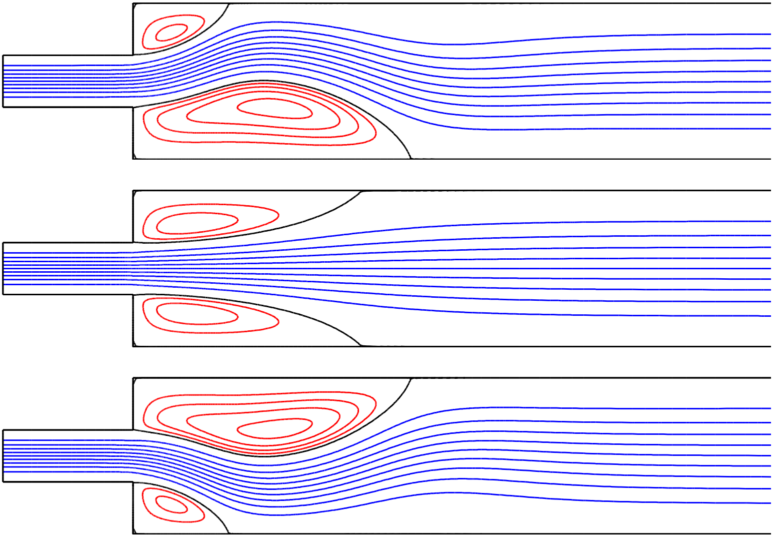

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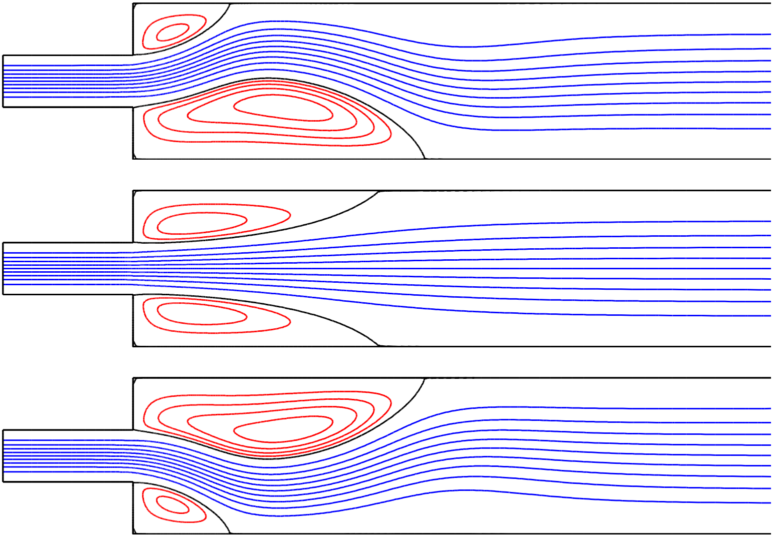

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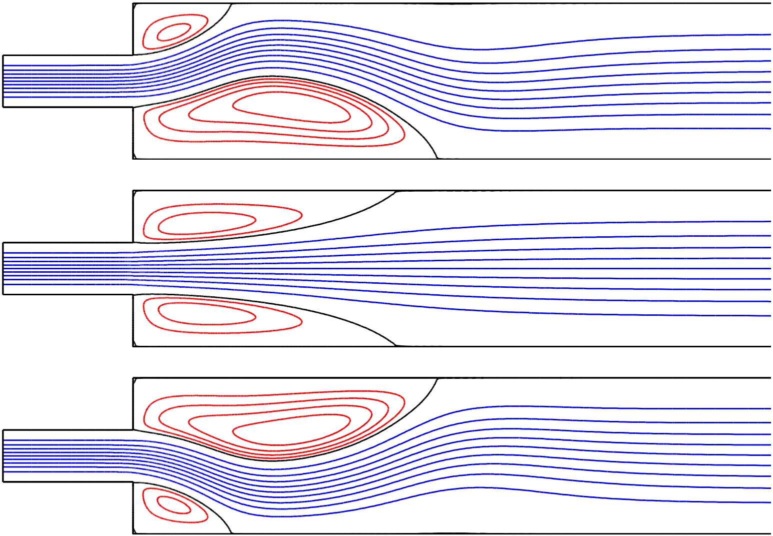
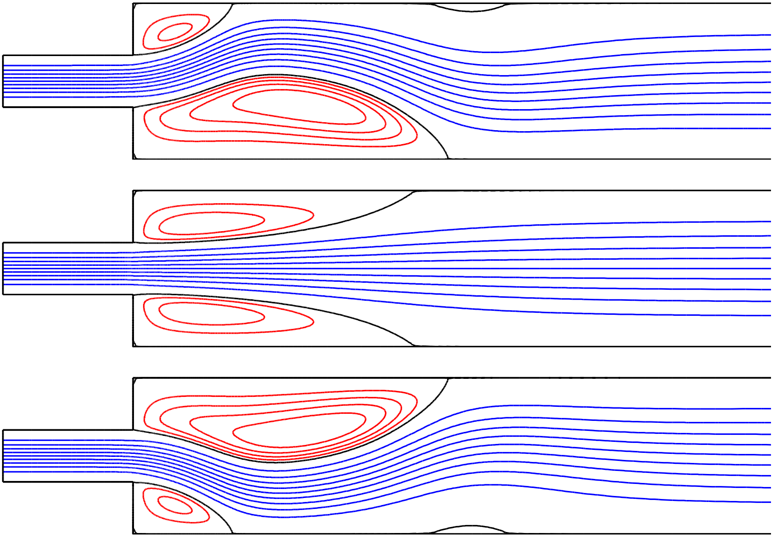


# Channel with a Sudden Expansion - Re = 80



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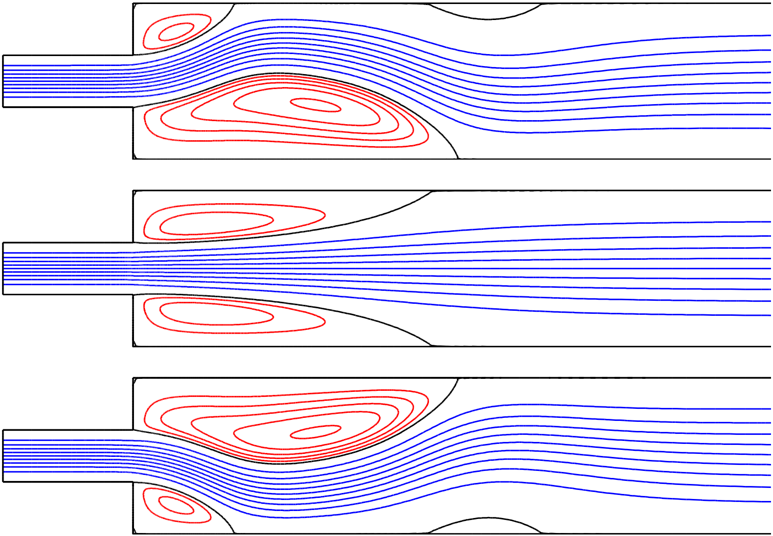
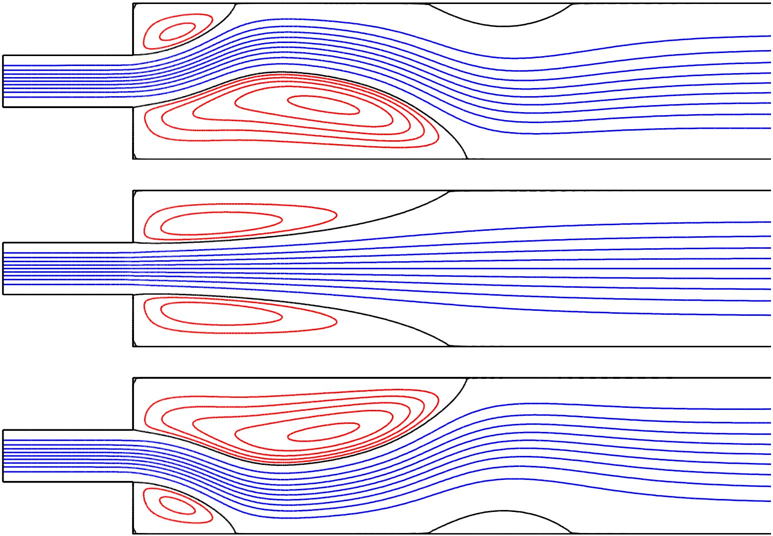
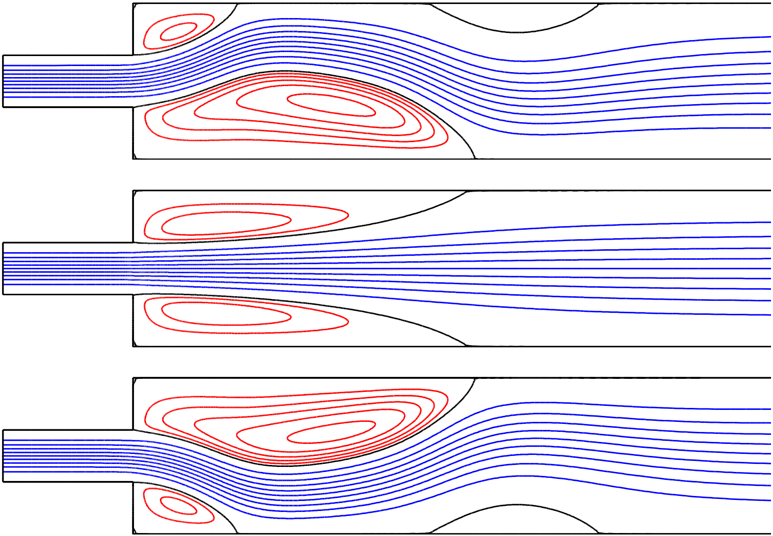


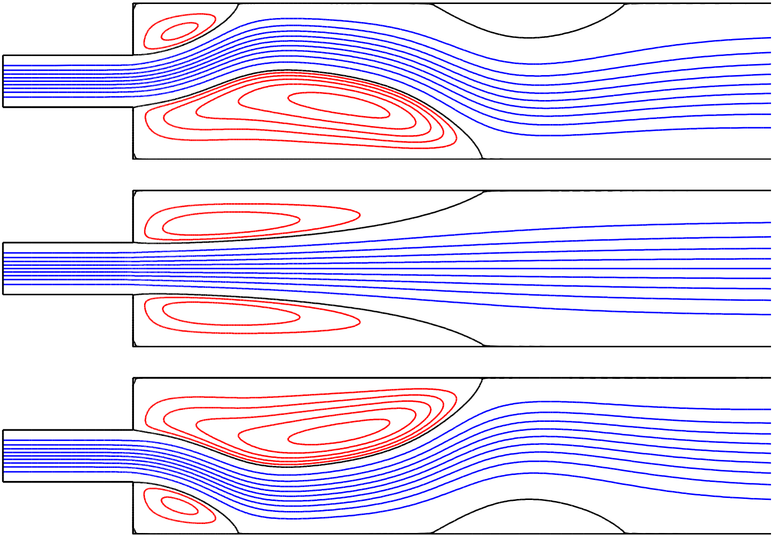
# Channel with a Sudden Expansion - Re = 90



# Channel with a Sudden Expansion - Re = 95



# Channel with a Sudden Expansion - Re = 100



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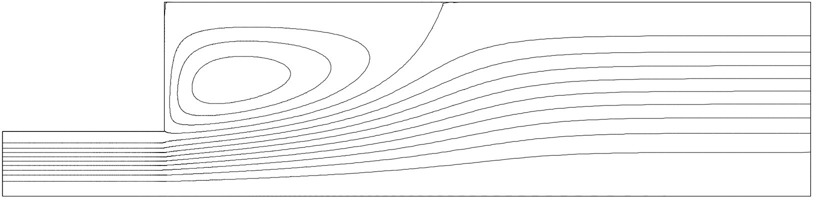

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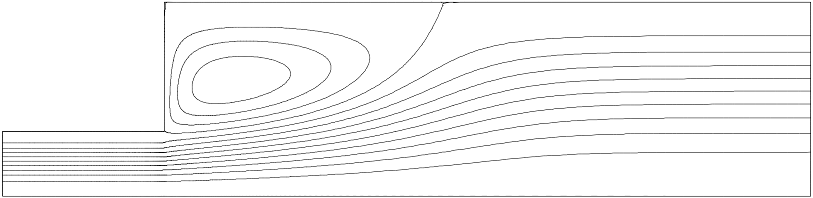

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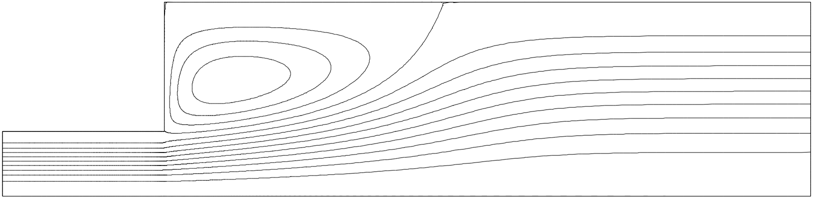

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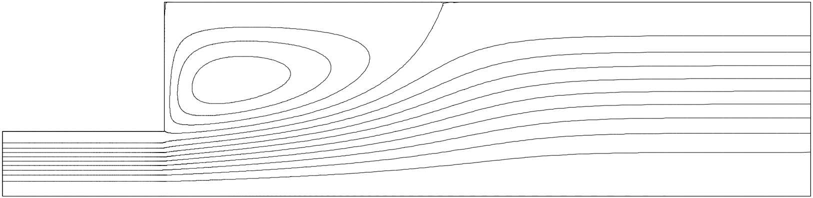

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- The mapping, $\rho$, that takes $\gamma$ to $\rho_\gamma$ is called a representation of $O(2)$ on $\mathbb{H}$.

# Bifurcation in the Presence of $O(2)$ Symmetry

- $O(2)$ equivariance

$$\rho_\gamma M = M\rho_\gamma, \quad \forall\gamma \in O(2)$$
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so that if $y \in \mathbb{H}^{O(2)}$ and $A = f_y(y_{O(2)},R)$ then

$$\rho_\gamma A = A\rho_\gamma.$$

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- Standard decomposition

$$\mathbb{H} = \sum_{m=0}^{\infty} \oplus \mathbb{V}_m, \quad \mathbb{V}_m \perp \mathbb{V}_l, \quad m \neq l.$$

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  decouples into the infinite set of simpler eigenvalue problems

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- Navier-Stokes in cylindrical coordinates

$$\mathbb{H} = W^{1,2}(\Omega)^3 \times L^2(\Omega),$$

$$y = \begin{pmatrix} u_r(r,\theta,z) \\ u_\theta(r,\theta,z) \\ u_z(r,\theta,z) \\ p(r,\theta,z) \end{pmatrix}.$$

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- Can study stability to three dimensional disturbances using a sequence of two dimensional problems.

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- Solve eigenvalue problem using modified Cayley transform and ARPACK (cf Alastair Spence's lectures).

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- Iterative solves of bordered systems.

# Solution of bordered systems

- Need to solve

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^T & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix},$$

  where

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- Rewrite as

$$\begin{pmatrix} \mathbf{D} & \mathbf{C}^T \\ \mathbf{B} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix}.$$

# Solution of bordered systems

- Use Householder QR algorithm

$$\begin{pmatrix} \mathbf{D}^{\mathsf{T}} \\ \mathbf{C} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix},$$

where

$$\mathbf{Q} = \mathbf{I} + \mathbf{U}\mathbf{T}\mathbf{U}^{\mathsf{T}},$$

and

$$\begin{aligned} \mathbf{Q}, \mathbf{I} &\in \mathbb{R}^{(n+m)\times(n+m)}, \quad \mathbf{Q} - \text{orthogonal}, \quad \mathbf{I} - \text{identity} \\ \mathbf{R}, \mathbf{T} &\in \mathbb{R}^{m\times m}, \quad \text{upper triangular} \\ \mathbf{0} &\in \mathbb{R}^{n\times m}, \\ \mathbf{U} &\in \mathbb{R}^{(n+m)\times m}. \end{aligned}$$

# Solution of bordered systems

- It follows that

$$\begin{pmatrix} \mathbf{D} & \mathbf{C}^{\mathsf{T}} \\ \mathbf{B} & \mathbf{A} \end{pmatrix} \mathbf{Q} = \begin{pmatrix} \mathbf{R}^{\mathsf{T}} & \mathbf{0} \\ \hat{\mathbf{B}} & \hat{\mathbf{A}} \end{pmatrix},$$

where

$$\hat{\mathbf{B}} = \mathbf{B} + (\mathbf{B}\mathbf{U}_1 + \mathbf{A}\mathbf{U}_2)\,\mathbf{T}\mathbf{U}_1^{\mathsf{T}},$$
$$\hat{\mathbf{A}} = \mathbf{A} + (\mathbf{B}\mathbf{U}_1 + \mathbf{A}\mathbf{U}_2)\,\mathbf{T}\mathbf{U}_2^{\mathsf{T}},$$

and

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- Note: $\hat{\mathbf{A}}$ is a rank $m$ modification of $\mathbf{A}$ that is non-singular.

# Solution of bordered systems

- Solve the block triangular system

$$\begin{pmatrix} \mathbf{R}^\top & \mathbf{0} \\ \hat{\mathbf{B}} & \hat{\mathbf{A}} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix}.$$

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$$\begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \mathbf{v} \\ \mathbf{u} \end{pmatrix}.$$

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- Large scale, distributed memory parallel implementation (LOCA, Trilinos).

# Sudden Expansion in a Channel: Error Effectivities

- $r : R = 3 : 1$
- $Re = 35$
- Eigenvalue = 0.00613553131999

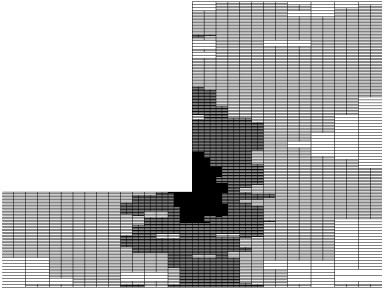
| Mesh No | No. Eles | Eig. Dof | Error | $\frac{\left\|\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa\right\|}{\text{Error}}$ | $\frac{\left\|\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa^m\right\|}{\text{Error}}$ |
|---|---|---|---|---|---|
| 1 | 760 | 16720 | 6.027E-05 | 1.92 | 0.14 |
| 2 | 1387 | 30514 | 1.540E-05 | 2.47 | 0.96 |
| 3 | 2479 | 54538 | 9.795E-06 | 1.98 | 1.16 |
| 4 | 4387 | 96514 | 6.327E-06 | 1.58 | 0.98 |
| 5 | 7645 | 168190 | 3.845E-06 | 1.33 | 0.80 |
| 6 | 13243 | 291346 | 2.231E-06 | 1.16 | 0.67 |
| 7 | 22585 | 496870 | 1.281E-06 | 1.00 | 0.56 |

# Sudden Expansion in a Channel: Mesh under Refinement

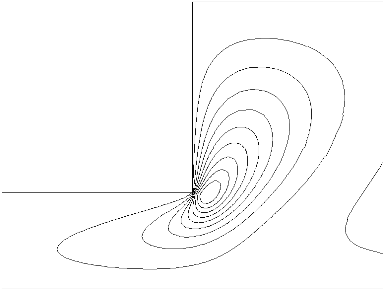
Mesh after 5 refinement steps

Contour plot of $\mathbf{z}_x^m$

# Sudden Expansion in a Channel: Mesh Detail under Refinement

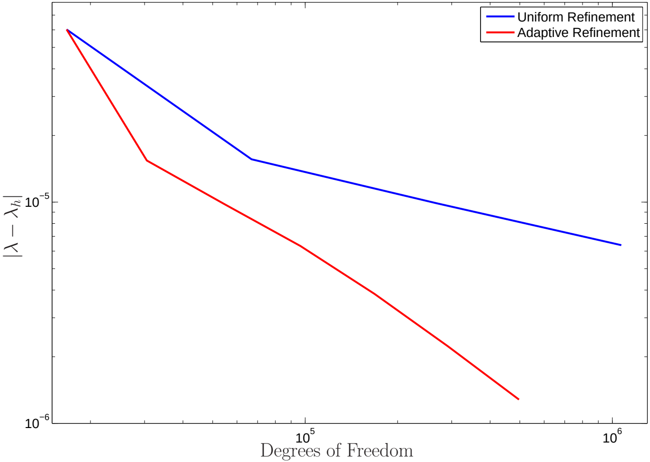


Mesh detail near expansion



Contour plot of $\mathbf{z}_y^0$ near expansion

# Sudden Expansion in a Channel: Error Convergence



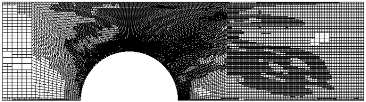
# Cylindrical Blockage in a Channel: Error Effectivities

- $r : R = 1 : 2$
- $Re = 100$
- Eigenvalue $= 0.114789963956350 + 2.116719676204527i$

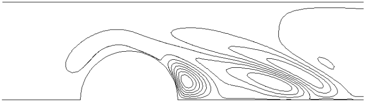
| Mesh No | No. Eles | Eig. Dof | Error | $\frac{\left\|\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa\right\|}{\text{Error}}$ | $\frac{\left\|\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa^m\right\|}{\text{Error}}$ |
|---|---|---|---|---|---|
| 1 | 816 | 17952 | 8.966E-02 | 1.08 | 4.51E-02 |
| 2 | 1443 | 31746 | 2.229E-03 | 1.54 | 0.55 |
| 3 | 2577 | 56694 | 1.455E-04 | 1.31 | 0.68 |
| 4 | 4590 | 100980 | 4.089E-05 | 0.980 | 0.53 |
| 5 | 8190 | 180180 | 1.033E-05 | 1.01 | 0.81 |
| 6 | 14400 | 316800 | 3.870E-06 | 0.946 | 0.51 |
| 7 | 24843 | 546546 | 1.060E-06 | 1.00 | 0.97 |

# Cylindrical Blockage in a Channel: Mesh under Refinement

Full Mesh

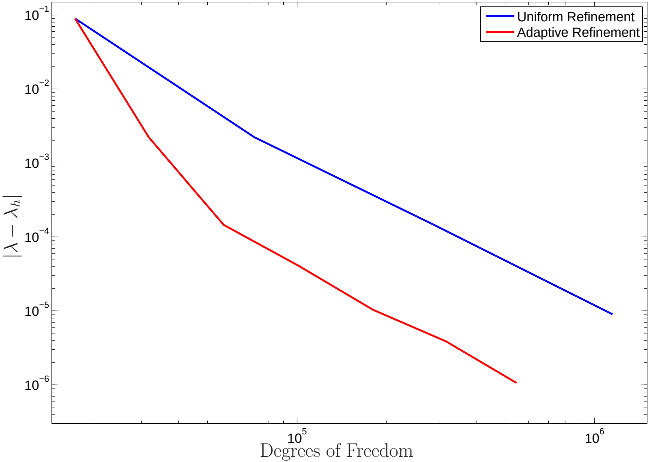


Mesh Detail near Blockage



Contour plot of $\mathbf{z}_y^0$ near blockage

# Cylindrical Blockage in a Channel: Error Convergence

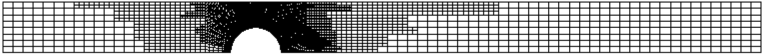


# Spherical Blockage in a Pipe: Error Effectivities

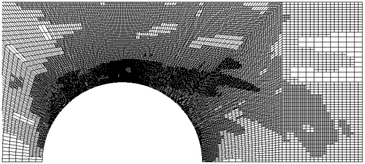
- $r : R = 1 : 2$
- $Re = 350$
- Eigenvalue $= 0.015358133759879$

| Mesh No | No. Eles | Eig. Dof | Error | $\frac{\lvert\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa\rvert}{\text{Error}}$ | $\frac{\lvert\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa^m\rvert}{\text{Error}}$ |
|---|---|---|---|---|---|
| 1 | 1016 | 31496 | 1.384E-01 | 1.68 | 1.14E-02 |
| 2 | 1793 | 55583 | 2.552E-03 | 2.01 | 2.70 |
| 3 | 3158 | 97898 | 4.877E-04 | 0.94 | 0.67 |
| 4 | 5624 | 174344 | 2.467E-05 | 1.02 | 1.14 |
| 5 | 10301 | 319331 | 2.111E-06 | 1.08 | 2.69 |

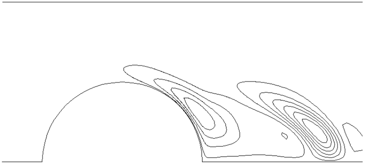
# Spherical Blockage in a Pipe: Mesh under Refinement



Full Mesh

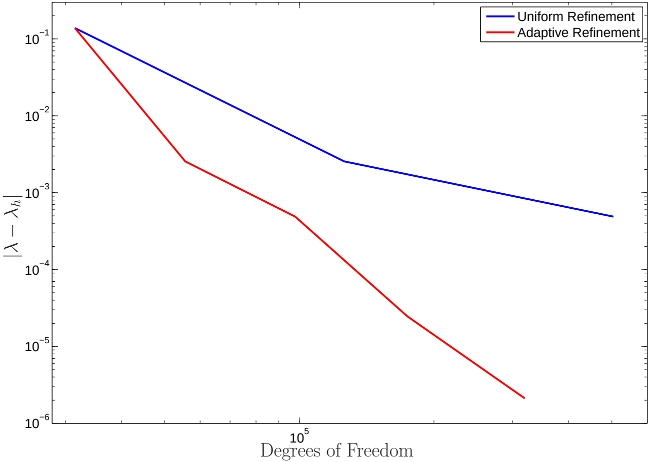


Mesh Detail near Blockage



Contour plot of $\mathbf{z}_r^0$ near blockage

# Spherical Blockage in a Pipe: Error Convergence



# Pipe with a Sudden Expansion: Experimental Results

- Tom Mullin and James Seddon, University of Manchester.

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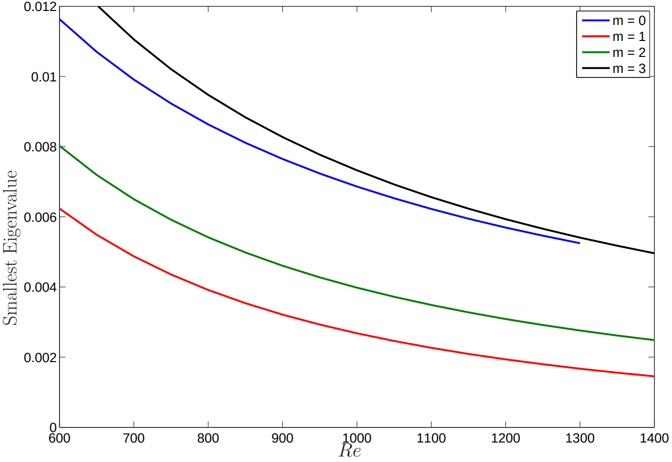
- New MRI flow visualisation techniques - MRRC in Cambridge.
- Preliminary experiments indicate presence of steady bifurcation at $Re \approx 1100$.
- Onset of time dependence at $Re \approx 1500$.

# Pipe with a Sudden Expansion: Eigenvalues with *Re*

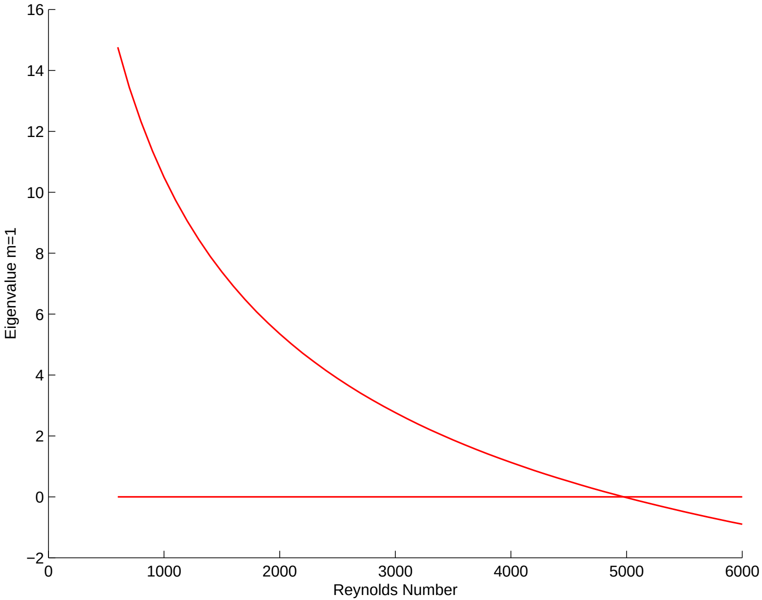


# Pipe with a Sudden Expansion: Eigenvalue Errors

- $Re = 1300$

| Mesh No. | No. Eles | Eig. Dofs | Eigenvalue | $\lvert \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa \rvert$ |
|---|---|---|---|---|
| 1 | 20000 | 420000 | 0.167241E-02 | 1.741E-06 |
| 2 | 34565 | 725865 | 0.167194E-02 | 1.914E-06 |
| 3 | 65909 | 1384089 | 0.167218E-02 | 9.771E-07 |
| 4 | 111956 | 2351076 | 0.167243E-02 | 5.765E-07 |

# Pipe with a Sudden Expansion: Eigenvalues with *Re*



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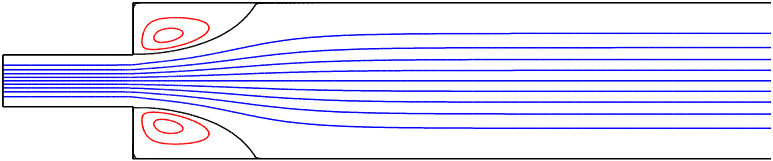
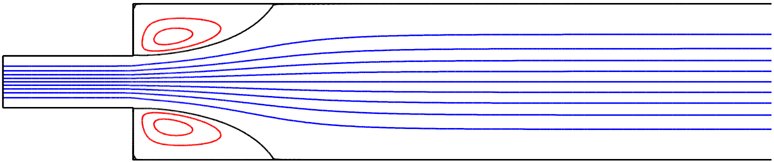
- Investigate effect of perturbations that destroy the $O(2)$ symmetry.
- Conclusion:
- In fluid mechanics we still need theory, computation and experiment!

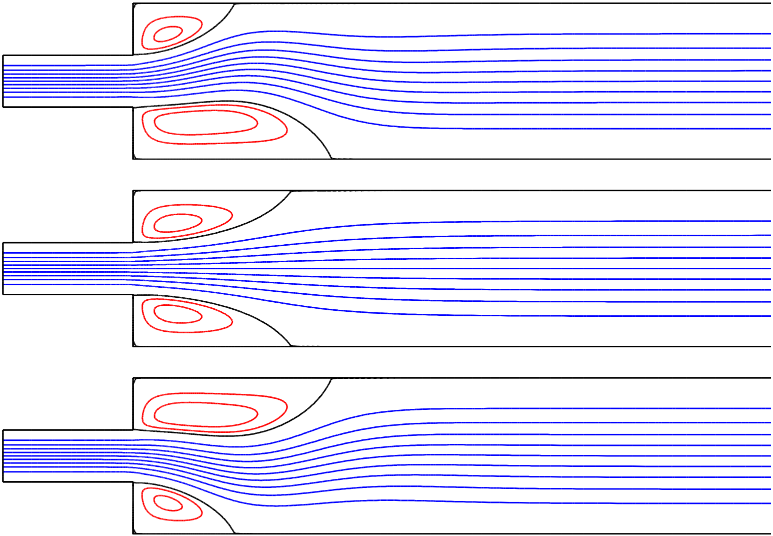
# Channel with a Sudden Expansion - Re = 35



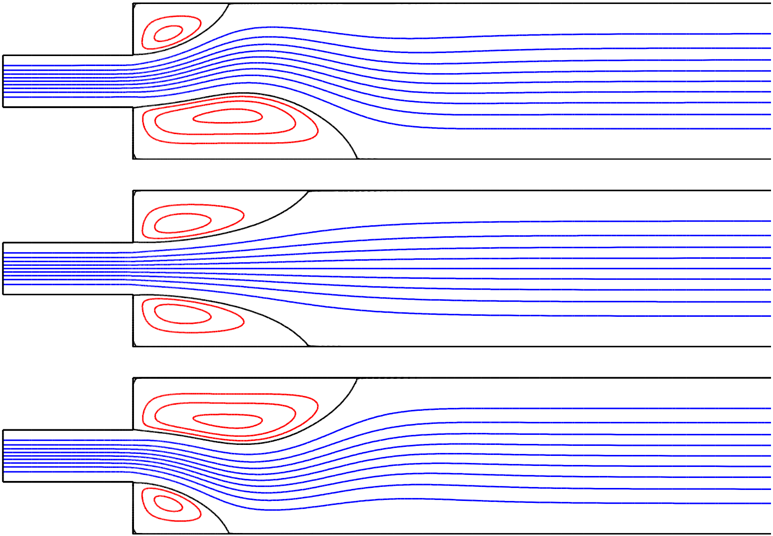
# Channel with a Sudden Expansion - Re = 40



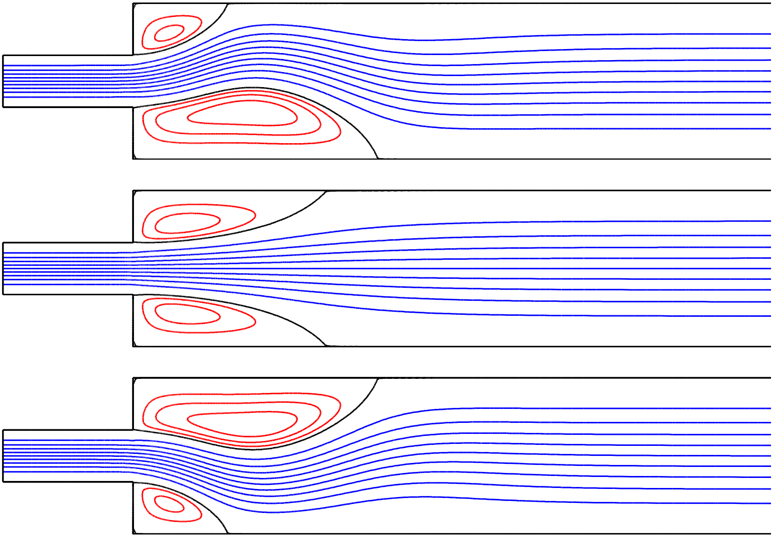
# Channel with a Sudden Expansion - Re = 45



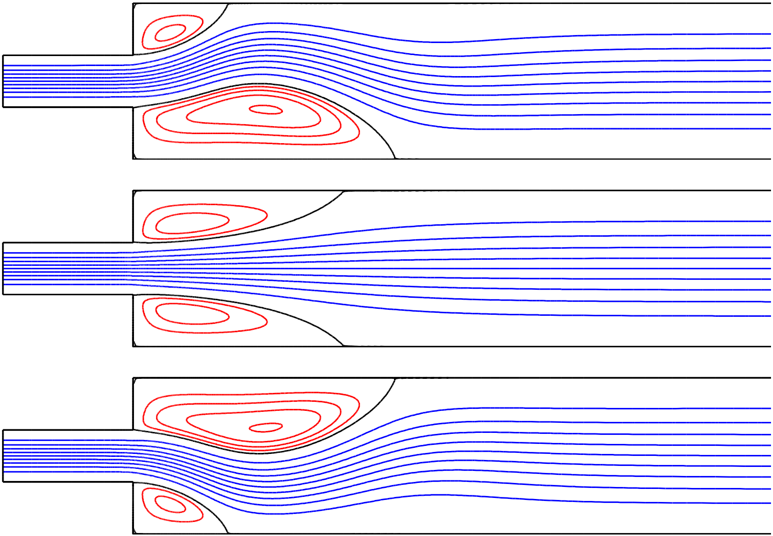
# Channel with a Sudden Expansion - Re = 50



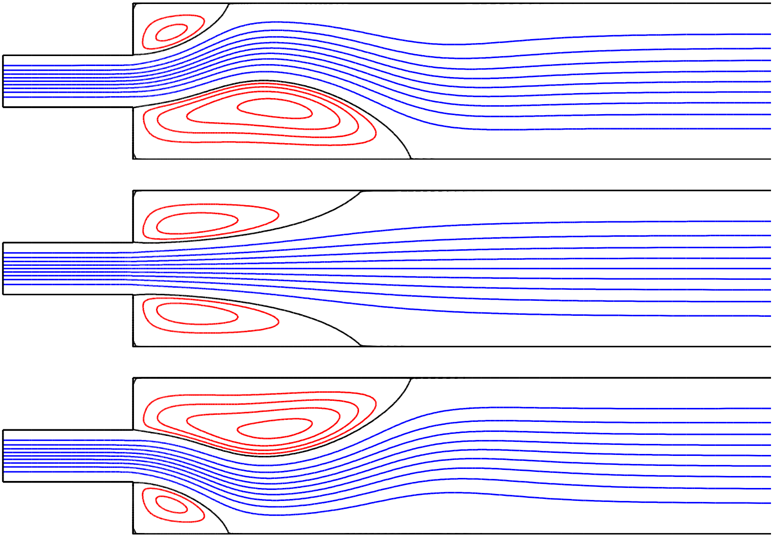
# Channel with a Sudden Expansion - Re = 55



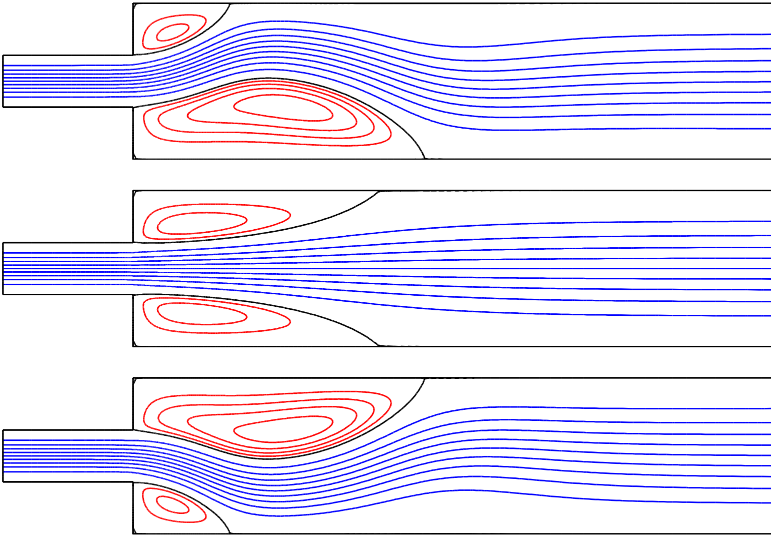
# Channel with a Sudden Expansion - Re = 60



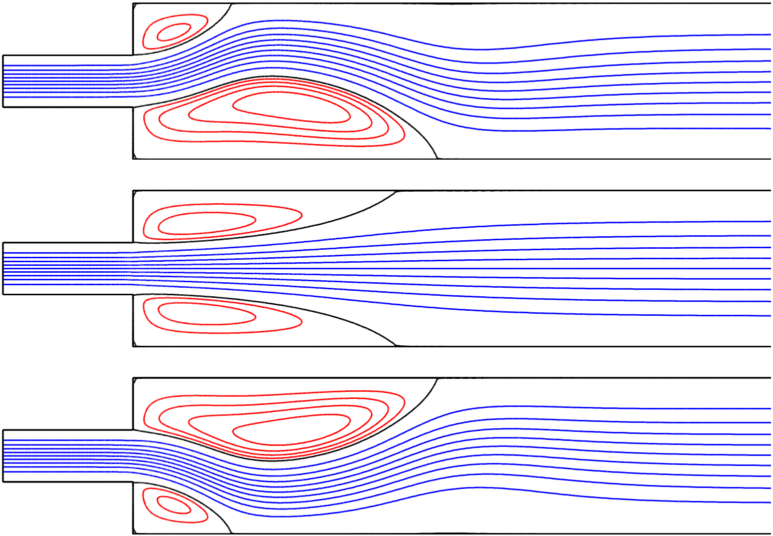
# Channel with a Sudden Expansion - Re = 65



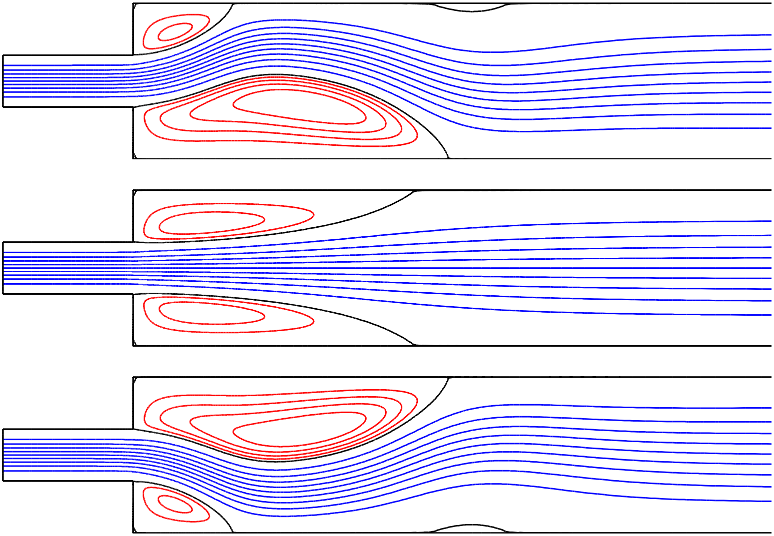
# Channel with a Sudden Expansion - Re = 70



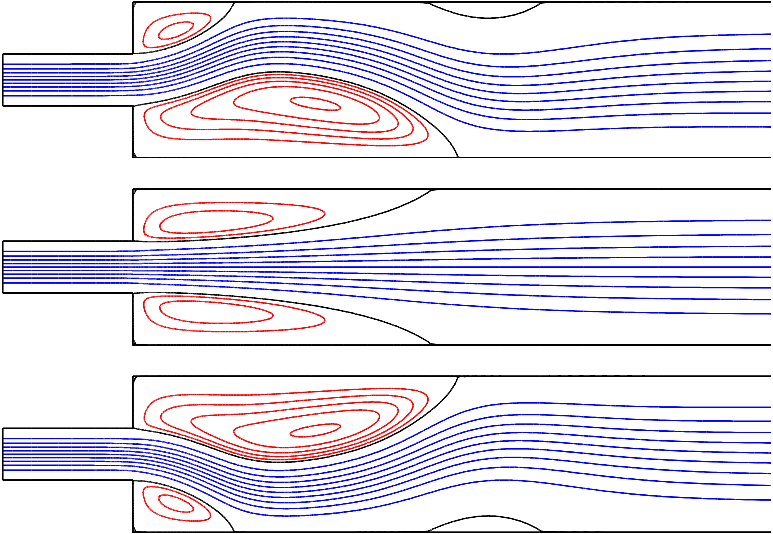
# Channel with a Sudden Expansion - Re = 75



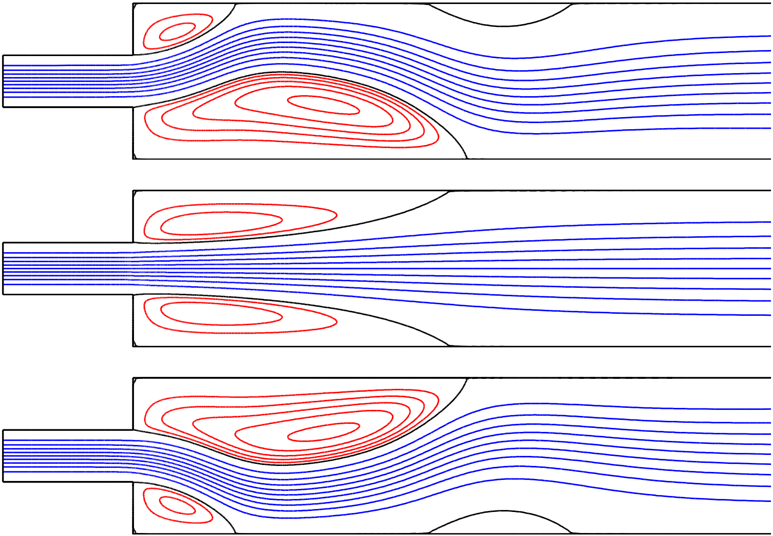
# Channel with a Sudden Expansion - Re = 80



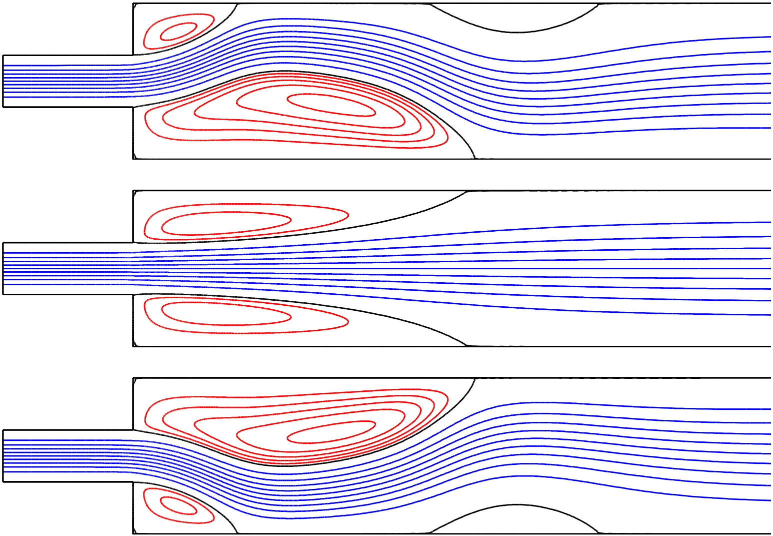
# Channel with a Sudden Expansion - Re = 85



# Channel with a Sudden Expansion - Re = 90



# Channel with a Sudden Expansion - Re = 95



# Channel with a Sudden Expansion - Re = 100