Effects of Boundary Conditions on Preconditioning Strategies for the Navier-Stokes Equations

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Navier-Stokes Equations

$$u_t - \nu \nabla^2 u + (u \cdot \text{grad}) u + \text{grad}\, p = f$$
$$- \text{div}\, u = 0$$

α=0 ⟶ steady state problem

α=1 ⟶ evolutionary problem

Discretization and linearization ⟶ Matrix equation

$$\begin{pmatrix} F & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \delta u \\ \delta p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \qquad \mathcal{A}x = b$$

In this study: *C=0*

Use preconditioner of form $\mathcal{Q} = \begin{pmatrix} Q_F & B^T \\ 0 & -Q_S \end{pmatrix}$

Solve right-preconditioned system

$$[\mathcal{A}\mathcal{Q}^{-1}][\hat{x}] = b, \quad x = \mathcal{Q}^{-1}\hat{x}$$

using Krylov subspace method (GMRES)

Key Component of Preconditioner

System: $\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} \delta u \\ \delta p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$ Preconditioner: $\begin{pmatrix} Q_F & B^T \\ 0 & -Q_S \end{pmatrix}$

$\mathcal{F} = -\nu\Delta + (\vec{w}\cdot grad)$ on velocity space

$F =$ discrete approximation of $\mathcal{F}$

$\mathcal{F}_p = -\nu\Delta + (\vec{w}\cdot grad)$ on pressure space

$F_p =$ discrete approximation of $\mathcal{F}_p$

B = discrete (–*div*), B^T = discrete *grad*

Key: approximation to Schur complement $Q_S \approx S = BF^{-1}B^T$

Derived using the commutator $\mathcal{F}\ grad - grad\ \mathcal{F}_p \approx 0$

Discrete form in finite element setting

$$M_v^{-1}FM_v^{-1}B^T - M_v^{-1}B^T M_p^{-1}F_p \approx 0$$

Two Preconditioning Strategies

E.,Kay, Loghin, Silvester, Wathen,Tuminaro, Howle, Shadid, Shuttleworth

1. Pressure Convection-Diffusion (PCD)

$$M_v^{-1} F M_v^{-1} B^T - M_v^{-1} B^T M_p^{-1} F_p \approx 0$$

$$\rightarrow B F^{-1} B^T \approx (B M_v^{-1} B^T) F_p^{-1} M_p = A_p F_p^{-1} M_p$$

$$(B F^{-1} B^T)^{-1} \approx M_p^{-1} F_p A_p^{-1}$$

2. Least-Squares Commutator (LSC)

Choose F_p to minimize

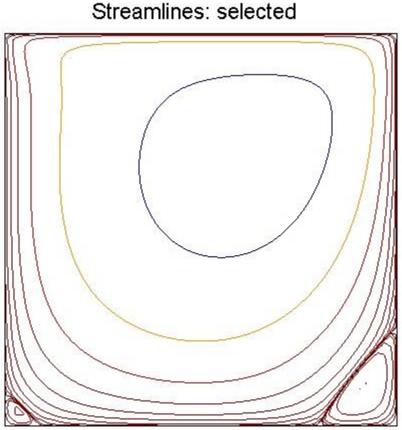
$$\| [M_v^{-1} F M_v^{-1} B^T]_j - M_v^{-1} B^T M_p^{-1} [F_p]_j \|_{M_v}$$

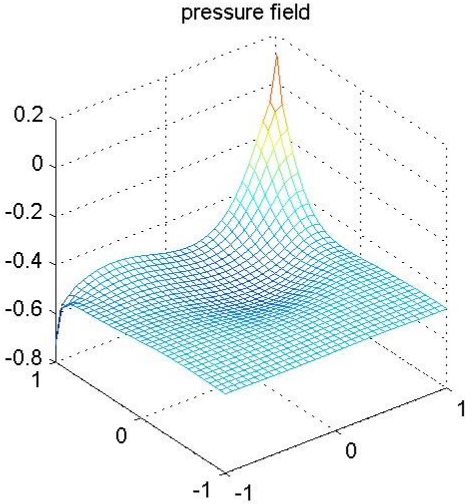
column by column

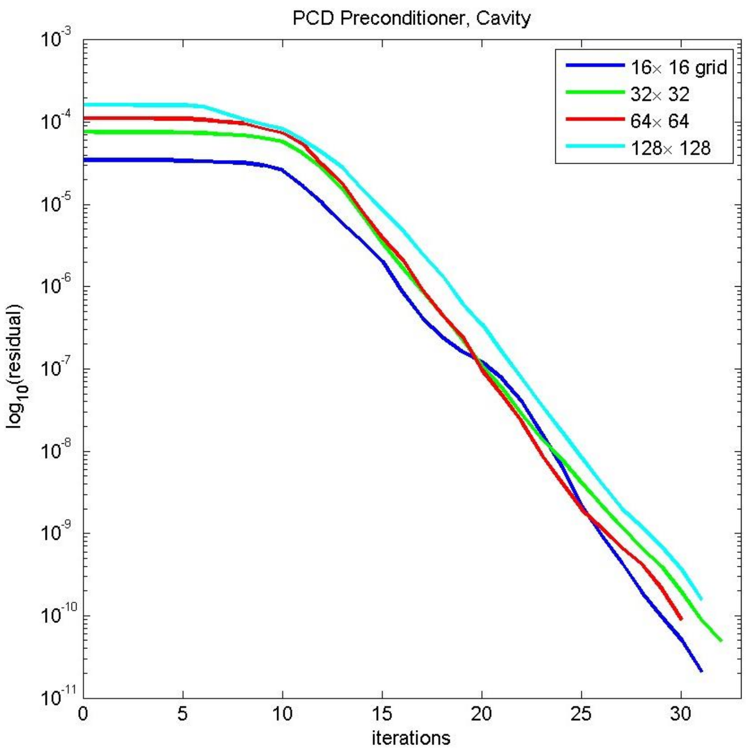
$$\rightarrow Q_S^{-1} \equiv (B M_v^{-1} B^T)^{-1} (B M_v^{-1} F M_v^{-1} B^T)(B M_v^{-1} B^T)^{-1}$$

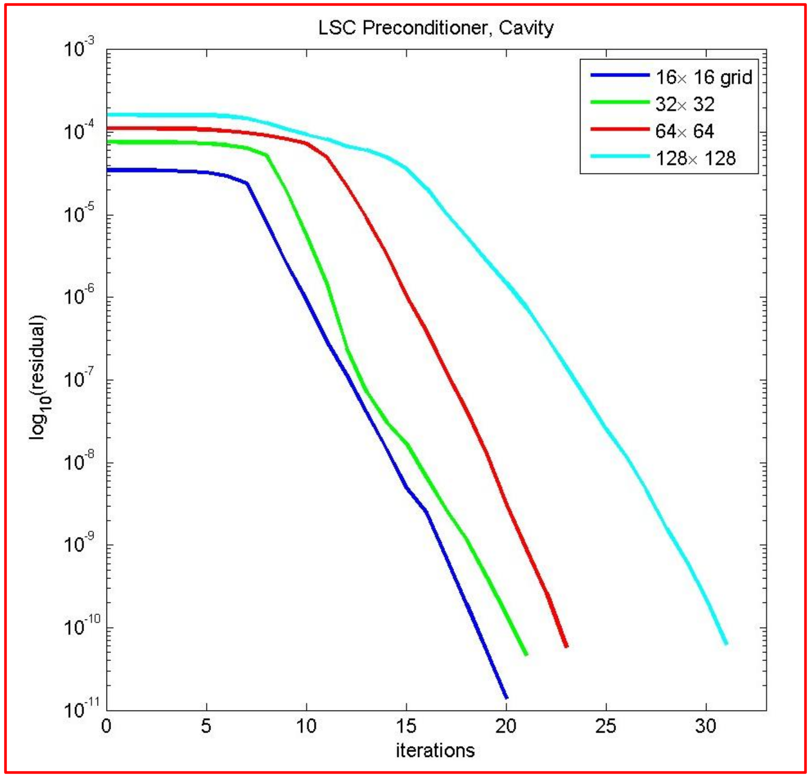
Representative Performance I

Driven cavity flow
Re=200



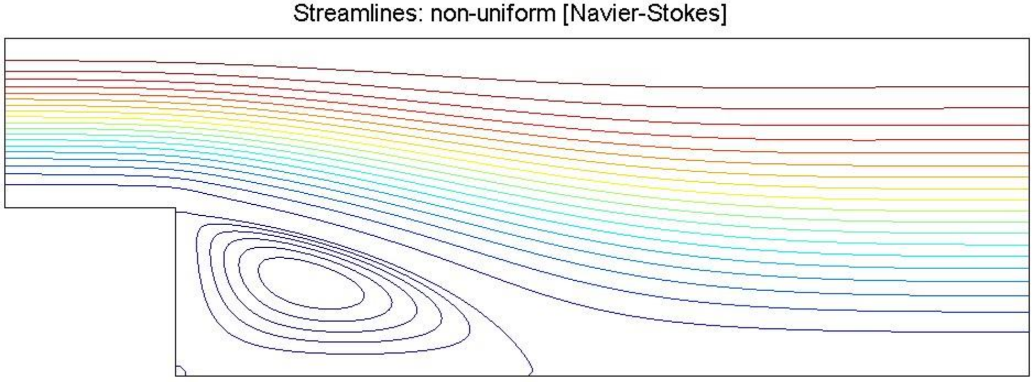


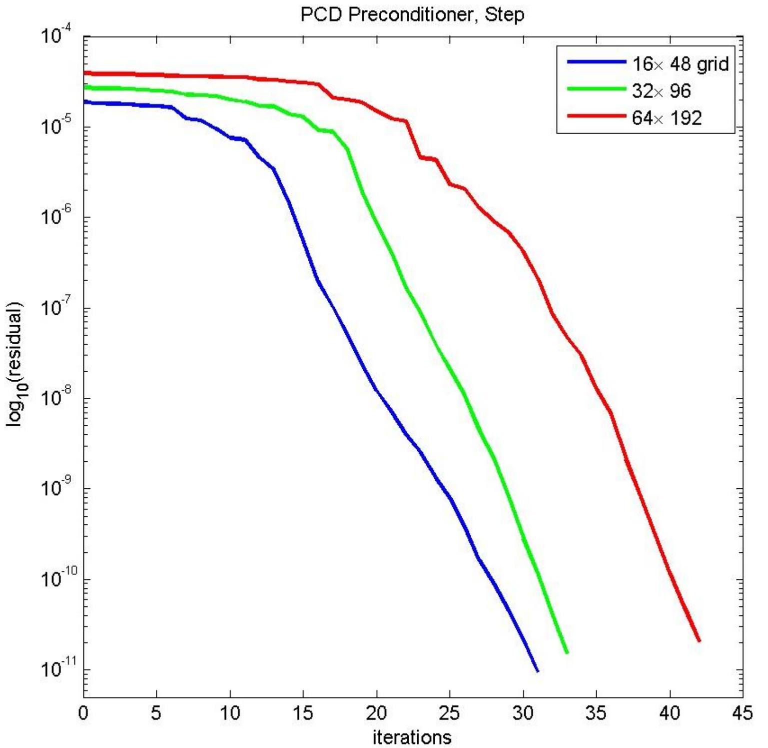


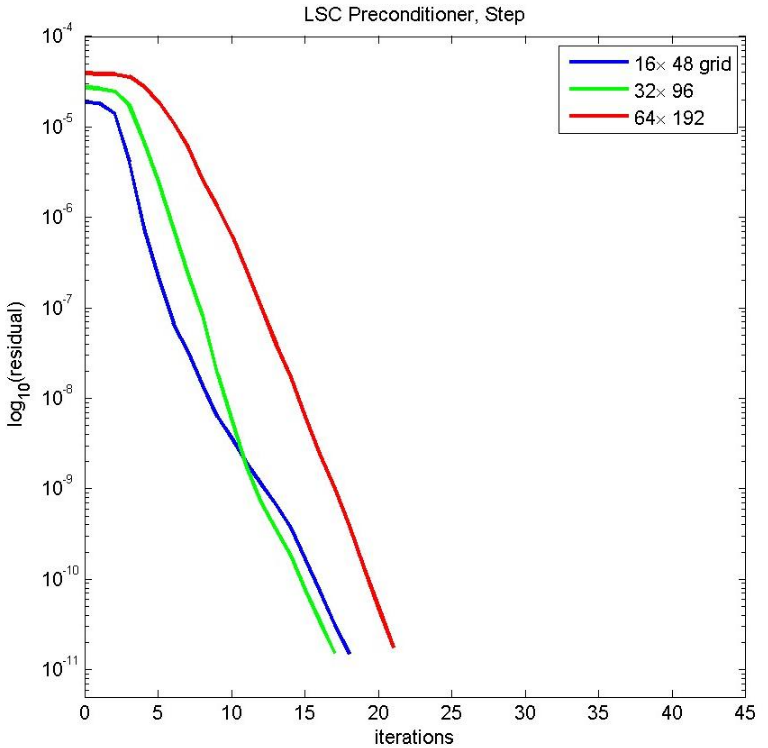


Representative Performance II

Backward facing step
Re=100







Issues

For PCD on step: Latency before asymptotic convergence rate is evident, on step

For LSC: Mesh-dependent convergence rate
Superior performance on step

Possible explanations:

Boundary conditions for F_p and A_p

Currently, for F_p in PCD:

Neumann conditions for cavity

Dirichlet at step inflow, Neumann otherwise

A_p matched with F_p

In LSC: boundary conditions are implicitly defined by

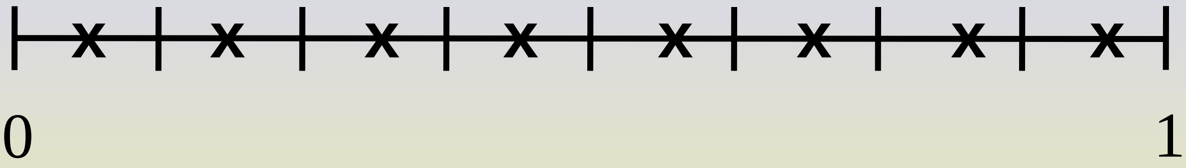
$$A_p = BM_v^{-1}B^T$$

Empirical observation: Neumann at step inflow

Operators in One Dimension

$$\text{Let } \mathcal{F} = \mathcal{F}_p = -\nu \frac{d^2}{dx^2} + w \frac{d}{dx}, \quad \mathcal{B} = \frac{d}{dx}, \quad w > 0$$

on Ω =



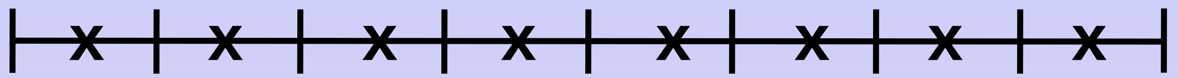
Discrete forms:

F, B defined on interval endpoints ~ velocities

F_p, B^T defined on interval centers ~ pressures

~ Marker-and-cell finite differences

Operators in One Dimension



Composite operators $\mathcal{B}\mathcal{F} = \mathcal{F}_p\mathcal{B} = -\nu\frac{d^3}{dx^3} + w\frac{d^2}{dx^2}$ **New order**

Assume: Dirichlet condition $u=0$ for $\mathcal{F}$ at inflow $x=0$

Dirichlet condition $u=0$ *for* $\mathcal{B}$ *at inflow* $x=0$

Look at
$$\mathrm{v} = \mathcal{F}u = \left(-\nu\frac{d}{dx} + w\right)\frac{du}{dx} = \left(-\nu\frac{d}{dx} + w\right)p,$$
$$p = \mathcal{B}u = \frac{du}{dx}$$

At inflow boundary $x=0$:

$v = 0$ for argument v of $\mathcal{B}$ in $\mathcal{B}\mathcal{F}$

$\equiv \left(-\nu\frac{d}{dx} + w\right)p = 0$ for argument p of $\mathcal{F}_p$ in $\mathcal{F}_p\mathcal{B}$

New, **Robin,** boundary condition at inflow

Matrix Analogue

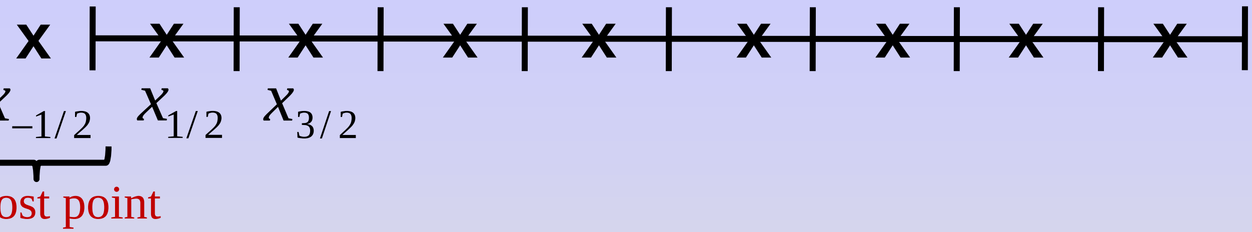
$$B = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 2\nu & -(\nu - wh/2) & & & \\ -(\nu + wh/2) & 2\nu & -(\nu - wh/2) & & \\ & -(\nu + wh/2) & \ddots & & \\ & & -(\nu + wh/2) & 2\nu & -(\nu - wh/2) \\ & & & -\nu & \nu \end{bmatrix}$$

$$F_p = \begin{bmatrix} \xi_1 & -(\nu - wh/2) & & & \\ -(\nu + wh/2) & 2\nu & -(\nu - wh/2) & & \\ & -(\nu + wh/2) & \ddots & & \\ & & -(\nu + wh/2) & 2\nu & -(\nu - wh/2) \\ & & & -(\nu + wh/2) & \xi_n \end{bmatrix}$$

$$[BF]_{11} = 2\nu = \xi_1 + (\nu - wh/2) = [F_p B]_{11}$$

$$\Rightarrow \xi_1 = \nu + wh/2$$

Interpret in Terms of Boundary Conditions



Discrete operator near inflow boundary:

$$[F_p p]_{1/2} = -(\nu + {}^{wh}\!/_2)\, p_{-1/2} + (2\nu)\, p_{1/2} - (\nu - {}^{wh}\!/_2)\, p_{3/2}$$

Discrete Robin boundary condition:

$$-\nu \frac{dp}{dx} + wp \approx -\nu\left(\frac{p_{1/2} - p_{-1/2}}{\mathrm{h}}\right) + \mathrm{w}\left(\frac{p_{1/2} + p_{-1/2}}{2}\right) = 0$$

Solve for ghost point: $p_{-1/2} = \dfrac{\nu - {}^{wh}\!/_2}{\nu + {}^{\mathrm{wh}}\!/_2}\, p_{1/2}$

$$\longrightarrow \quad [F_p p]_{1/2} = \underbrace{(\nu + {}^{wh}\!/_2)}\, p_{1/2} - (\nu - {}^{wh}\!/_2)\, p_{3/2}$$

Previous page: ξ_1 to make discrete commutator zero

At Outflow (Right) Boundary:

$$F_p = \begin{bmatrix} \xi_1 & -(\nu - {wh}/{2}) & & & & \\ -(\nu + {wh}/{2}) & 2\nu & -(\nu - {wh}/{2}) & & & \\ & -(\nu + {wh}/{2}) & & & & \\ & & & \ddots & & \\ & & & -(\nu + {wh}/{2}) & 2\nu & -(\nu - {wh}/{2}) \\ & & & & -(\nu + {wh}/{2}) & \xi_n \end{bmatrix}$$

Can force $BF\text{-}F_pB=0$ ⟶ $\xi_n = 2\nu - {wh}/{2}$

Result: $BF^{-1}B^T = F_p^{-1}(BB^T)$, perfect preconditioner

But: does not have interpretation in terms of outflow b.c. for $\mathcal{F}_p$

Mesh dependent condition at right:

$$-\nu p' + 2(w - {\nu}/{h}) = 0$$

⟶ Reduces coercivity of F_p

For Problems in Higher Dimensions

$$\mathcal{F} = -\nu\Delta + \vec{w}\cdot grad = \begin{bmatrix} \mathcal{F}^{(u_1)} & \\ & \mathcal{F}^{(u_2)} \end{bmatrix}$$

$\mathcal{B} = [\mathcal{B}_x, \mathcal{B}_y]$, divergence operator **New!**

Commutator: $\mathcal{E} = \mathcal{B}\mathcal{F} - \mathcal{F}_p\mathcal{B}$

$$= [\mathcal{B}_x\mathcal{F}^{(u_1)} - \mathcal{F}_p\mathcal{B}_x, \mathcal{B}_y\mathcal{F}^{(u_2)} - \mathcal{F}_p\mathcal{B}_y]$$

Further refinement: split by coordinate $\mathcal{F} = \mathcal{F}_x + \mathcal{F}_y$

$$\mathcal{F}_x = -\nu\frac{\partial^2}{\partial x^2} + w_1\frac{\partial}{\partial x}, \quad \mathcal{F}_y = -\nu\frac{\partial^2}{\partial y^2} + w_2\frac{\partial}{\partial y}$$

Similarly for $\mathcal{F}_p$

Component Splittings of Commutator

$$(1)\ \mathcal{B}_x \mathcal{F}_x^{(u_1)} - \mathcal{F}_x^{(p)} \mathcal{B}_x \qquad (2)\ \mathcal{B}_x \mathcal{F}_y^{(u_1)} - \mathcal{F}_y^{(p)} \mathcal{B}_x$$

$$(3)\ \mathcal{B}_y \mathcal{F}_x^{(u_2)} - \mathcal{F}_x^{(p)} \mathcal{B}_y \qquad (4)\ \mathcal{B}_y \mathcal{F}_y^{(u_2)} - \mathcal{F}_y^{(p)} \mathcal{B}_y$$

Commutator satisfies

$$\begin{aligned} \mathcal{E} &= [\mathcal{B}_x \mathcal{F}^{(u_1)} - \mathcal{F}_p \mathcal{B}_x, \mathcal{B}_y \mathcal{F}^{(u_2)} - \mathcal{F}_p \mathcal{B}_y] \\ &= [(1)+(2), (3)+(4)] \end{aligned}$$

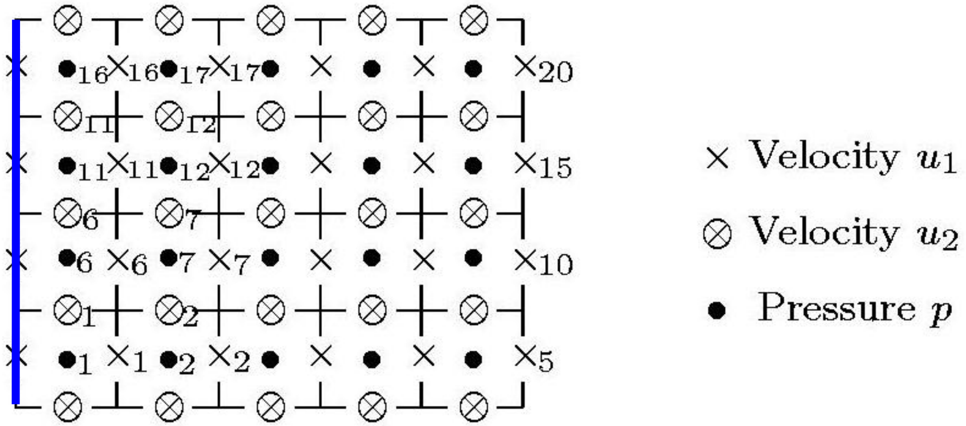
Motivation for this:

- Discrete version of commutator **cannot** be zero
  $\Longrightarrow$ no “perfect” preconditioner is possible
- Components above behave more like one-dimensional operators
- Perhaps: some are more important than others

Examine Commutator Components at Inflow

$$\mathcal{F} = -\nu\Delta + w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y}$$

$$w_1 > 0,\ \ w_2 = 0$$



Assume: Dirichlet b.c. along left (inflow)

Periodic b.c. *u(x,0)=u(x,1)* along bottom and top

$$\nu \frac{\partial u_1}{\partial x} - p = 0, \quad \frac{\partial u_2}{\partial x} = 0$$ at right (outflow)

(1) $\mathcal{B}_x \mathcal{F}_x^{(u_1)} - \mathcal{F}_x^{(p)} \mathcal{B}_x$ (2) $\mathcal{B}_x \mathcal{F}_y^{(u_1)} - \mathcal{F}_y^{(p)} \mathcal{B}_x$

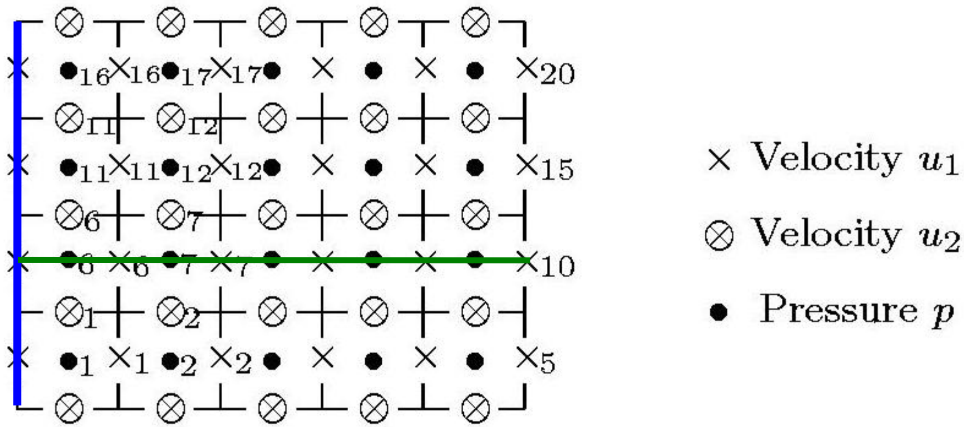
(3) $\mathcal{B}_y \mathcal{F}_x^{(u_2)} - \mathcal{F}_x^{(p)} \mathcal{B}_y$ (4) $\mathcal{B}_y \mathcal{F}_y^{(u_2)} - \mathcal{F}_y^{(p)} \mathcal{B}_y$

Commutator Components at Inflow

(1) $\mathcal{B}_x \mathcal{F}_x^{(u_1)} - \mathcal{F}_x^{(p)} \mathcal{B}_x$

$$-\nu \frac{\partial^2}{\partial x^2} + w_1 \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x}$$



For each fixed y: same as 1D commutator ⟹
want Robin condition for $\mathcal{F}_x^{(p)}$ at inflow

(2) $\mathcal{B}_x \mathcal{F}_y^{(u_1)} - \mathcal{F}_y^{(p)} \mathcal{B}_x$

$$-\nu \frac{\partial^2}{\partial y^2} + w_1 \frac{\partial}{\partial y}$$

No requirements on $\mathcal{F}_y^{(p)}$

Robin condition for $\mathcal{F}^{(p)}$ at inflow: $-\nu p_x + w_1 p = 0$

A Difficulty with This

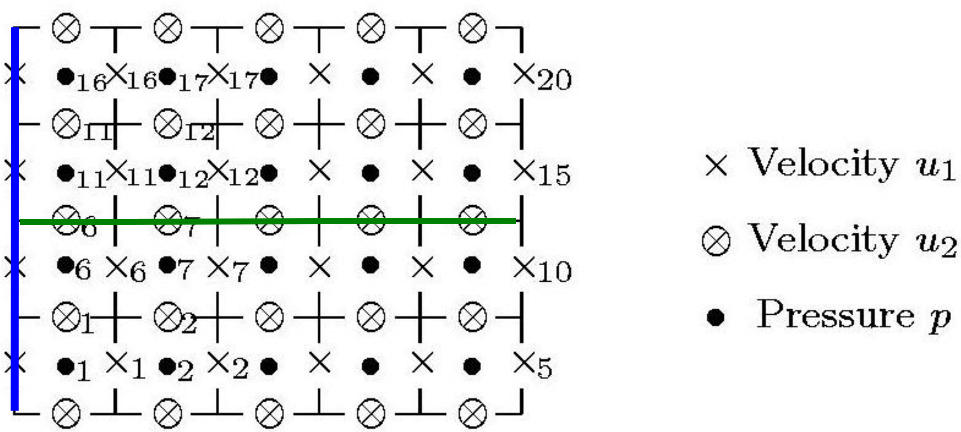
We just showed: for component (1) $\mathcal{B}_x \mathcal{F}_x^{(u_1)} - \mathcal{F}_x^{(p)} \mathcal{B}_x$

to be zero, need Robin b.c. for $\mathcal{F}^{(p)}$

Now look at (3) $\mathcal{B}_y \mathcal{F}_x^{(u_2)} - \mathcal{F}_x^{(p)} \mathcal{B}_y$

$$-\nu \frac{\partial^2}{\partial x^2} + w_1 \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial y}$$



$\mathcal{F}^{(u_2)}$ has a Dirichlet condition, implies $\mathcal{F}^{(p)}$ must also have a Dirichlet condition to make (3) equal zero

Not compatible with Robin condition imposed by (1)

Summarizing: At Inflow

$$(1)\ \mathcal{B}_x\mathcal{F}_x^{(u_1)} - \mathcal{F}_x^{(p)}\mathcal{B}_x \qquad (2)\ \mathcal{B}_x\mathcal{F}_y^{(u_1)} - \mathcal{F}_y^{(p)}\mathcal{B}_x$$

$$(3)\ \mathcal{B}_y\mathcal{F}_x^{(u_2)} - \mathcal{F}_x^{(p)}\mathcal{B}_y \qquad (4)\ \mathcal{B}_y\mathcal{F}_y^{(u_2)} - \mathcal{F}_y^{(p)}\mathcal{B}_y$$

Zero commutator components (1) and (3) are incompatible
Zero for (2) and (4): compatible with each other
and with (1) or (3)

We must *choose* either (1) or (3), i.e., choose either Robin or Dirichlet boundary conditions for $\mathcal{F}^{(p)}$

Previously: used Dirichlet conditions

Will show: Robin conditions are better

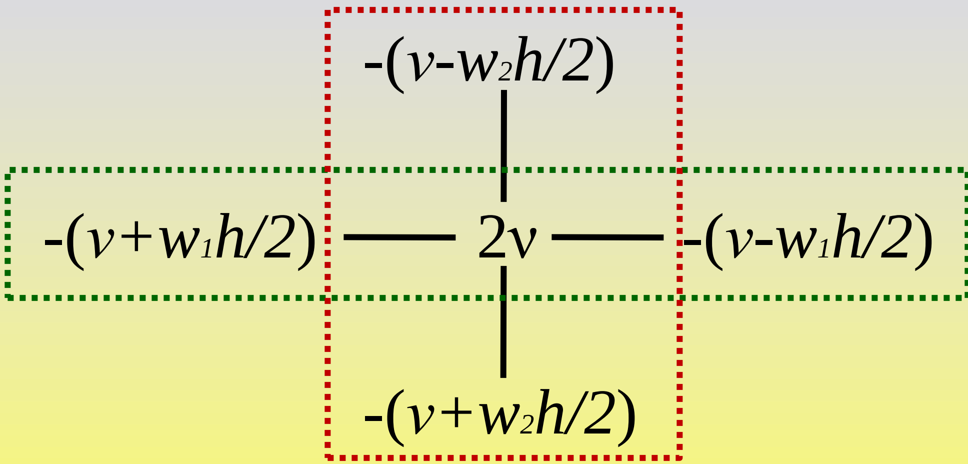
Matrix Versions of these Results (MAC discretization)

Components of discrete commutator:

$$(1)\ B_x F_x^{(u_1)} - F_x^{(p)} B_x \qquad (2)\ B_x F_y^{(u_1)} - F_y^{(p)} B_x$$

$$(3)\ B_y F_x^{(u_2)} - F_x^{(p)} B_y \qquad (4)\ B_y F_y^{(u_2)} - F_y^{(p)} B_y$$

$F\sim$



$= F_x + F_y$

$$F_x^{(u_1)} = diag(F_1, \ldots, F_1)$$

$$F_x^{(u_2)} = diag(F_2, \ldots, F_2)$$

1D tridiagonal matrices,
~ Dirichlet b.c. at left

$$F_x^{(p)} = diag(F_p, \ldots, F_p)$$

1D tridiagonal, **b.c. needed**

Matrix Versions of these Results (II)

Component (1)

$$B_x = diag(B_1, \ldots, B_1), \qquad B_1 = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & & & & \\ & & \ddots & & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \end{bmatrix}$$

$$\Rightarrow (1) \quad B_x F_x^{(u_1)} - F_x^{(p)} B_x$$
$$= diag(B_1 F_1 - F_p B_1, \ldots, B_1 F_1 - F_p B_1)$$

Each block is identical to 1D matrices:

Discrete Robin conditions on left for F_p makes component (1) equal zero

Matrix Versions of these Results (III)

Component (3)

$$B_y = \begin{bmatrix} I & & & & & I \\ -I & I & & & & \\ & -I & & & & \\ & & \ddots & & & \\ & & & -I & I & \\ & & & & -I & I \end{bmatrix}$$

$$\Rightarrow (3)\ \ B_y F_x^{(u_2)} - F_x^{(p)} B_y =$$

$$\begin{bmatrix} F_2 - F_p & & & & & -(F_2 - F_p) \\ -(F_2 - F_p) & F_2 - F_p & & & & \\ & -(F_2 - F_p) & & & & \\ & & \ddots & & & \\ & & & -(F_2 - F_p) & & \\ & & & & -(F_2 - F_p) & F_2 - F_p \end{bmatrix}$$

Discrete Dirichlet conditions on left for F_p makes component (3) equal zero

(1) and (3) are not compatible

Other Matrix Results

$$(1)\ B_x F_x^{(u_1)} - F_x^{(p)} B_x \qquad (2)\ B_x F_y^{(u_1)} - F_y^{(p)} B_x$$

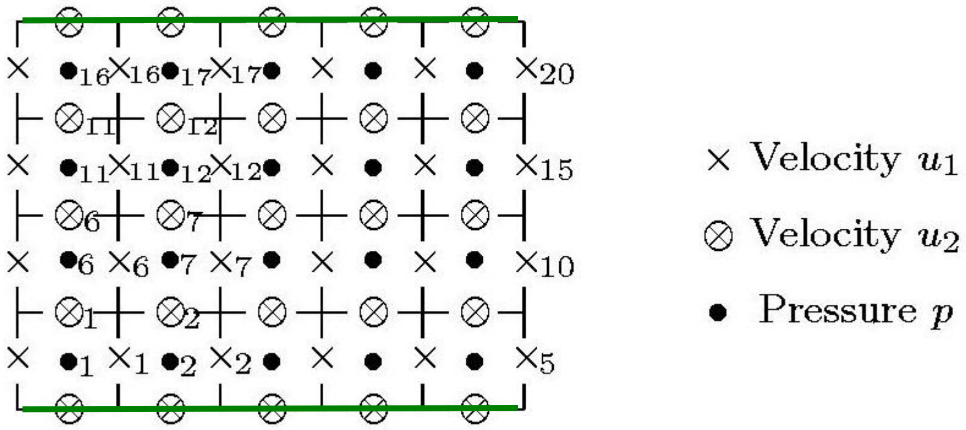
$$(3)\ B_y F_x^{(u_2)} - F_x^{(p)} B_y \qquad (4)\ B_y F_y^{(u_2)} - F_y^{(p)} B_y$$

Components (2) and (4):

- can be made zero with no requirements on $F_y^{(p)}$
- compatible with each other
- compatible with (1) and (3)

Details omitted

For Characteristic Boundaries



Suppose b.c. of form $w=(w_1,0)$ are specified at top and bottom

The analogue of Robin condition is

$$-\nu\, p_y + w_2\, p = 0 \quad \Rightarrow \quad p_y = 0$$

Pure Neumann condition, choice made previously for characteristic boundaries

General condition: $-\nu\, p_n + (w \cdot n)\, p = 0$

Cf. Achdou, LeTallec, Nataf, Vidrascu

What about outflow boundaries?

Have seen: 1D commutator $\equiv 0$, but coercivity of F_p is reduced

Adopt strategy: Neumann conditions for F_p at outflow

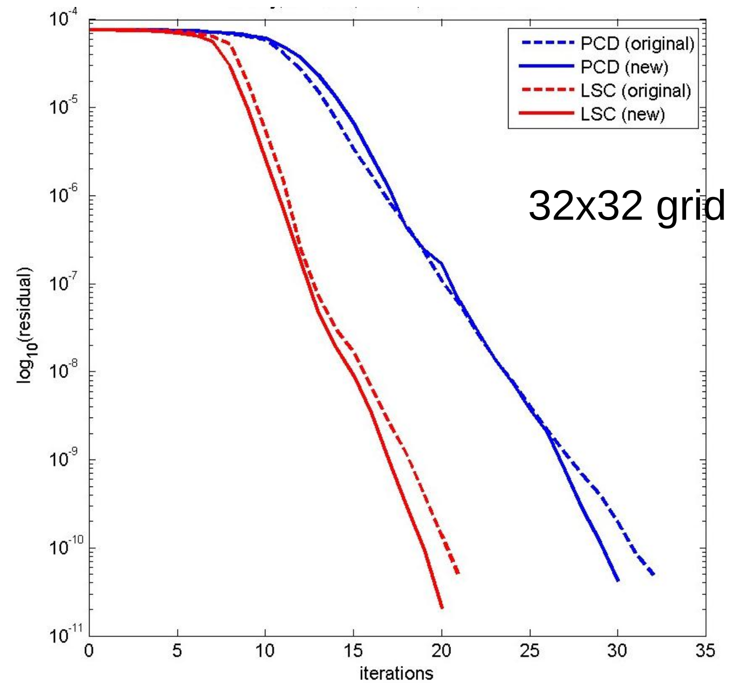
Recapitulating: for PCD Preconditioning

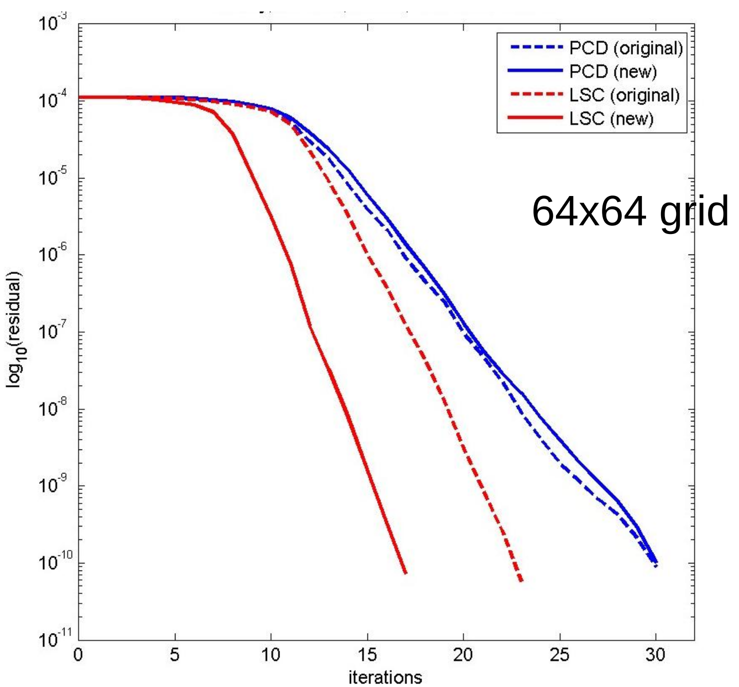
	Original	New
Q_S^{-1}	$M_p^{-1} F_p A_p^{-1}$	$A_p^{-1} F_p M_p^{-1}$
F_p inflow b.c.	Dirichlet	Robin or Dirichlet
F_p characteristic b.c.	Neumann	Neumann or Dirichlet
F_p outflow b.c.	Neumann	Neumann
A_p	User-defined, b.c. compatible with F_p	$BM_v^{-1}B^T$ b.c. inherited

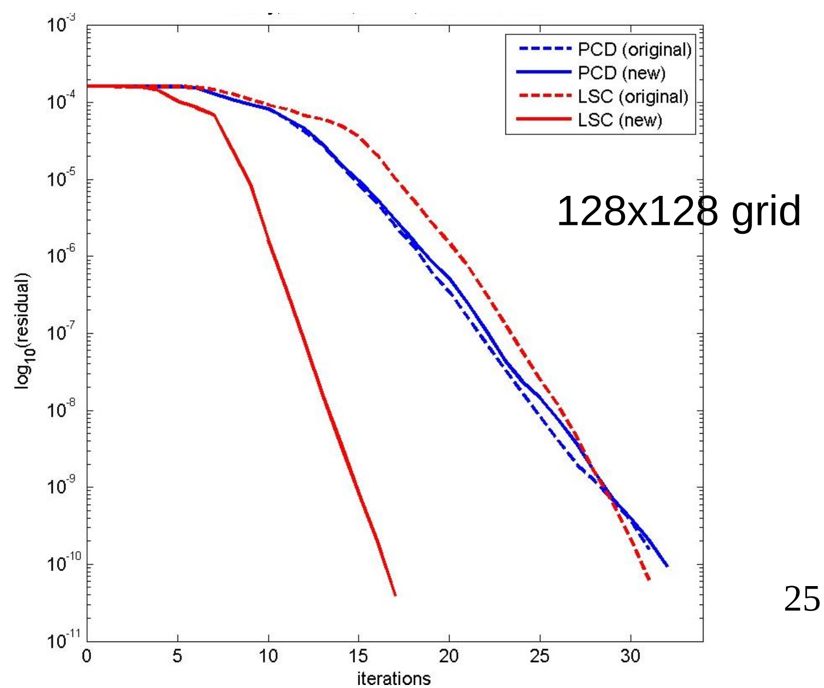
Performance I

Driven cavity flow
Re=200
Various grid sizes

Biquadratic velocities
Bilinear pressures



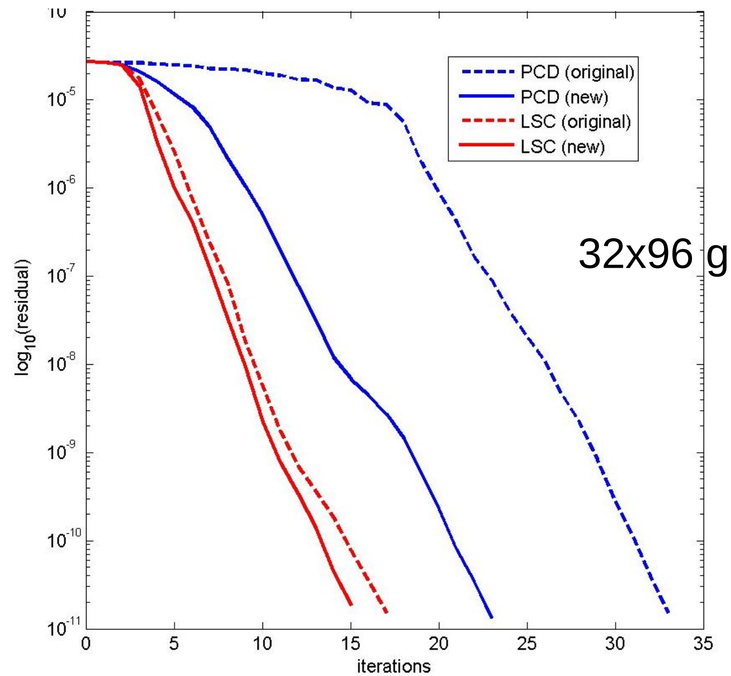




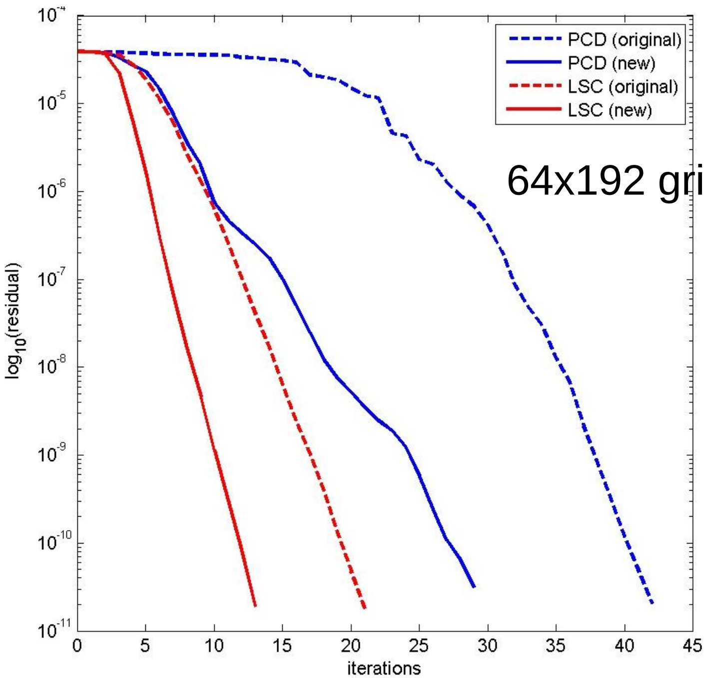
Performance II

Backward facing step
Re=100
Various grid sizes

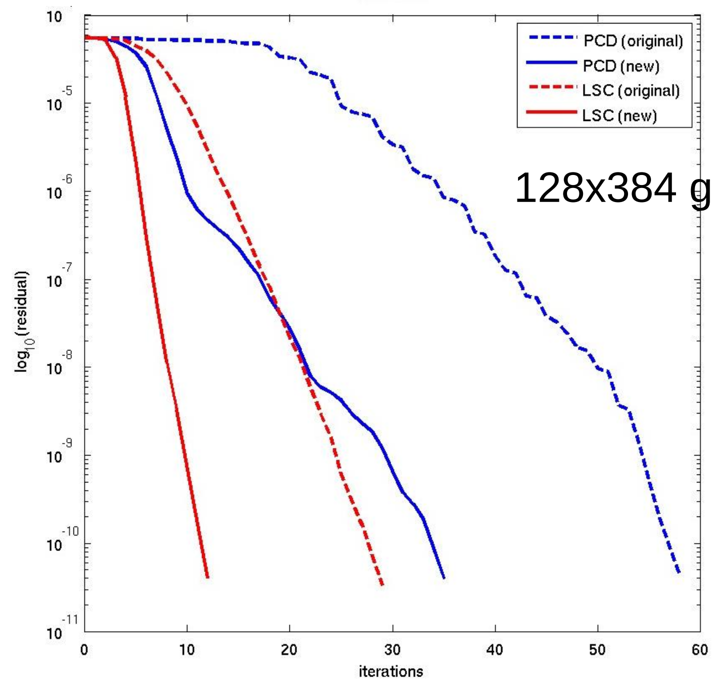
Biquadratic velocities
Bilinear pressures



32x96 grid



64x192 grid

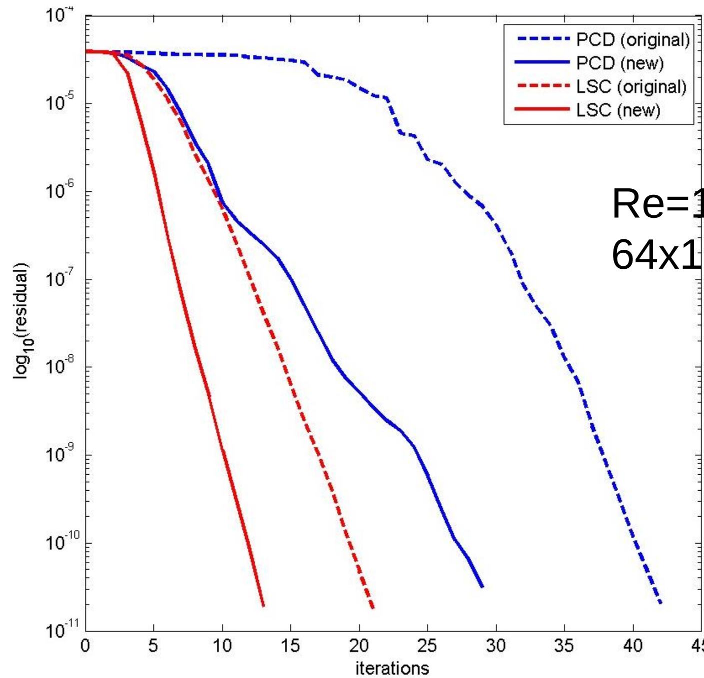


128x384 grid

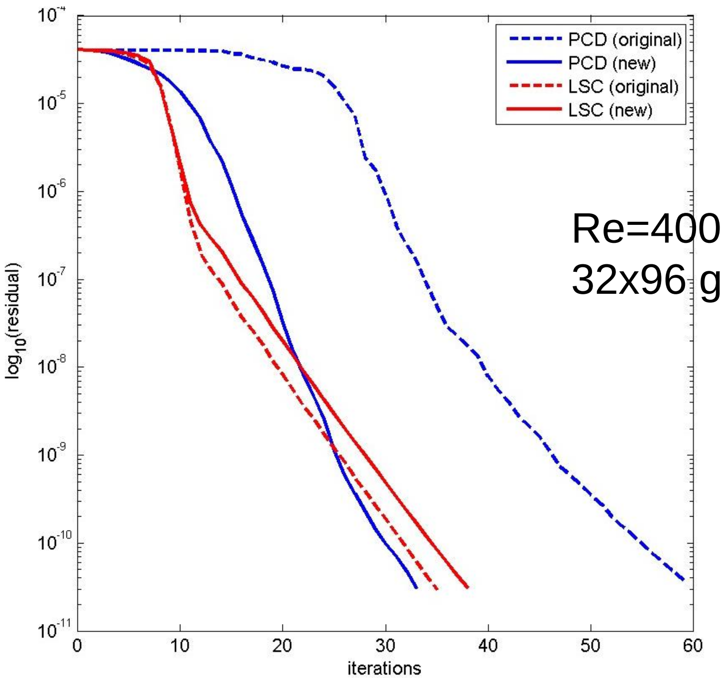
Performance III

Backward facing step
Re=100, 400
Various grid sizes

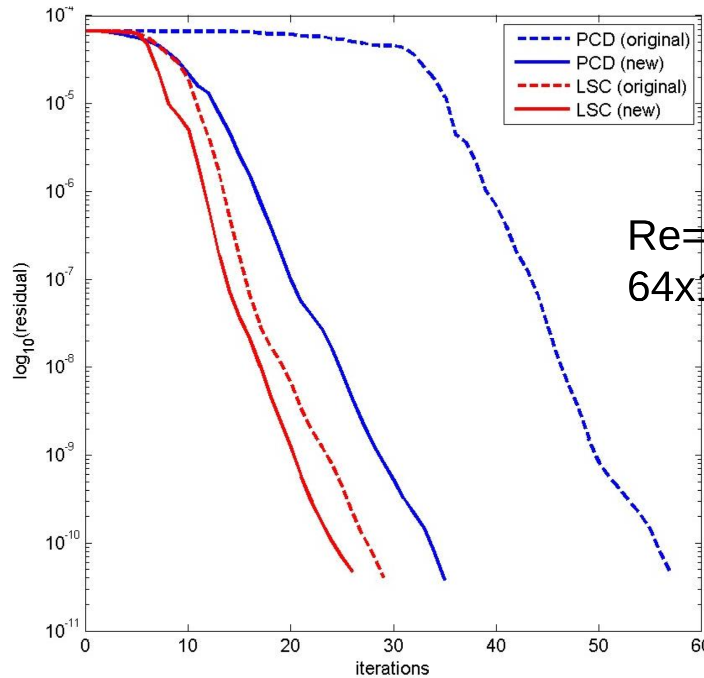
Biquadratic velocities
Bilinear pressures



Re=100,
64x192 grid



Re=400,
32x96 grid



Re=400,
64x192 grid

Revised Version of Least-Squares Commutator Preconditioning

Commutator: $\mathcal{E} = \mathcal{B}\mathcal{F} - \mathcal{F}_p\mathcal{B}$

$$= [\mathcal{B}_x\mathcal{F}^{(u_1)} - \mathcal{F}_p\mathcal{B}_x, \mathcal{B}_y\mathcal{F}^{(u_2)} - \mathcal{F}_p\mathcal{B}_y]$$

Finite element discretization

$$E = M_p^{-1}BM_v^{-1}F - M_p^{-1}F_pM_p^{-1}B$$
$$= M_p^{-1}[B_xM_{v_1}^{-1}F^{(u_1)} - F_pM_p^{-1}B_x, \underline{B_yM_{v_2}^{-1}F^{(u_2)} - F_pM_p^{-1}B_y}]$$

In finite difference setting:

Robin condition adjusts rows of F_p for commutator with component of B orthogonal to boundary

Here:

Mimic this by row-wise weighting to *de-emphasize* the part of commutator from the component of B tangent to the boundary

Revised LSC Preconditioning

Find $X \equiv F_p M_p^{-1}$ one row at a time such that

$$\|([BM_v^{-1}F]_{i,:} - X_{i,:}B)H^{1/2}\| = \| B^T[X^T]_{:,i} - [FM_v^{-1}B^T]_{:,i} \|_H$$

is minimized in a least squares sense, where

$$H = W^{1/2} M_v^{-1} W^{1/2}$$

and W is a diagonal weighting matrix with small value $W_{jj} = \varepsilon$ for all indices j such that:

$$B_{ij} \neq 0$$

where i is a pressure index near boundary, and j is the index of a velocity component tangent to the boundary

Otherwise: $W_{jj} = 1$

Revised LSC Preconditioning

Resulting preconditioning operator:

$$Q_S^{-1} = (BM_v^{-1}B^T)^{-1}(BM_v^{-1}FHB^T)(BM_v^{-1}B^T)^{-1}$$

where

$$H = W^{1/2}M_v^{-1}W^{1/2}$$

The only difference from the original version (where W=I)

ε=*1/100* in experiments

Concluding Remarks

1. Boundary conditions influence PCD preconditioning
2. Robin boundary conditions for F_p enhance performance for problems with Dirichlet (inflow or characteristic) conditions: reduce transient period of slow convergence.
3. Outflow conditions are less well understood. They also appear to have less impact.
4. Results for PCD lead to modified LSC with better properties: convergence rate independent of mesh.
5. Results clarify poorly understood aspect of these ideas: weaknesses previously displayed were caused by boundary conditions. (Cf. projection methods.)