Development of a High Order Discontinuous Galerkin Method for the Simulation of Elastohydrodynamic Lubrication Phenomena

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Outline

- Elastohydrodynamic Lubrication (EHL) problems.
- Governing equations.
- Challenges of solving EHL problems.
- Steady-state line contact.
- Transient line contact.
- Steady-state point contact.
- Discussion.

Lubrication Problems

The use of lubricants to control friction and wear has long been recognised as being of enormous importance throughout society...

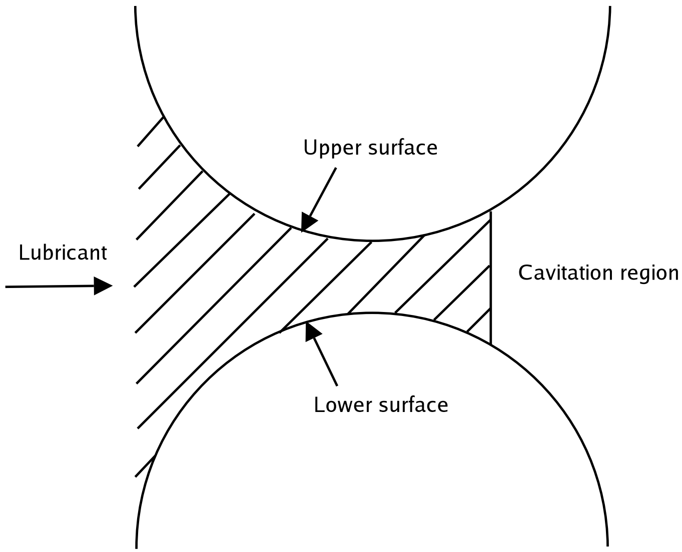
- Jost Report (1966) estimated that 1.3-1.6% of GDP could be saved via good tribological principles.
- First “oil crisis” (1973) led to fuel economy becoming an issue.
- Kyoto Protocol (1997) required all signatories to reduce CO_2 emissions.
- Current price of oil makes these issues as topical as ever!

Lubrication Problems

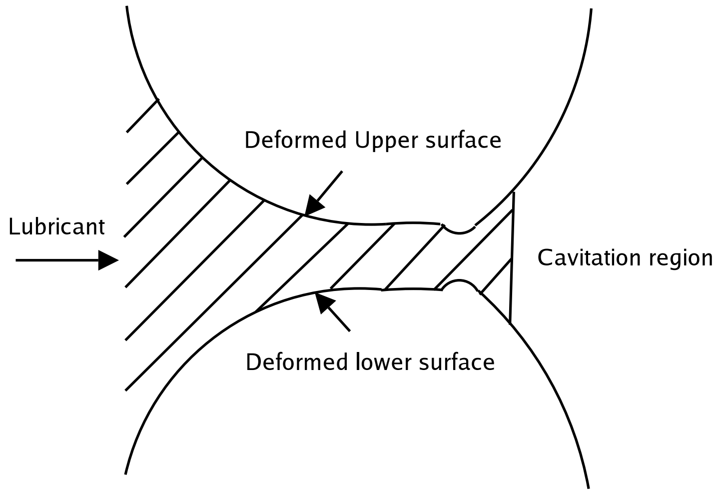
- Lubricant manufacture represents a multi-billion dollar market worldwide.
- Shell are one of the market leaders in this field.
- Challenge is to develop new lubricants to maximize efficiency and/or minimize wear as components evolve or new components are designed.
- Physical manufacture and testing is very expensive.
- Goal is to be able to predict (and optimize) performance without the need for physical experiments!
- The Scientific Computing group at Leeds has worked with Shell on these problems for the past 15 years and have provided a range of bespoke software...

Lubrication Problems

This talk focuses on the elastohydrodynamic (EHL) problem...



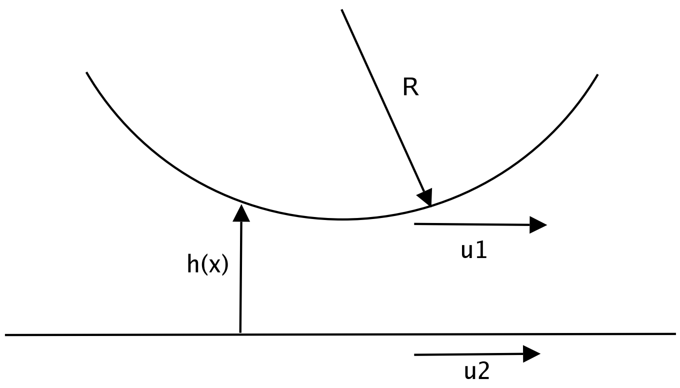
(a) Hydrodynamic Lubrication



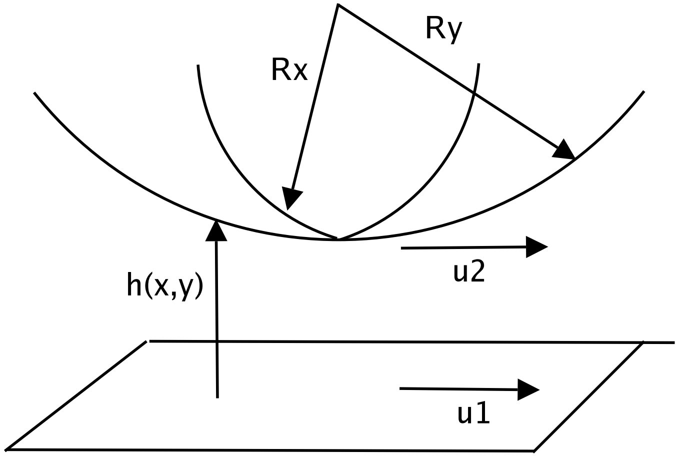
(b) Elastohydrodynamic Lubrication

EHL Problems

Two cases will be considered...



(a) Line Contact



(b) Point Contact

Equations (1d)

Reynolds equation:

$$\frac{\partial}{\partial X}\left(\varepsilon\frac{\partial P}{\partial X}\right)-\frac{\partial(\bar{\rho}H)}{\partial X}-\frac{\partial(\bar{\rho}H)}{\partial T}=0,$$

where

$$\varepsilon=\frac{\bar{\rho}H^3}{\bar{\eta}\lambda}$$

$$\bar{\eta}(P)=e^{\left\{\frac{\alpha P_0}{z}\left[-1+\left(1+\frac{PP_h}{P_0}\right)^z\right]\right\}}$$

$$\bar{\rho}(P)=\frac{0.59\times10^9+1.34PP_h}{0.59\times10^9+PP_h}$$

Equations (1d)

Film thickness equation:

$$H(X,T) = H_{00}(T) + \frac{X^2}{2} - \mathscr{R}(X,T) - \frac{1}{\pi}\int_{X_{in}}^{X_{cavi}} \ln\left|X - X'\right| P(X',T)\,\mathrm{d}X'$$

Force balance equation:

$$\int_{X_{in}}^{X_{cavi}} P(X)\,\mathrm{d}X - \frac{\pi}{2} = 0$$

Cavitation boundary: X_{cavi} is an unknown position where

$$\frac{\partial P}{\partial X} = P = 0.$$

Challenges

- High nonlinearity.
- Variation of ε changes the character of the Reynolds equation.
- Free boundary.
- Steep pressure spike.
- Stability.
- Adaptivity.

Features of DG

- High degree on each element.
- Only impose continuity weakly.
- Only impose Dirichlet boundary conditions weakly.
- Implement "upwinding" through the numerical flux over element boundaries.

Find P such that

$$a(P,v) = l(P,v) \quad \forall v \in \{N_i^e\} \ ,$$

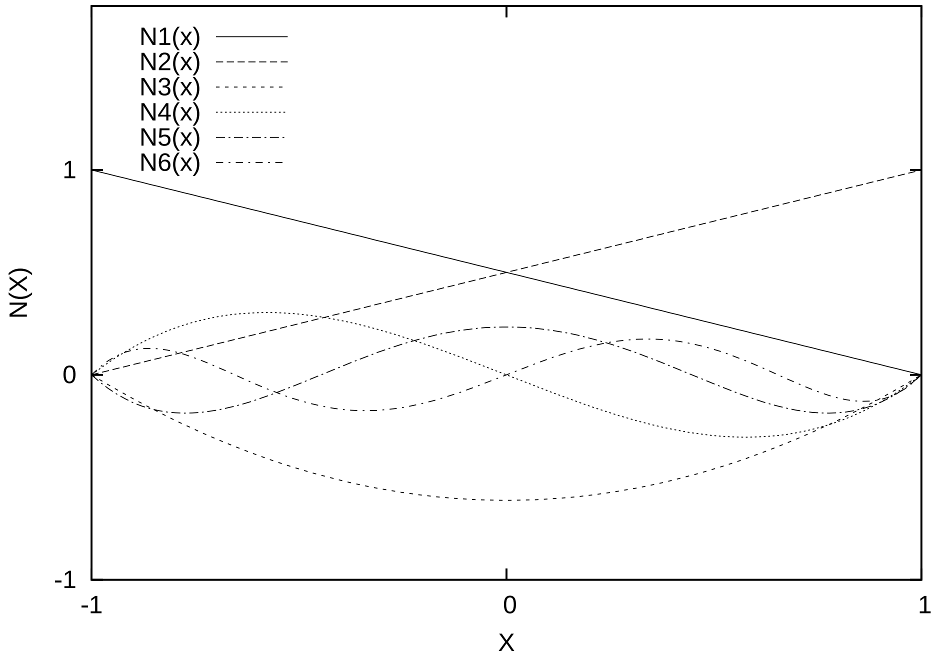
where...

DG Spatial Discretization (1d)

In each element e, P is given by :

$$P^e(X) = \sum_{i=1}^{p^e+1} u_i^e N_i^e(X).$$

In the reference element, the basis function are:



DG Spatial Discretization (1d)

$$a(P,v) = \sum_{e\in\Omega_h}\left(\int_e \varepsilon \frac{\partial P}{\partial X}\frac{\partial v}{\partial X}\,\mathrm{d}X\right) + \sum_{\Gamma_{int}}\left([v]\left\langle \varepsilon \frac{\partial P}{\partial X}\right\rangle - [P]\left\langle \varepsilon \frac{\partial v}{\partial X}\right\rangle\right)$$

$$+ \left(v\varepsilon \frac{\partial P}{\partial X}\right)|_{X_{inlet}} - \left(v\varepsilon \frac{\partial P}{\partial X}\right)|_{X_{outlet}} - \left(P\varepsilon \frac{\partial v}{\partial X}\right)|_{X_{inlet}} + \left(P\varepsilon \frac{\partial v}{\partial X}\right)|_{X_{outlet}}$$

$$l(P,v) = \sum_{e\in\Omega_h}\left(\int_e \rho H \frac{\partial v}{\partial X}\,\mathrm{d}X\right) + \sum_{\Gamma_{int}}[v]\langle \rho(P^-)H\rangle$$

$$+ (\rho H v)|_{X_{inlet}} - (\rho H v)|_{X_{outlet}} - \left(g_{inlet}\varepsilon \frac{\partial v}{\partial X}\right)|_{X_{inlet}} + \left(g_{outlet}\varepsilon \frac{\partial v}{\partial X}\right)|_{X_{outlet}}$$

where $P^- = \lim_{\sigma\to 0} P(x-\sigma),$ for $x \in \Gamma_{int}$.

Handling Cavitation Condition

1. Penalty method (e.g. Wu(1986)):

$$L(P,v) = a(P,v) + \frac{1}{\delta}\int_{X_{in}}^{X_{cavi}} P_{-}v\,\mathrm{d}X - l(P,v) = 0,$$

   where δ is an arbitrary positive number and

$$P_{-} = min(P,0)$$

2. Other techniques are possible (e.g. local r-refinement).

Spatial Discretization (cont.)

Film thickness equation:

$$H(X,T) = H_{00}(T) + \frac{X^2}{2} - \mathscr{R}(X,T) - \frac{1}{\pi}\sum_{e=1}^{N}\sum_{i=1}^{p^e+1} K_i^e(X)\,u_i^e(T),$$

where

$$K_i^e(X) = \int_e \ln\left|X - X'\right| N_i^e(X')\,\mathrm{d}X'.$$

Force balance equation:

$$\sum_{e=1}^{N}\sum_{i=1}^{p^e+1} G_i^e u_i^e - \frac{\pi}{2} = 0,$$

where

$$G_i^e = \int_e N_i^e(X)\,\mathrm{d}X.$$

Relaxation (1d)

The discrete steady-state Reynolds equation may be written as:

$$L(U) = A(U)U - b(U) = 0.$$

U is relaxed according to:

$$U \leftarrow U + \left(\frac{\partial L(U)}{\partial U}\right)^{-1} (-L(U)),$$

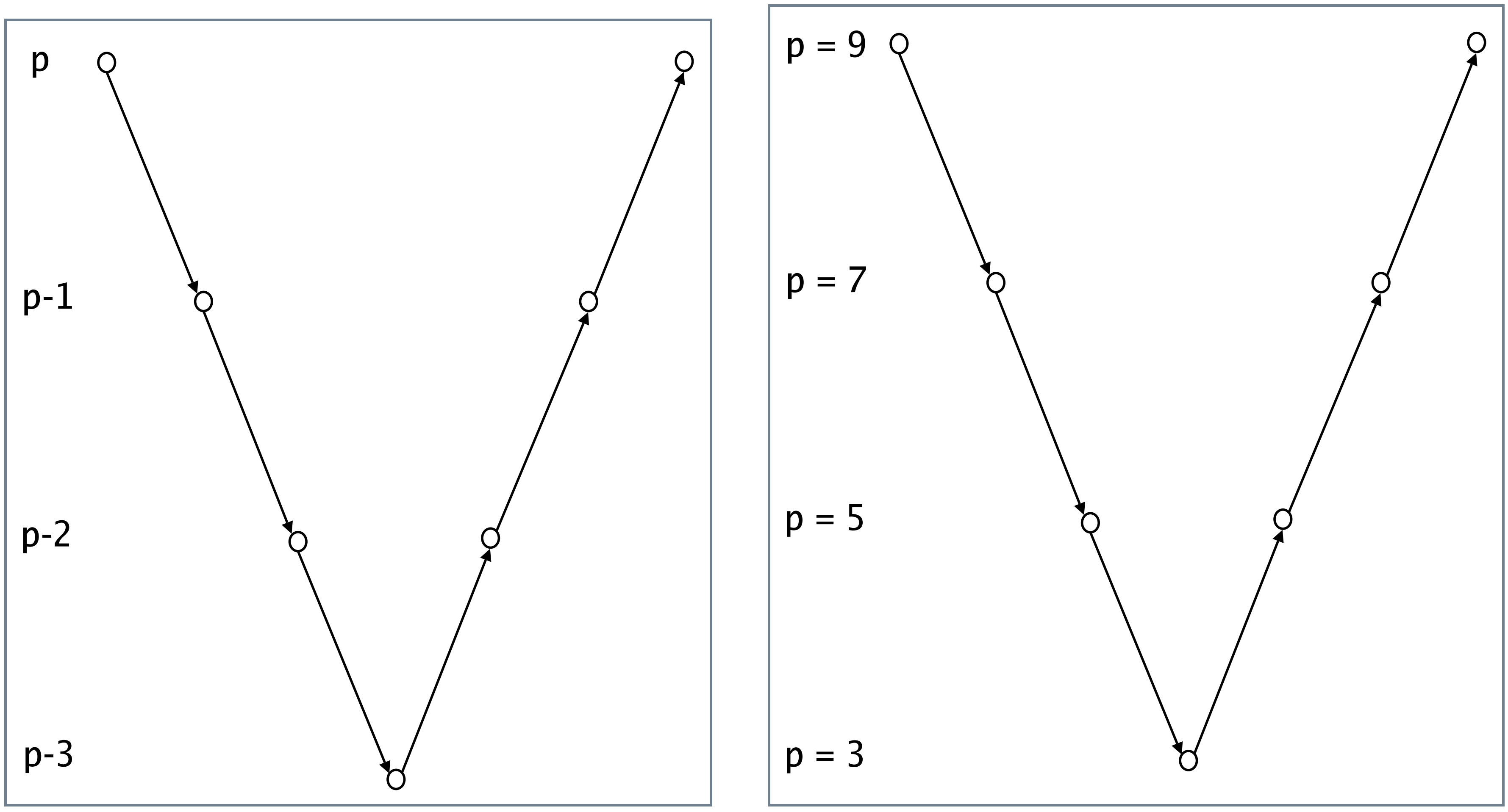
where

$$\frac{\partial L(U)}{\partial U} = \frac{\partial}{\partial U}(A(U)U) - \frac{\partial b(U)}{\partial U} \approx A(U) - \frac{\partial b(U)}{\partial U}.$$

P-multigrid

- The previous quasi-Newton (banded-matrix) iteration can be repeated until convergence but it is quite slow and requires a good initial guess.
- Could apply exact Newton to accelerate convergence but expensive and still need good initial guess.
- Instead we apply a p-multigrid approach to accelerate convergence.
- This is based upon the FAS nonlinear multigrid scheme with the smoother given above.
- Note that the p-multigrid does not generally have optimal (linear) complexity: nevertheless it provides a robust and efficient solver (in our experience).

P-multigrid



(a) $\triangle p = 1$ (b) $\triangle p = 2$

Figure 1: Four level V-cycles for p-multigrid

P-multigrid: Two-Level FAS

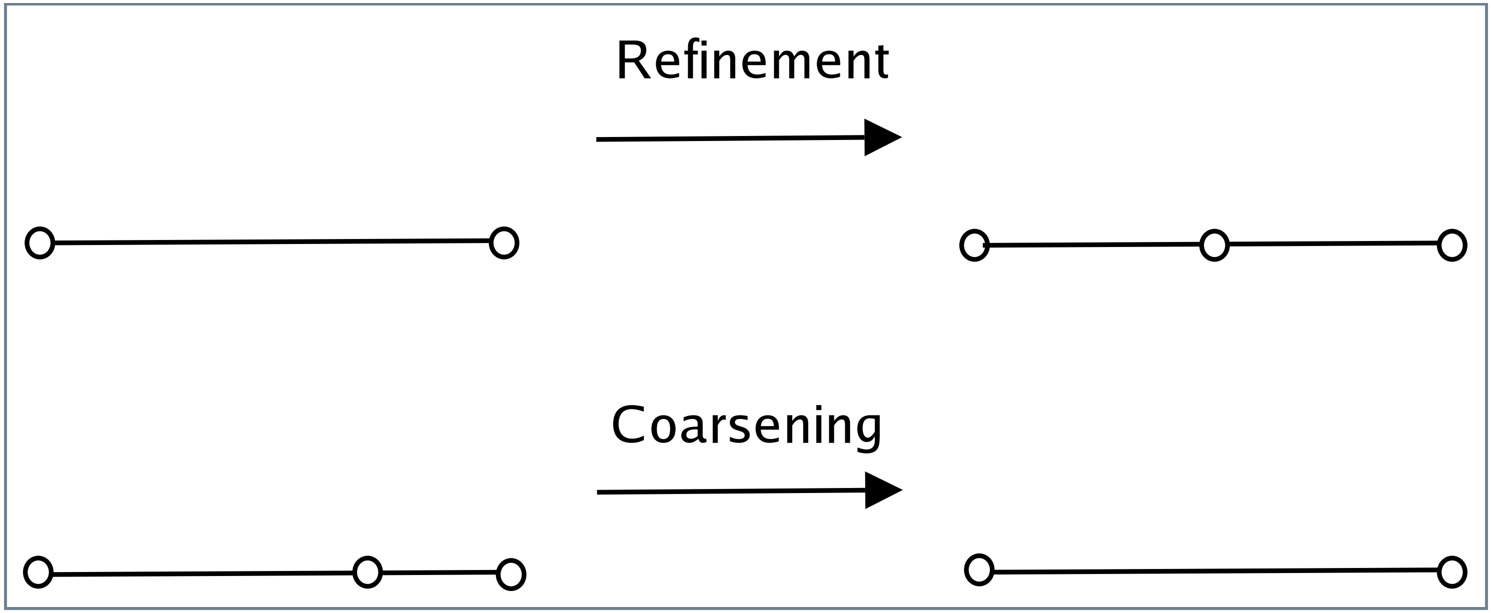
Consider fine ($\mathscr{L}^p(v^p) = 0$) and coarse ($\mathscr{L}^q(v^q) = 0$) discretizations of a problem, where degree $p > q$...

1. Pre-smooth at fine level: u^p.
2. Get residual at fine level: $r^p := f^p - \mathscr{L}^p(u^p)$ [$f^p = 0$ at the finest level].
3. Restrict to coarse level: $u_0^q := \tilde{I}_p^q u^p \; ; \quad r^q := I_p^q r^p$.
4. Solve coarse grid correction: $\mathscr{L}^q(u^q) = f^q := r^q + \mathscr{L}^q(u_0^q)$.
5. Interpolate correction to fine level: $e^p := I_q^p(u^q - u_0^q)$.
6. Update fine level solution: $u^p += e^p$.
7. Post-smooth at fine level.

Note that multigrid is obtained by applying this algorithm recursively at step 4!

H-Adaptivity

Based on error indicator: e.g. discontinuity between elements and/or the highest order contributions to the solution.

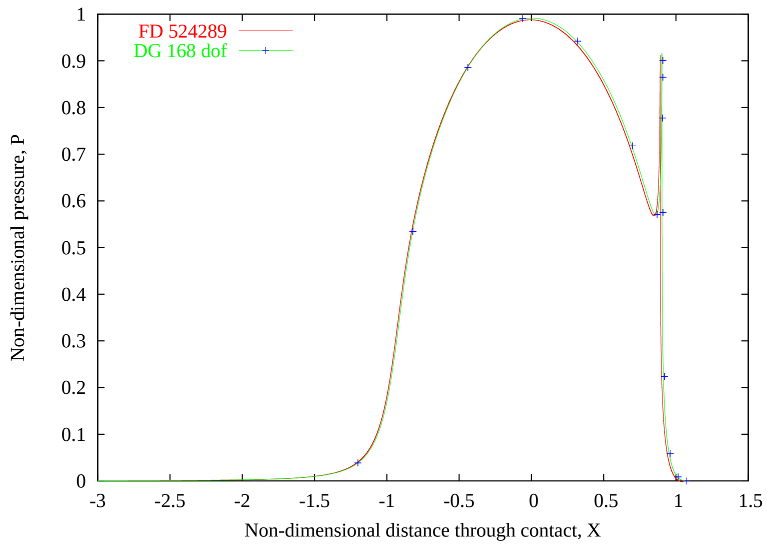


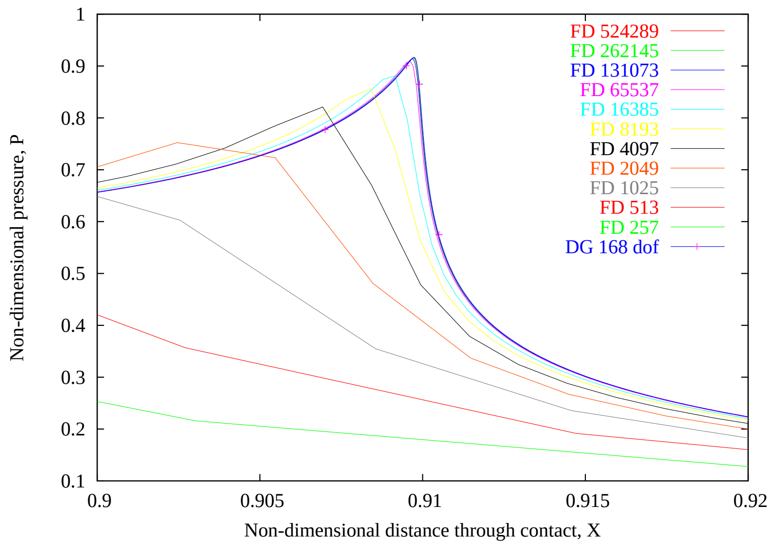
Solution Procedure

1. Give an initial grid and ensure that this grid covers the pressurized domain.
2. Initialize the pressure on the given grid and calculate the kernels.
3. Update the pressure on the current grid until it is nearly converged.
4. Check if the grid needs to be adapted.
5. Stop if the grid does not need to be adapted any more and the numerical residual is smaller than some final converged value (10^{-10} say).
6. Adapt the grid if needed and transfer the current pressure profile from the old grid onto the new grid. Calculate the kernels related to those new elements. Go to 3.

Example Results (1d)

$$U = 1.0 \times 10^{-11}, W = 1.0 \times 10^{-4}, G = 5000$$





1. Variable mesh size and order.
2. Half a million FD unknowns *VS* $\sim$200.

Example Results (1d)

Method	Unknowns	Peak Pressure	Peak Position	Cavitation Position
FD	4097	0.8212	0.9069	1.0693
FD	8193	0.8566	0.9084	1.0701
FD	16385	0.8810	0.9092	1.0704
FD	65537	0.9095	0.9066	1.0705
FD	131073	0.9138	0.9097	1.0707
FD	262145	0.9158	0.9097	1.0706
FD	524289	0.9164	0.9097	1.0706
DG	252	0.9166	0.9097	1.0707

Comparison of FD with DG + penalty method

Temporal Discretization

Using the Crank-Nicolson method, the 1D transient Reynolds equation is discretized to be:

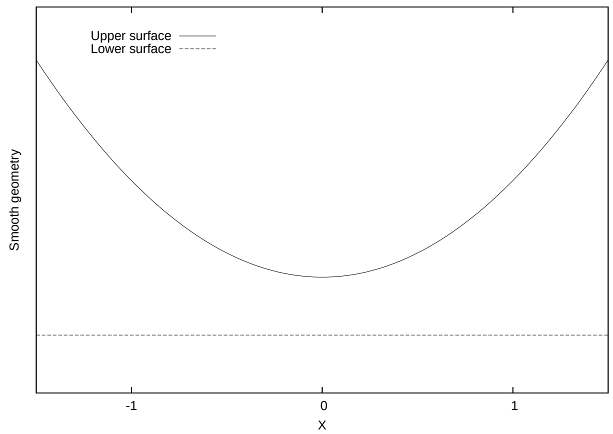
$$-\sum_{e\in\Omega_h}\left(\int_e \rho H v \mathrm{d}x\right)^T + \sum_{e\in\Omega_h}\left(\int_e \rho H v \mathrm{d}x\right)^{T+\triangle T} + \theta \triangle T L(P,v)^T + (1-\theta)\triangle T L(P,v)^{T+\triangle T} = 0$$

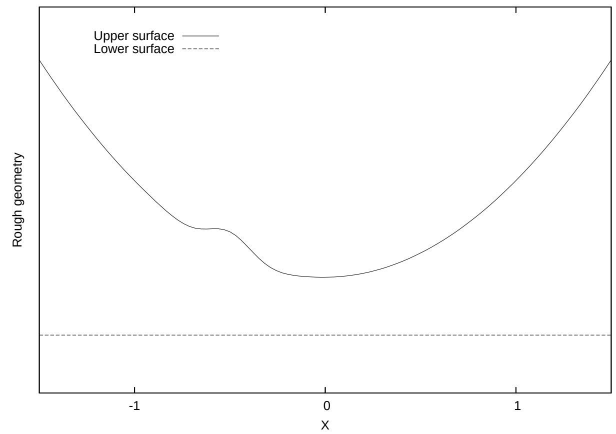
Each (implicit) time step requires a "steady-like" nonlinear system to be solved for $u^{T+\triangle T}$ (using u^T as the initial guess).

Example Transient Problem

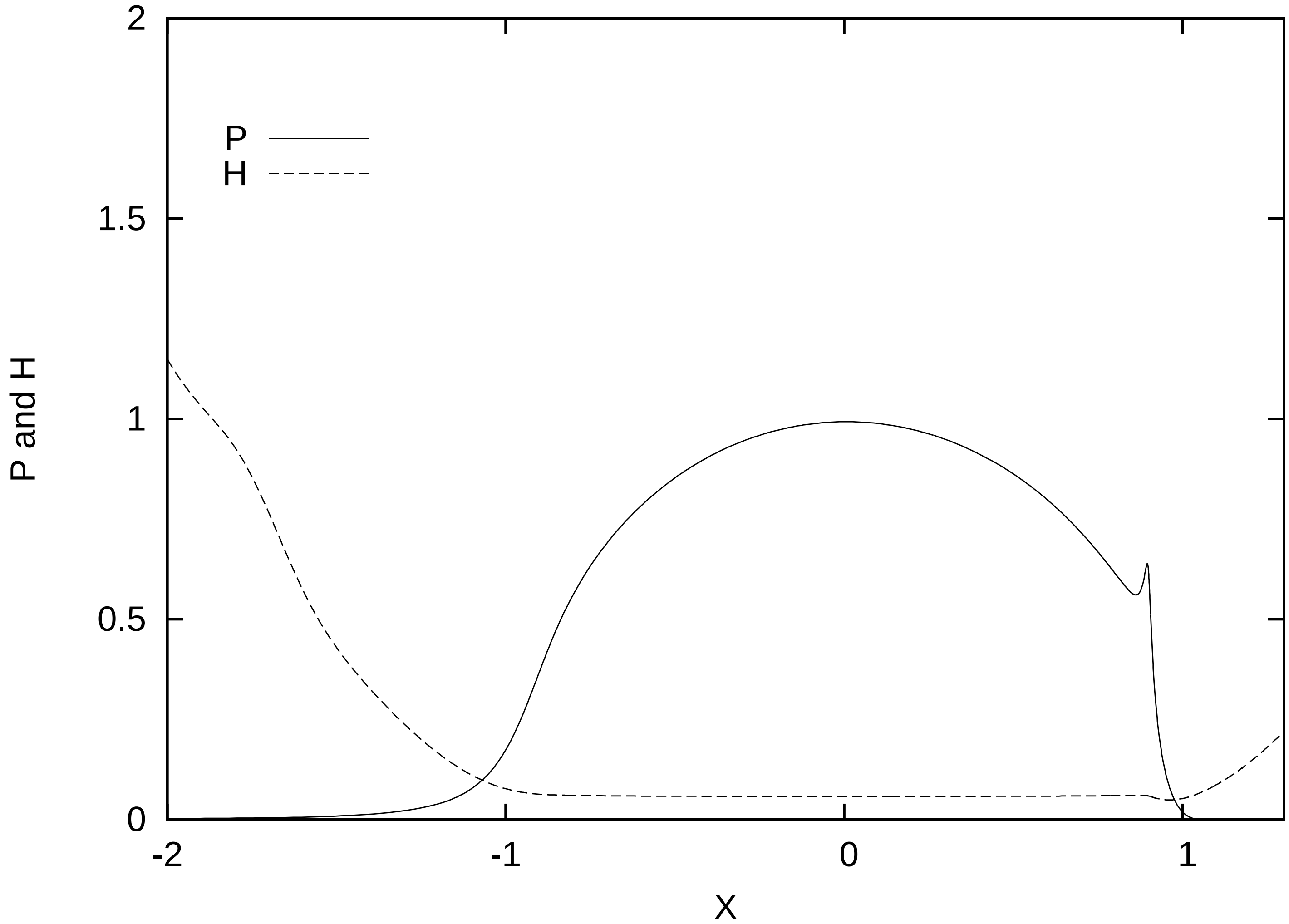
We adopt the same dimensionless model of the roughness used by Venner(1994):

$$\mathscr{R}(X,T) = \alpha 10^{-10\left(\frac{X-X_d}{\mathscr{W}}\right)^2} \cos\left(2\pi \frac{X-X_d}{\mathscr{W}}\right), \qquad (1)$$





Transient Results



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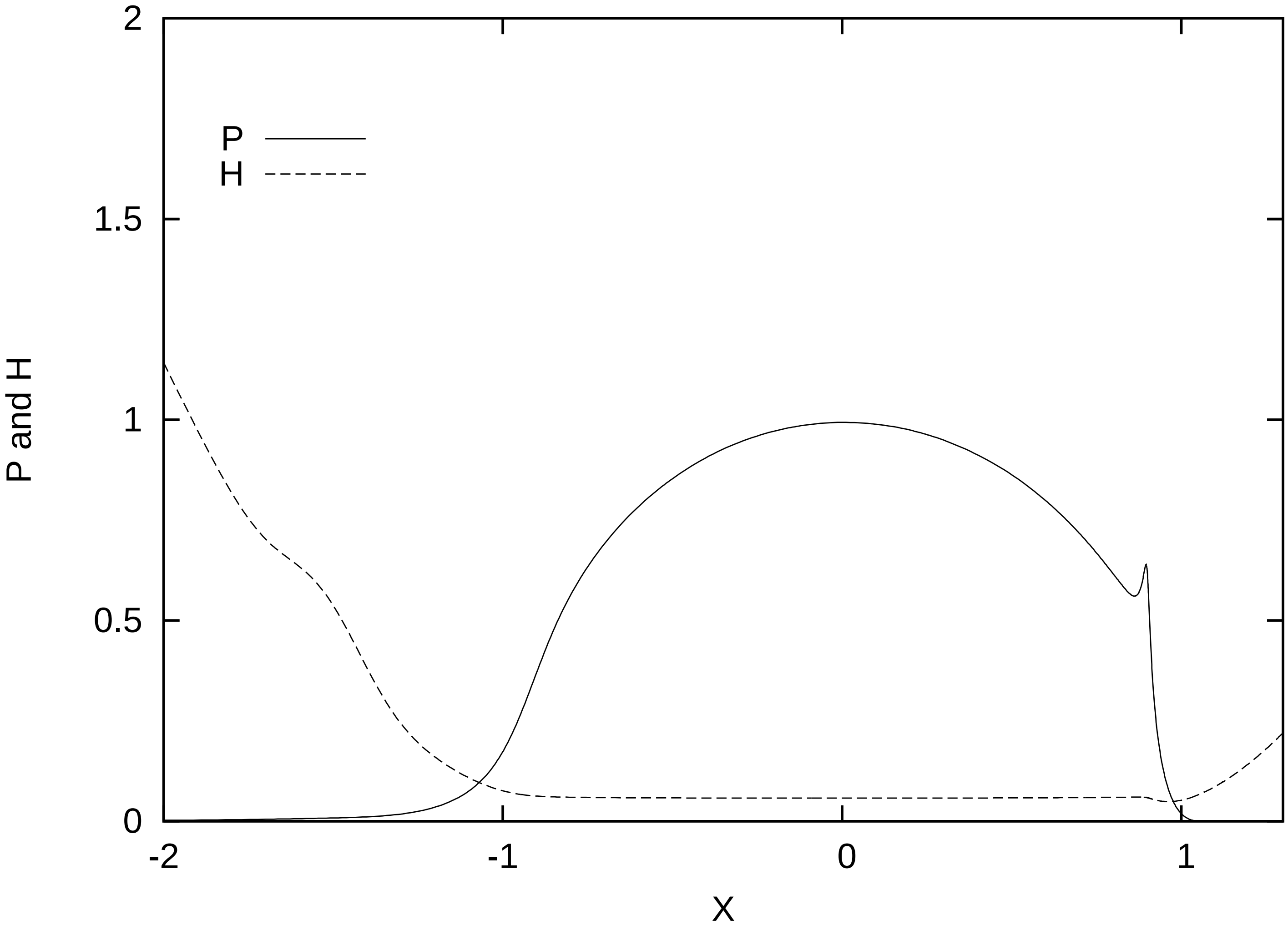

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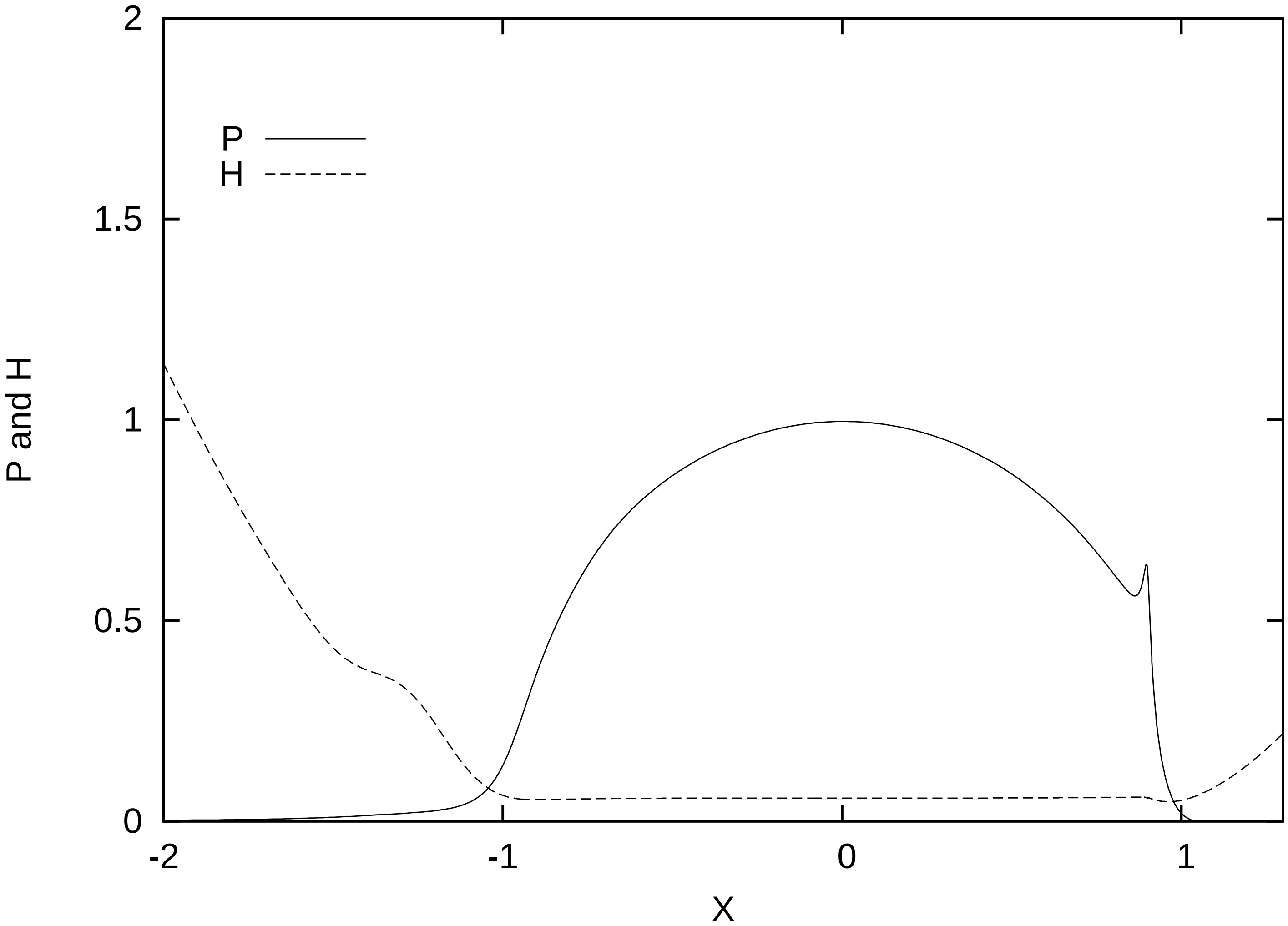

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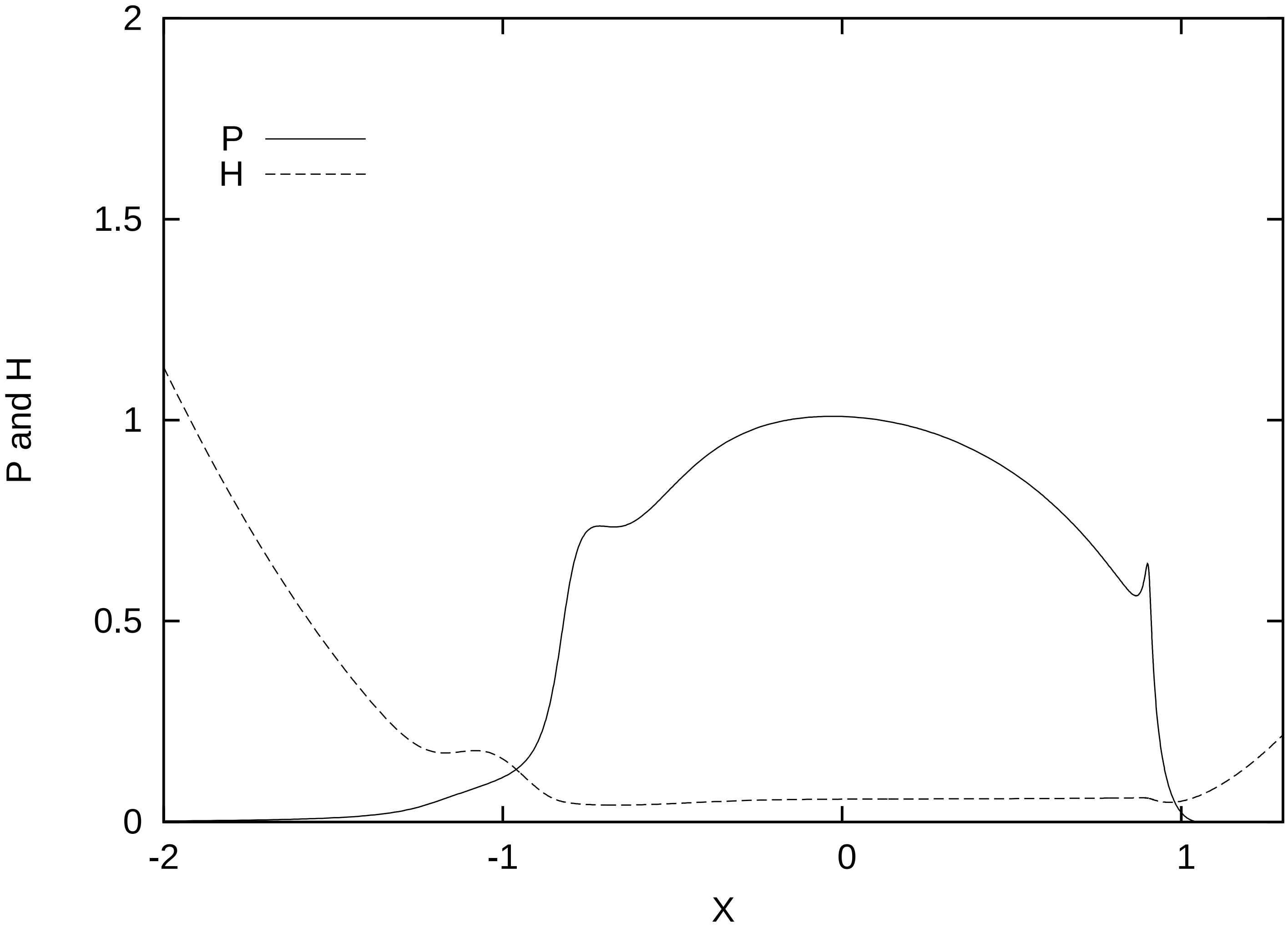

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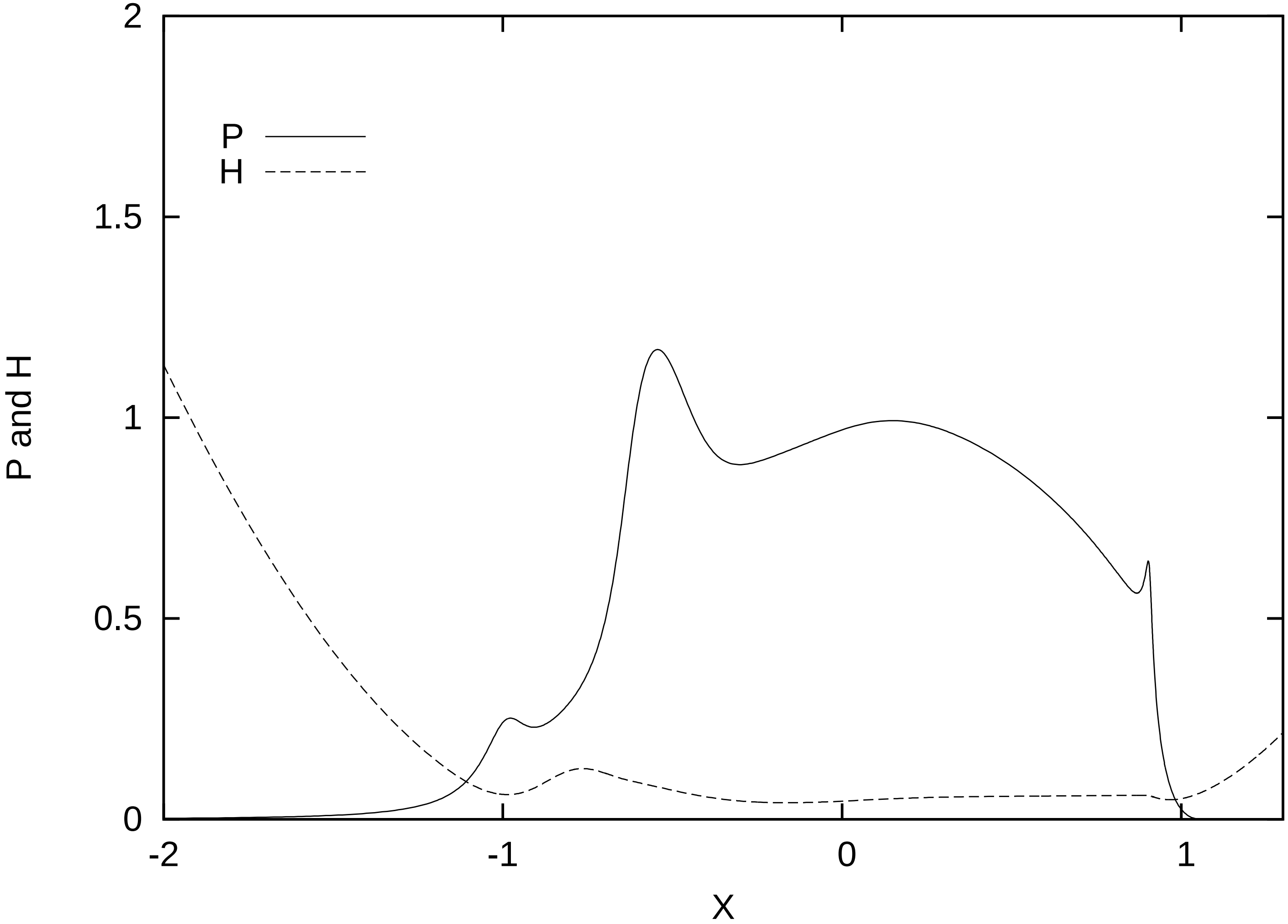

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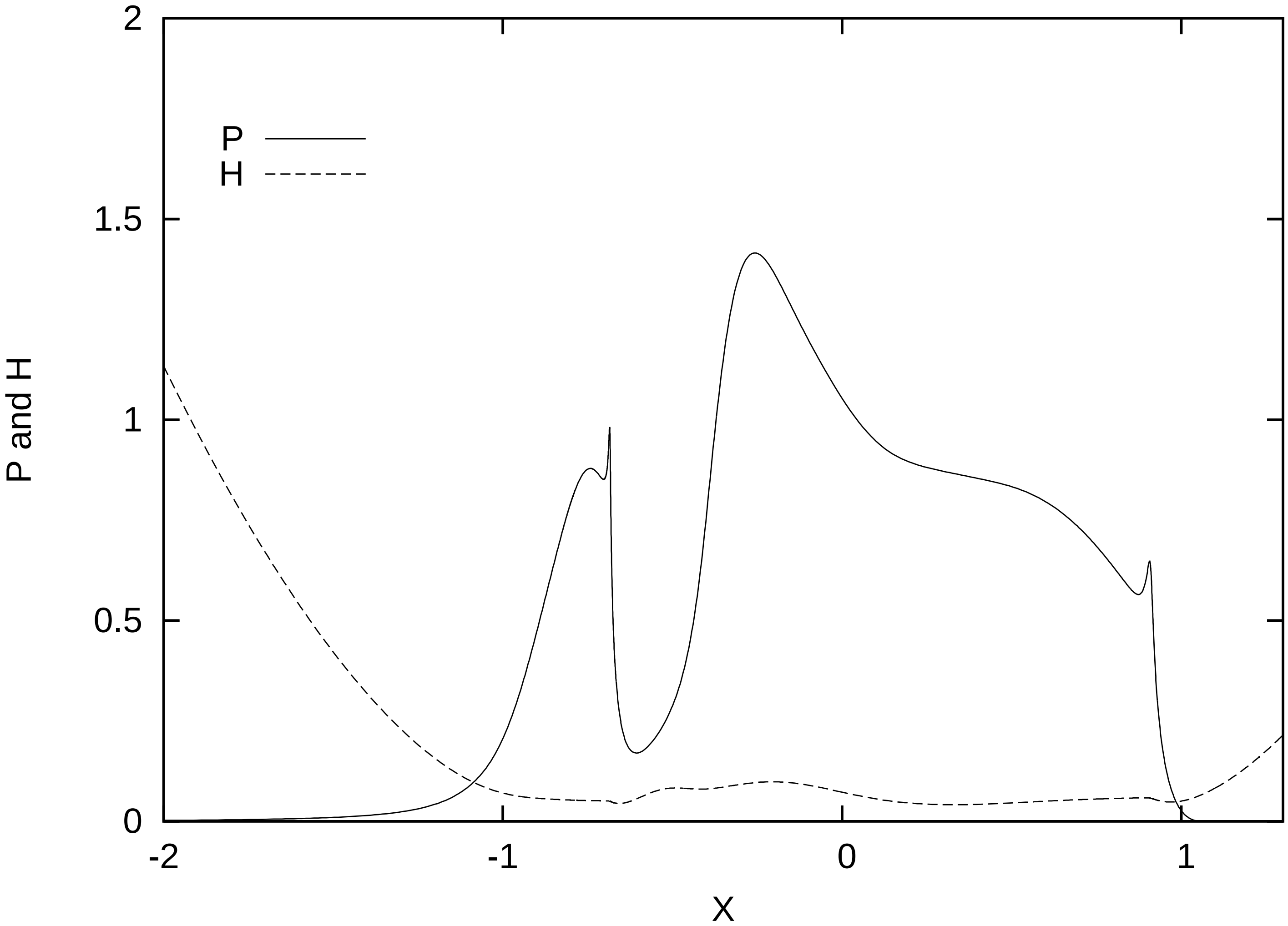

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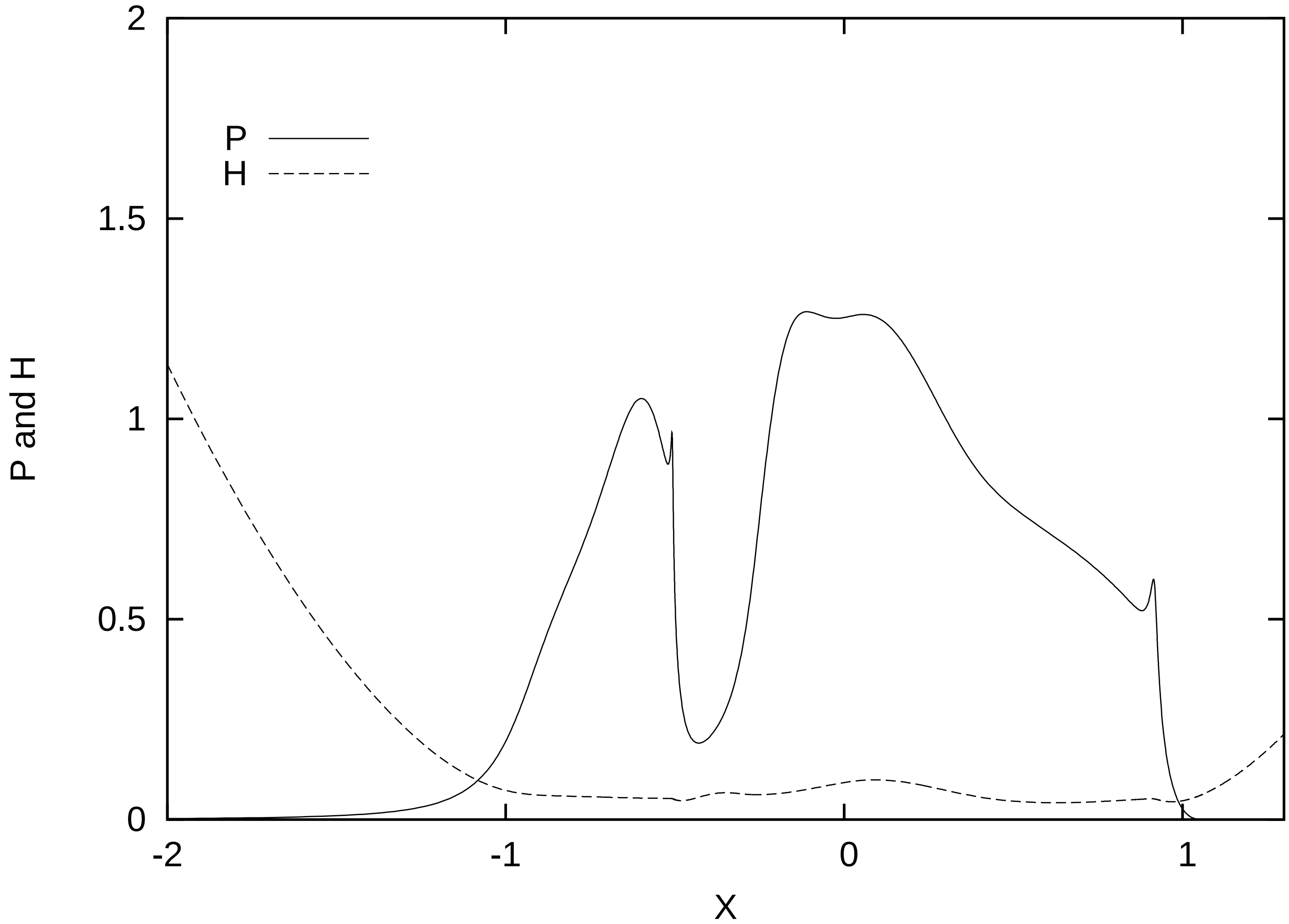

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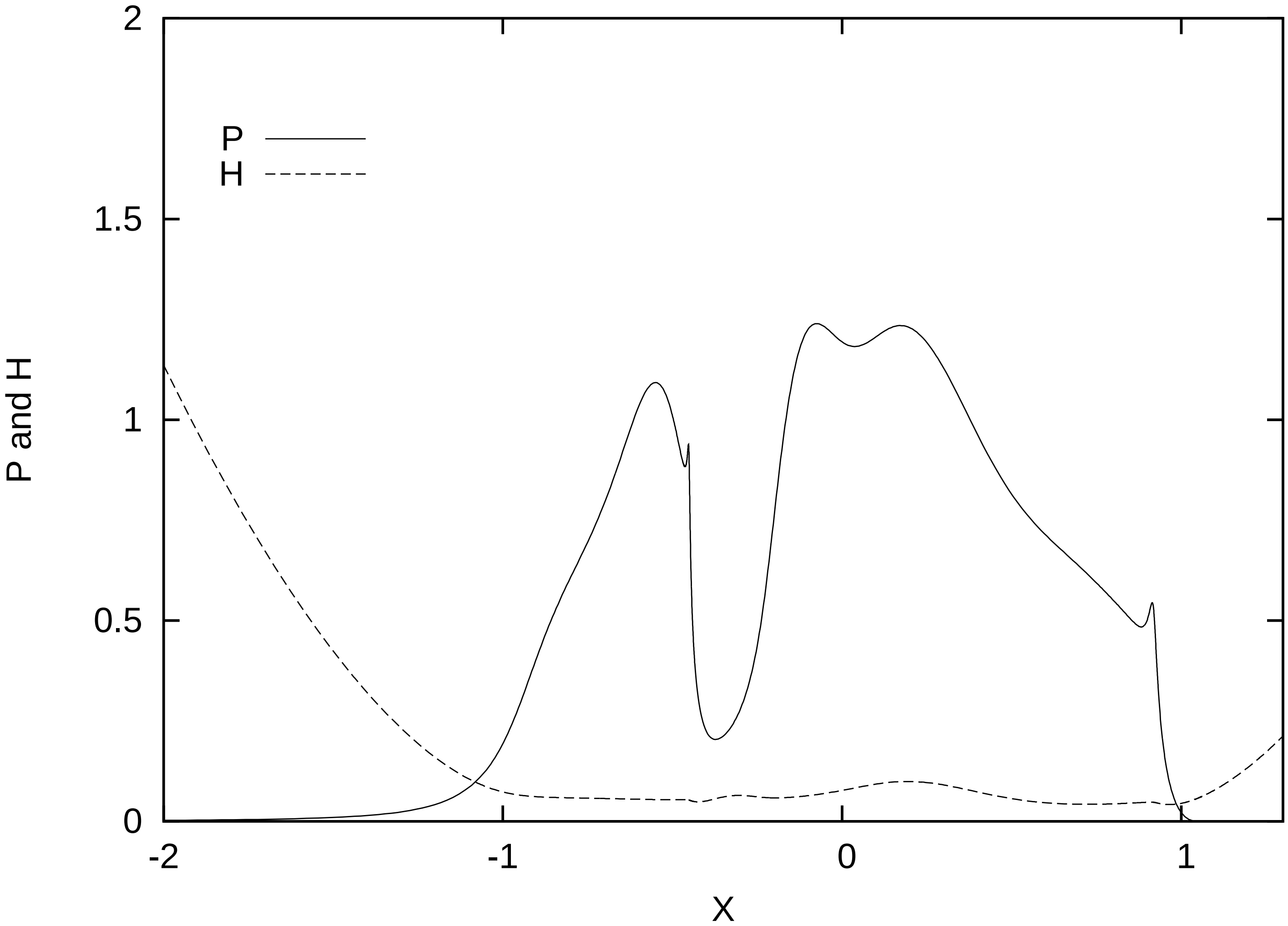

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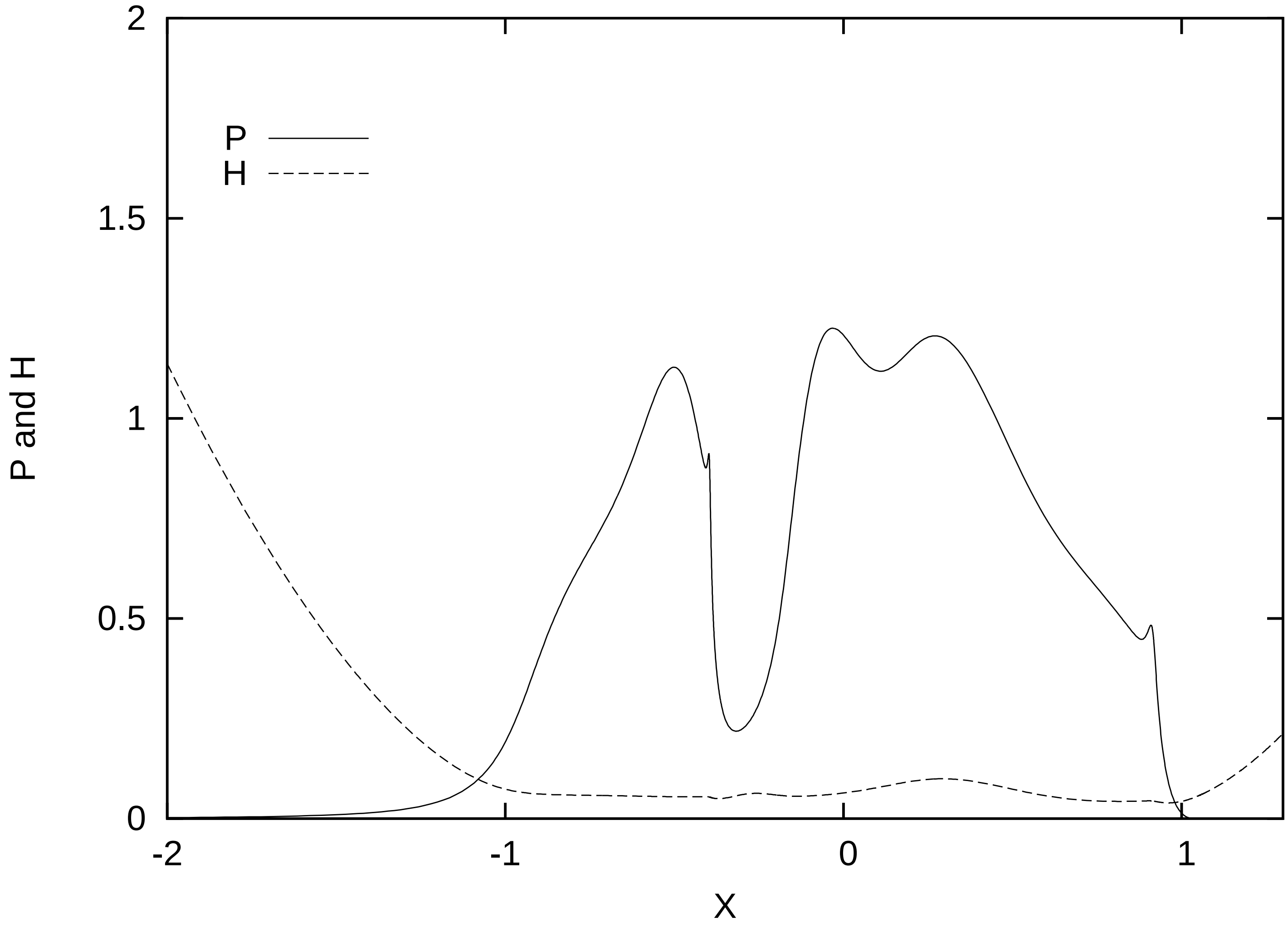

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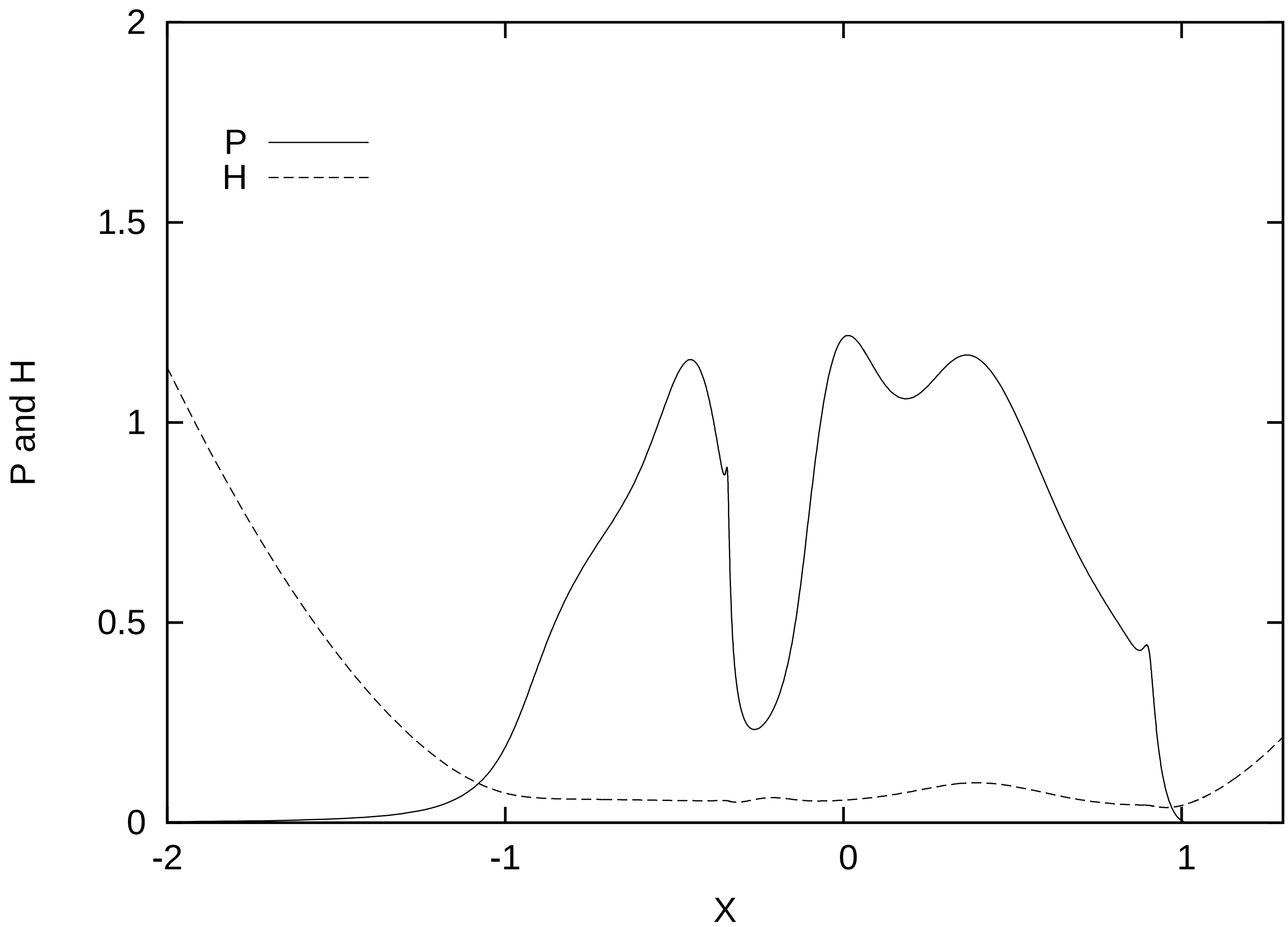

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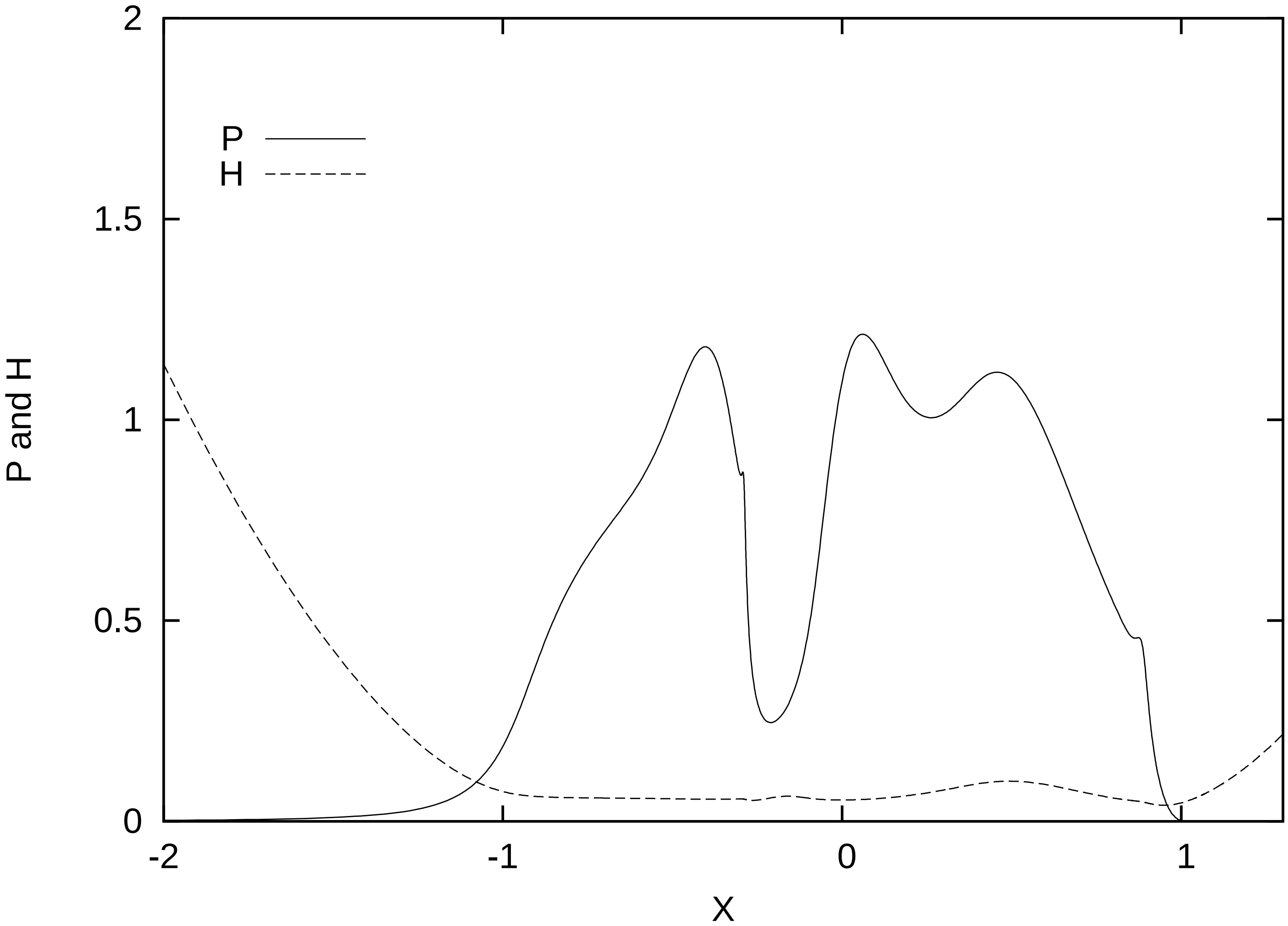

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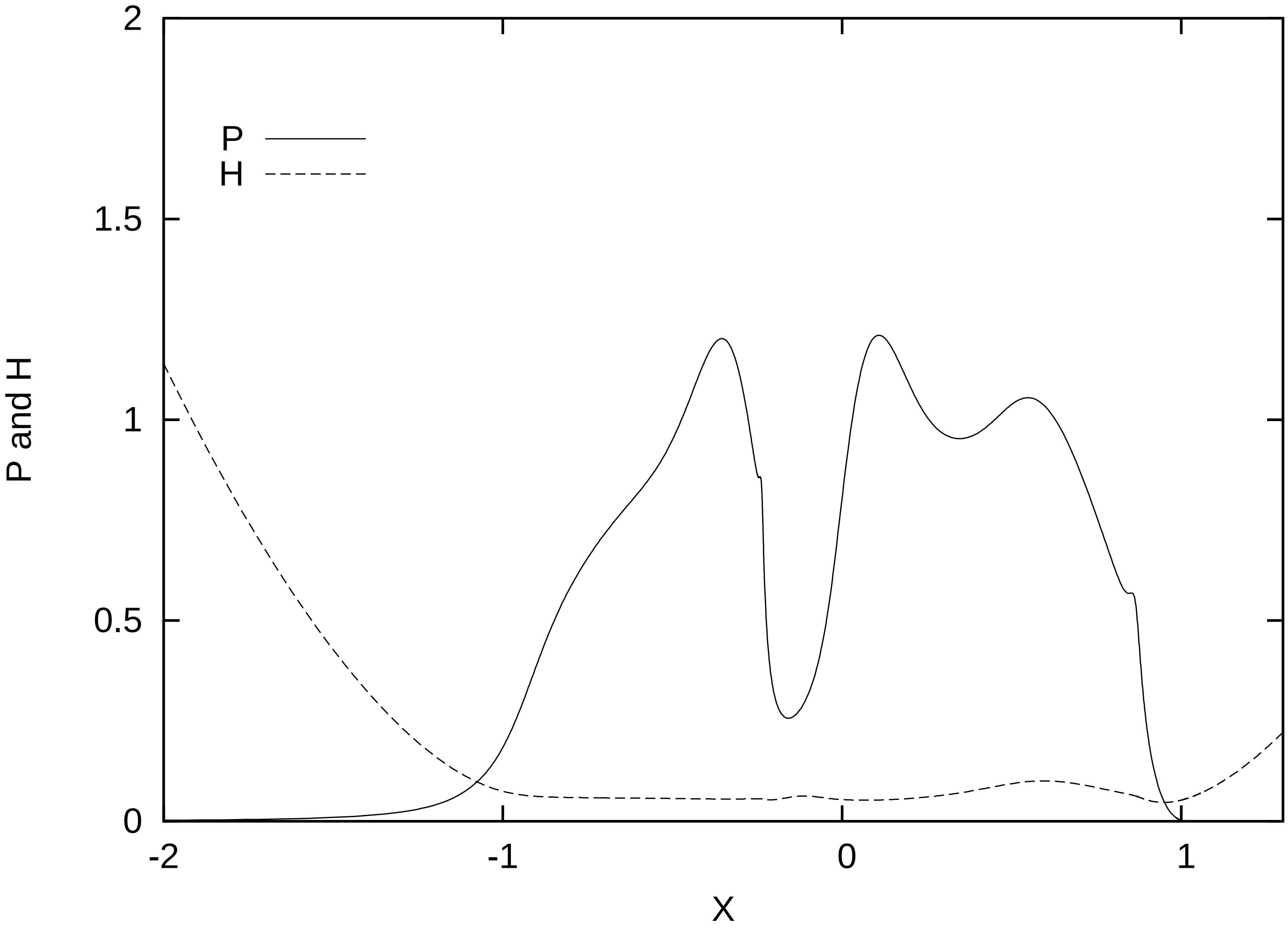

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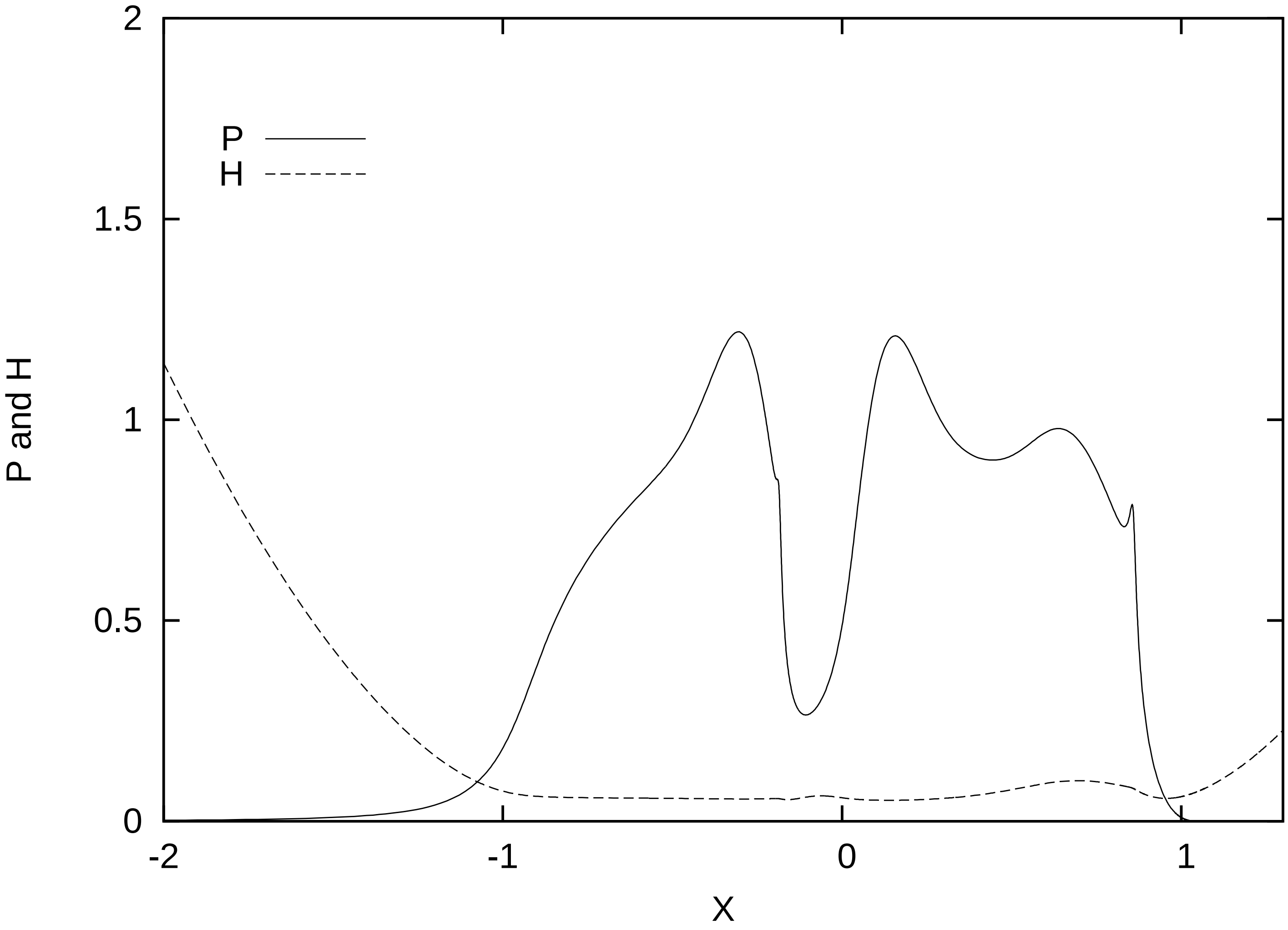

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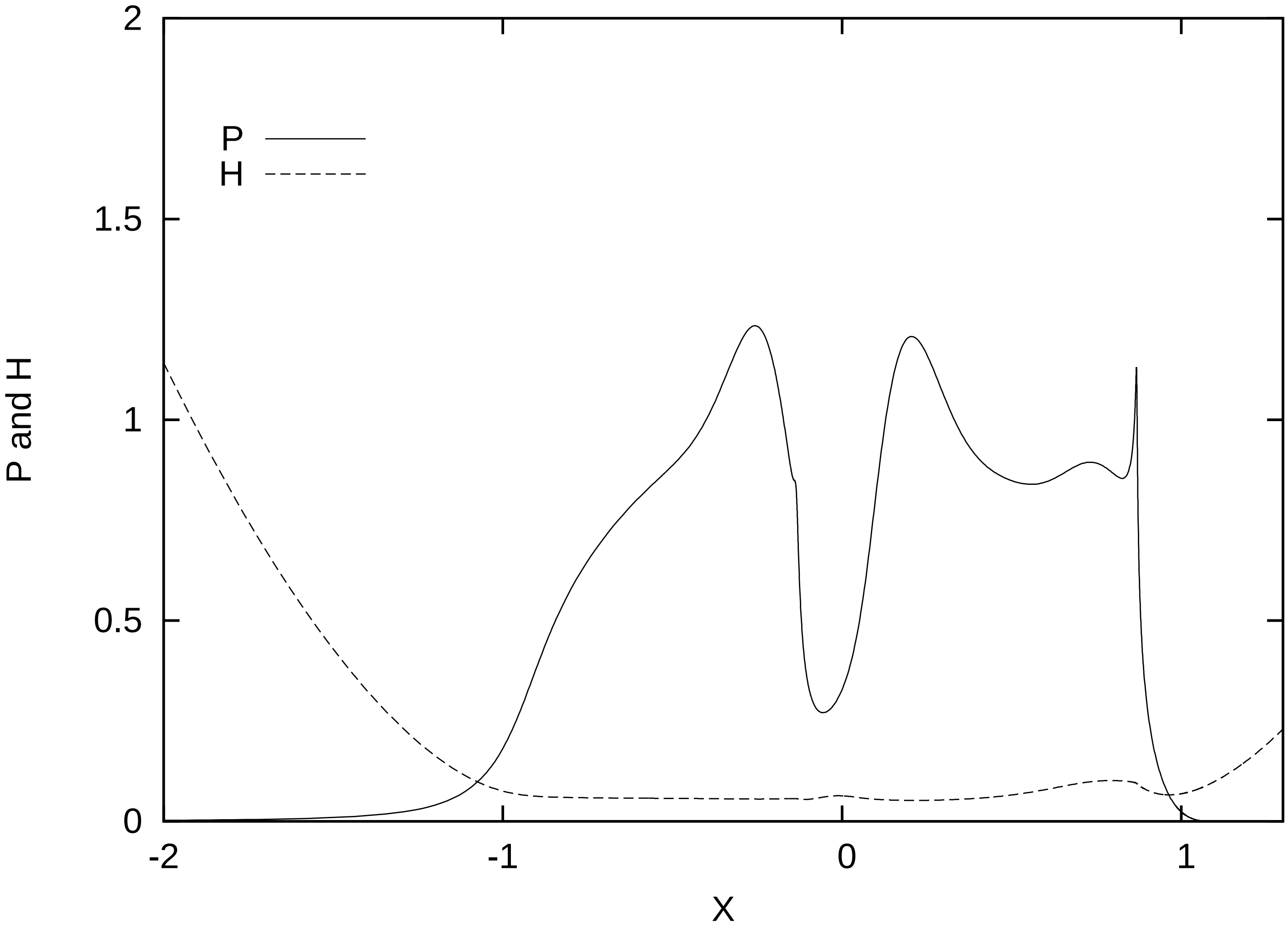

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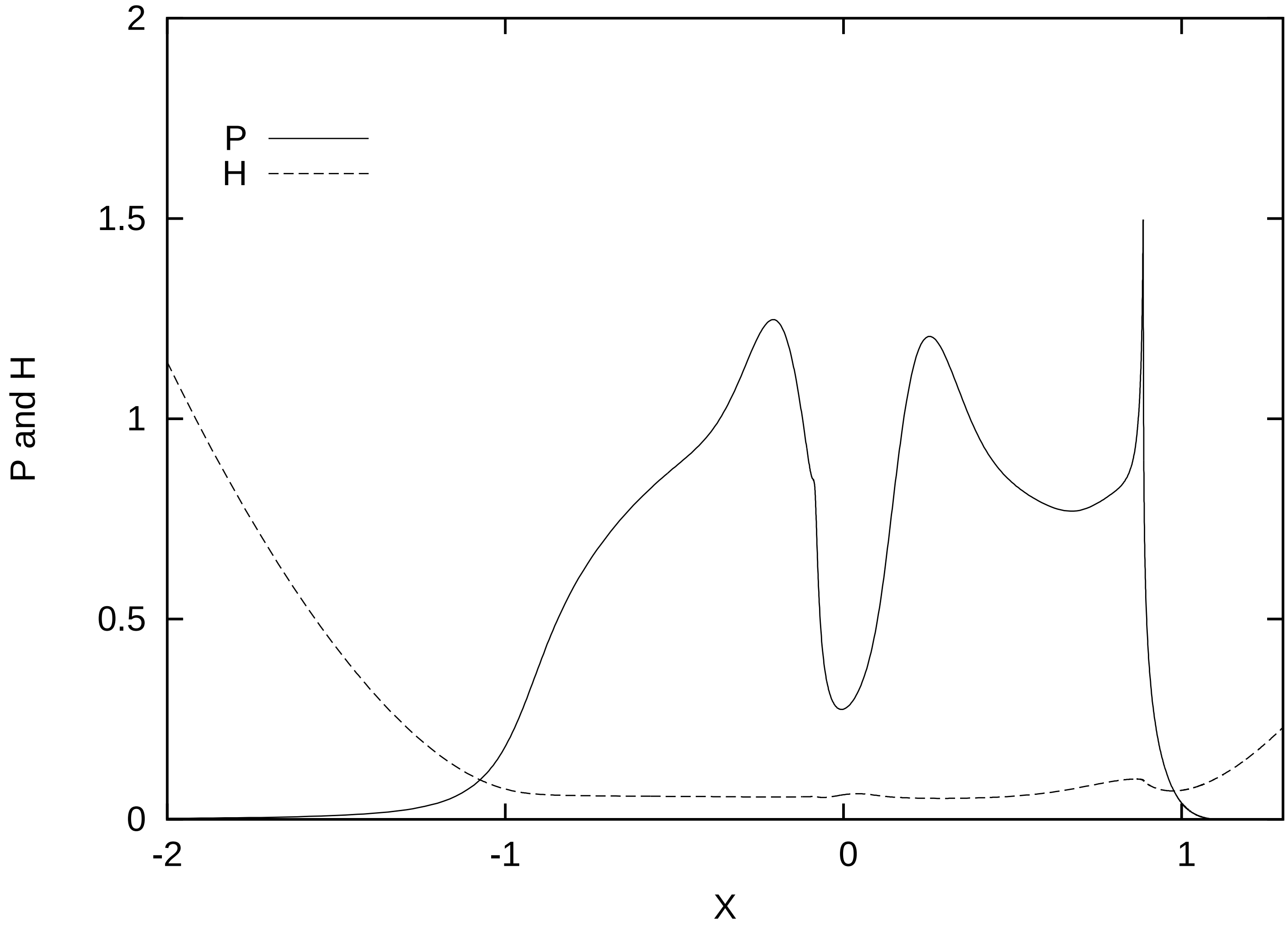

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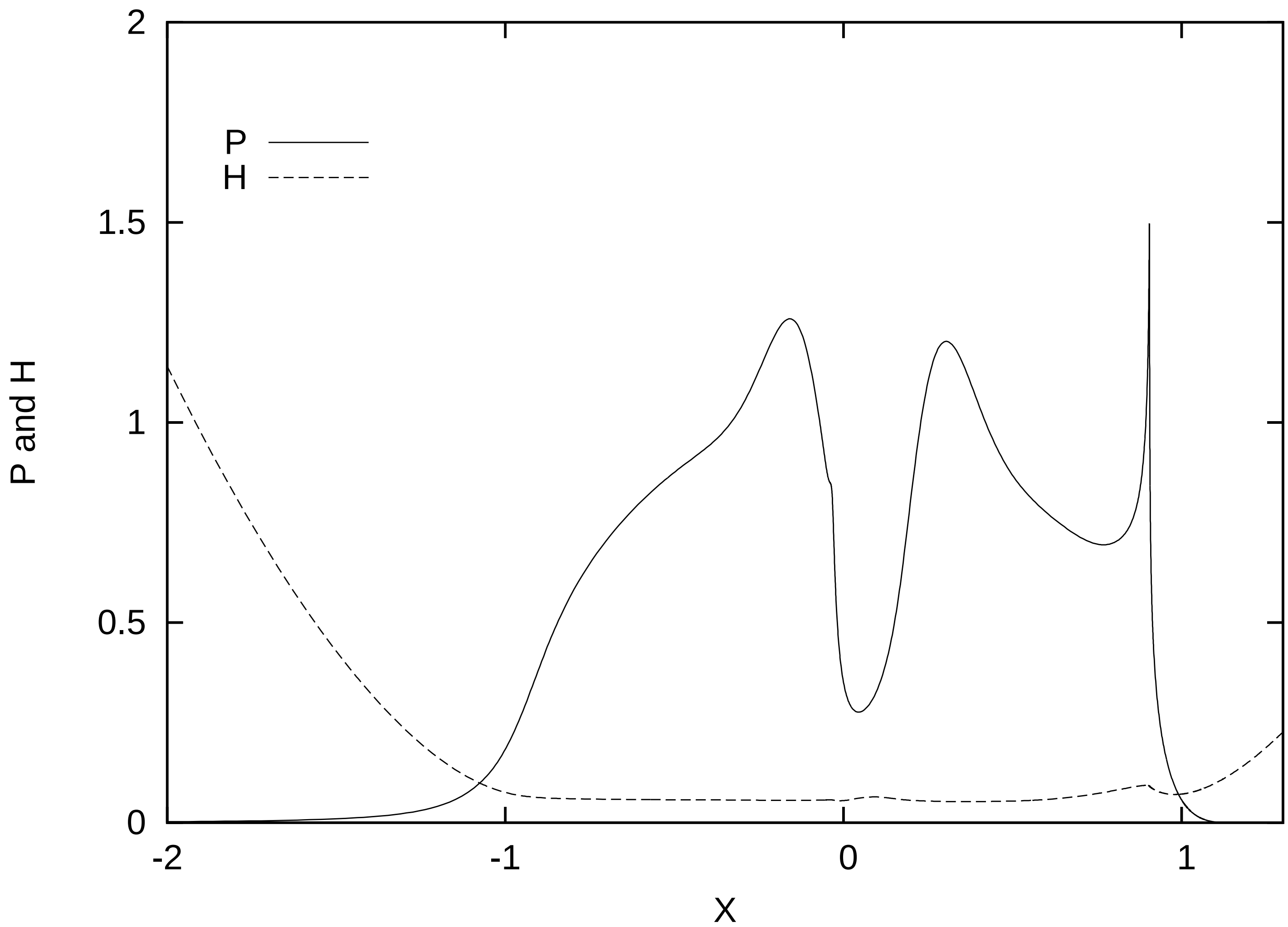

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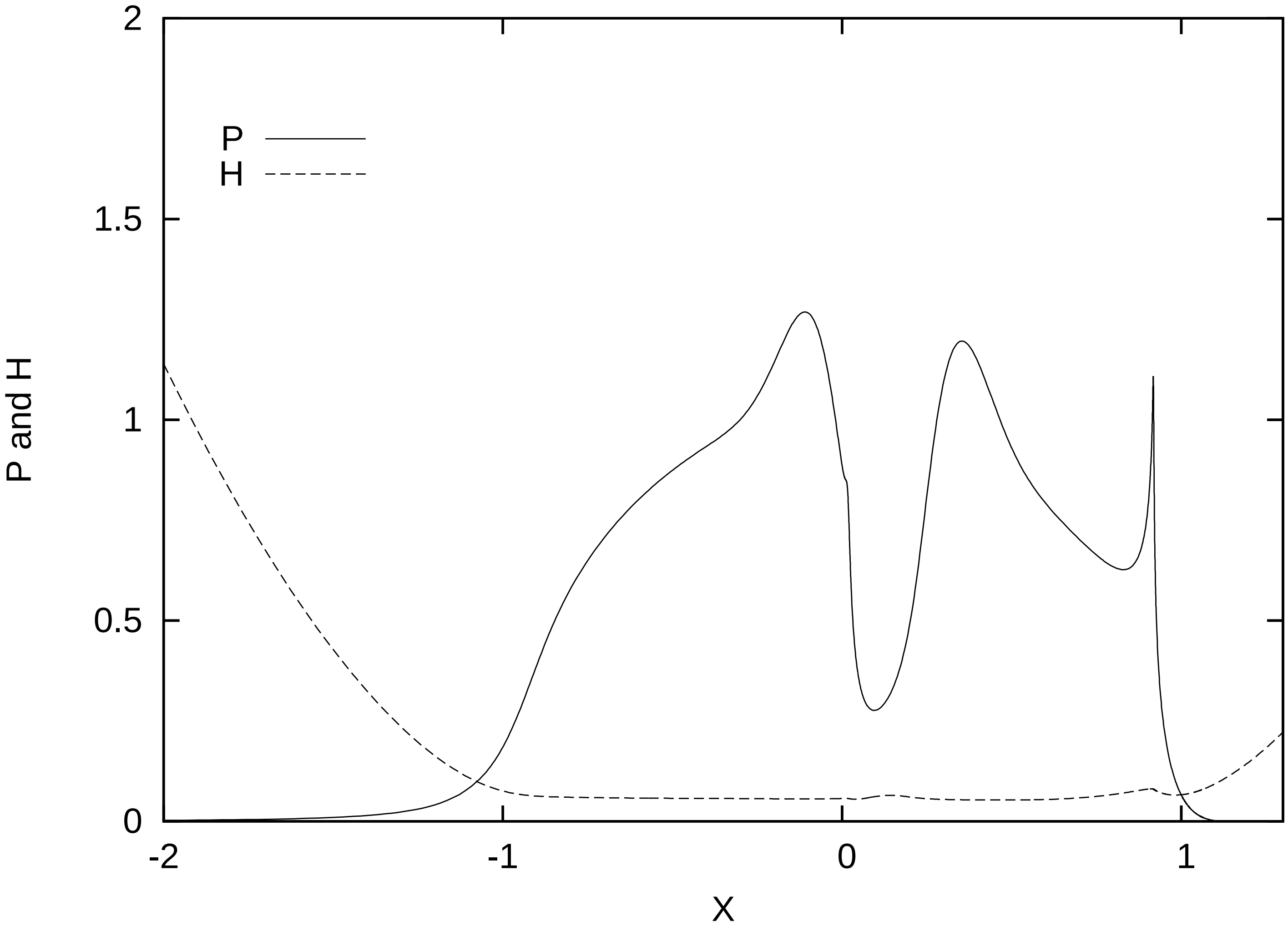

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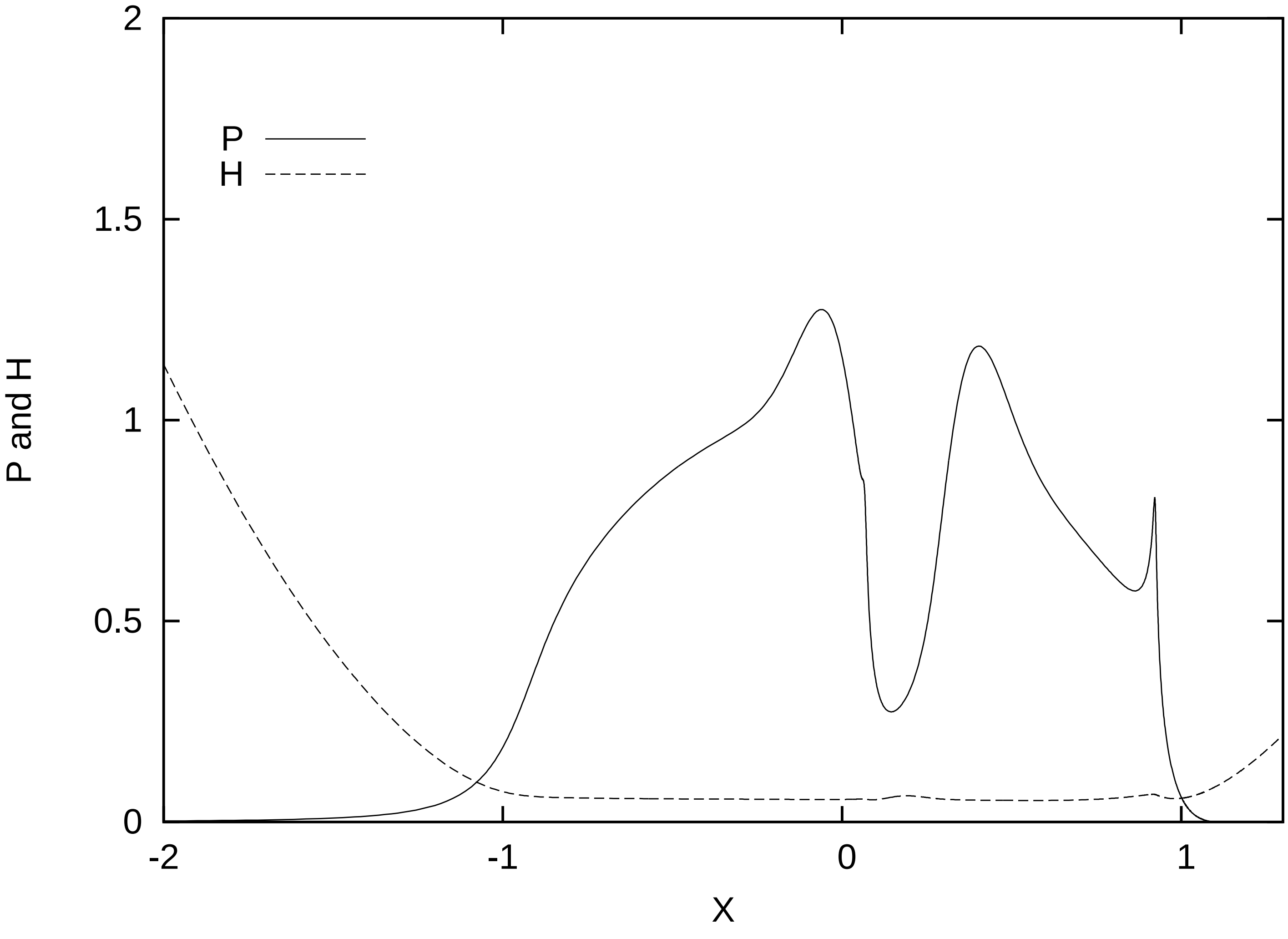

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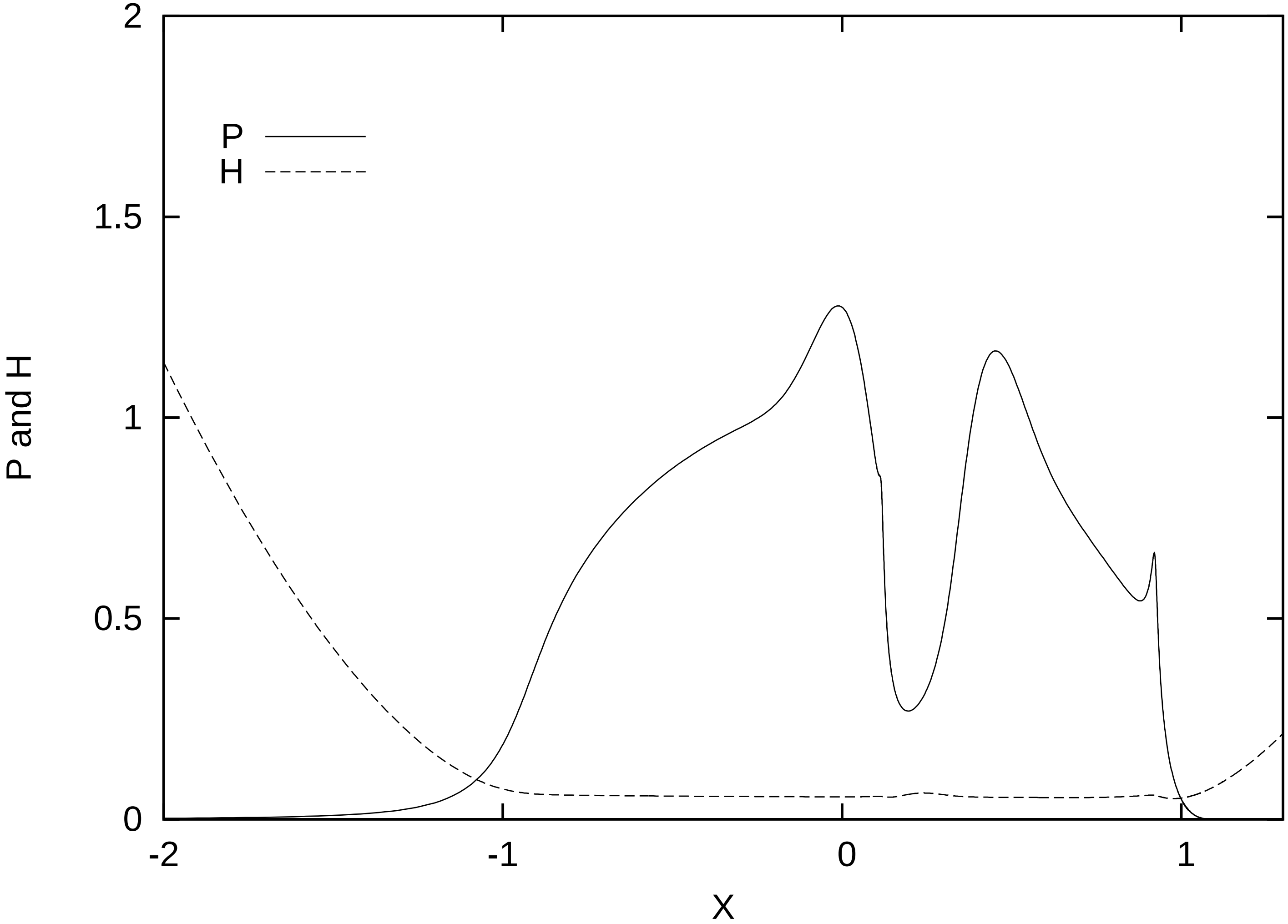

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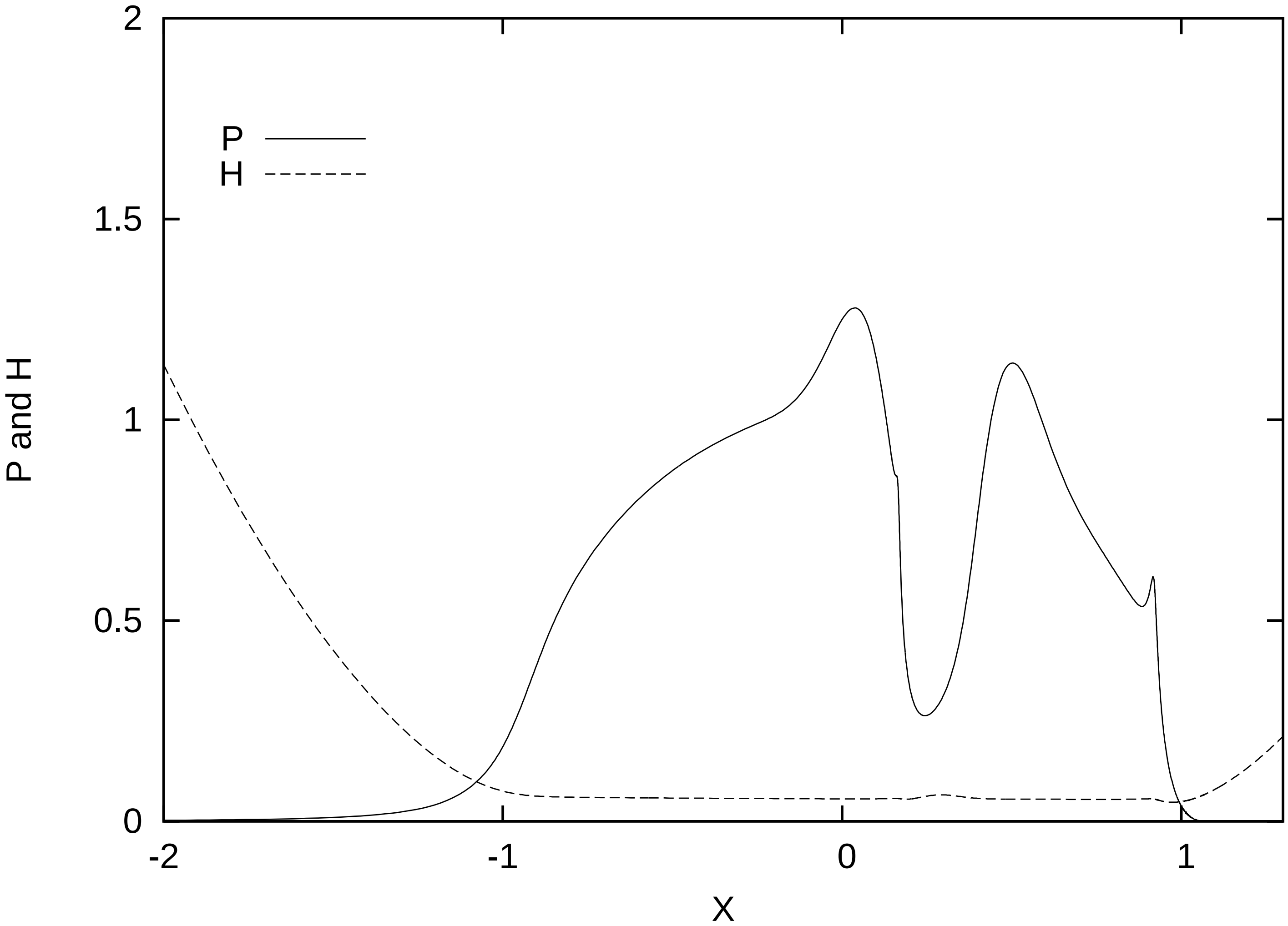

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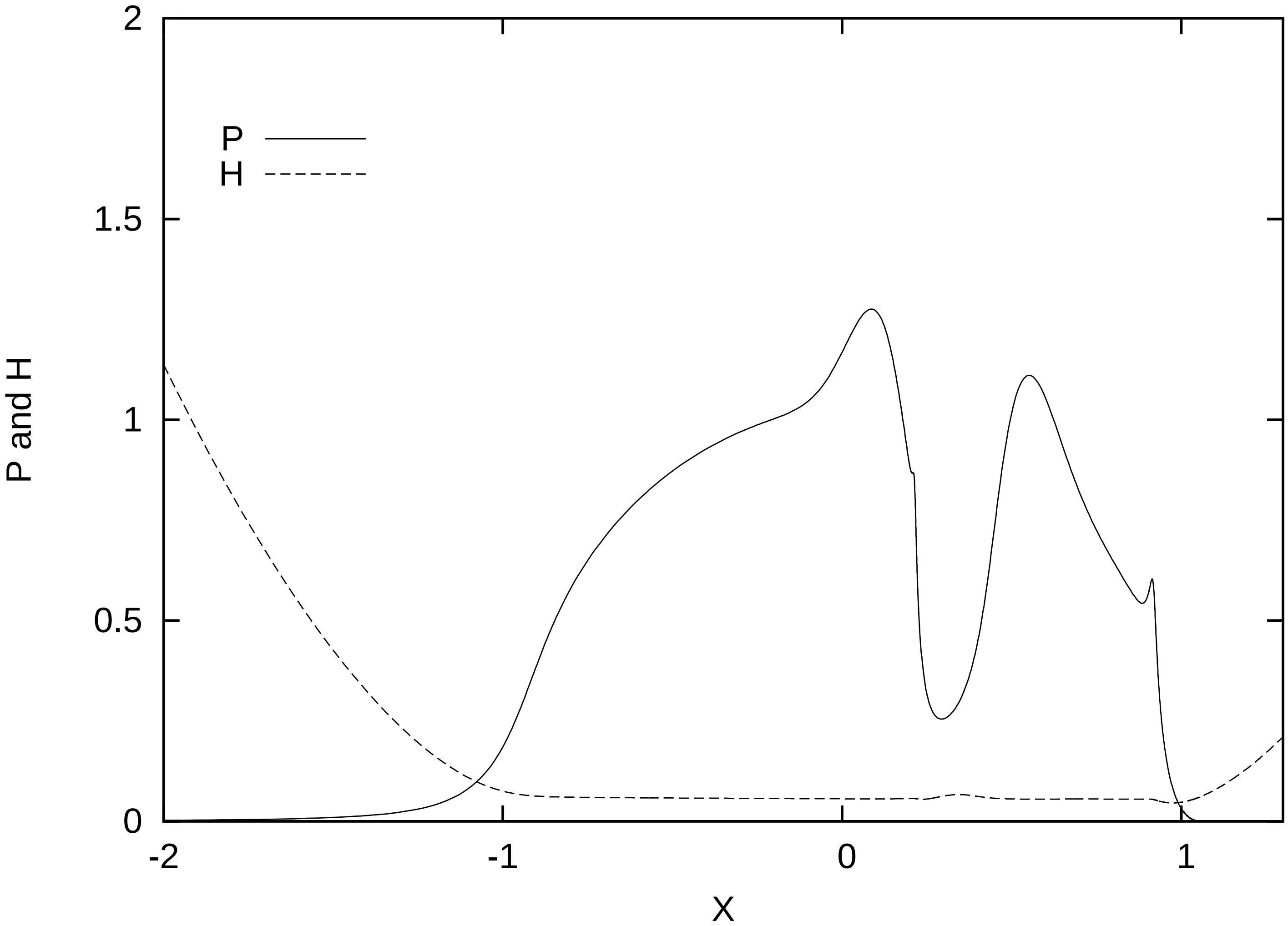

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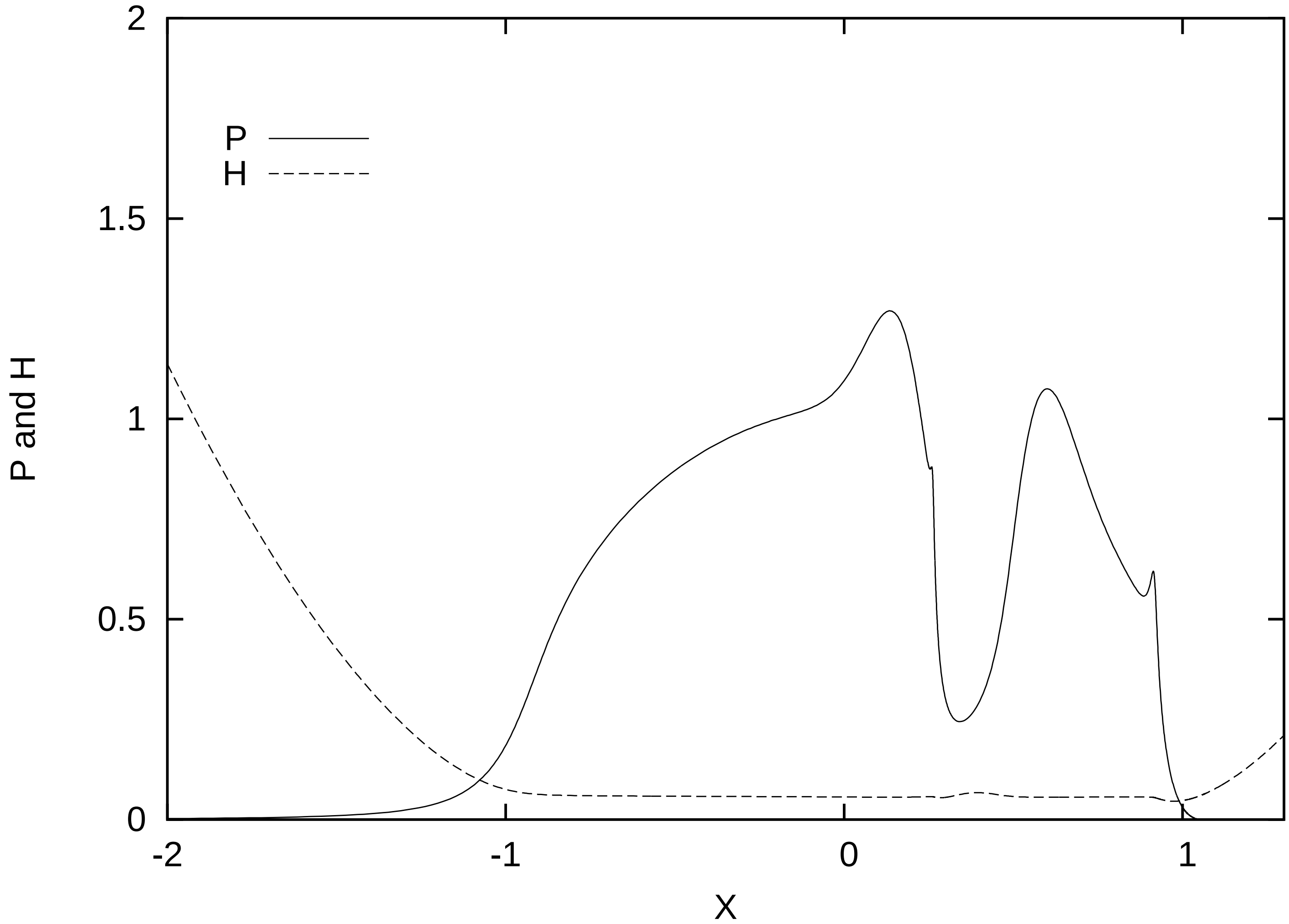

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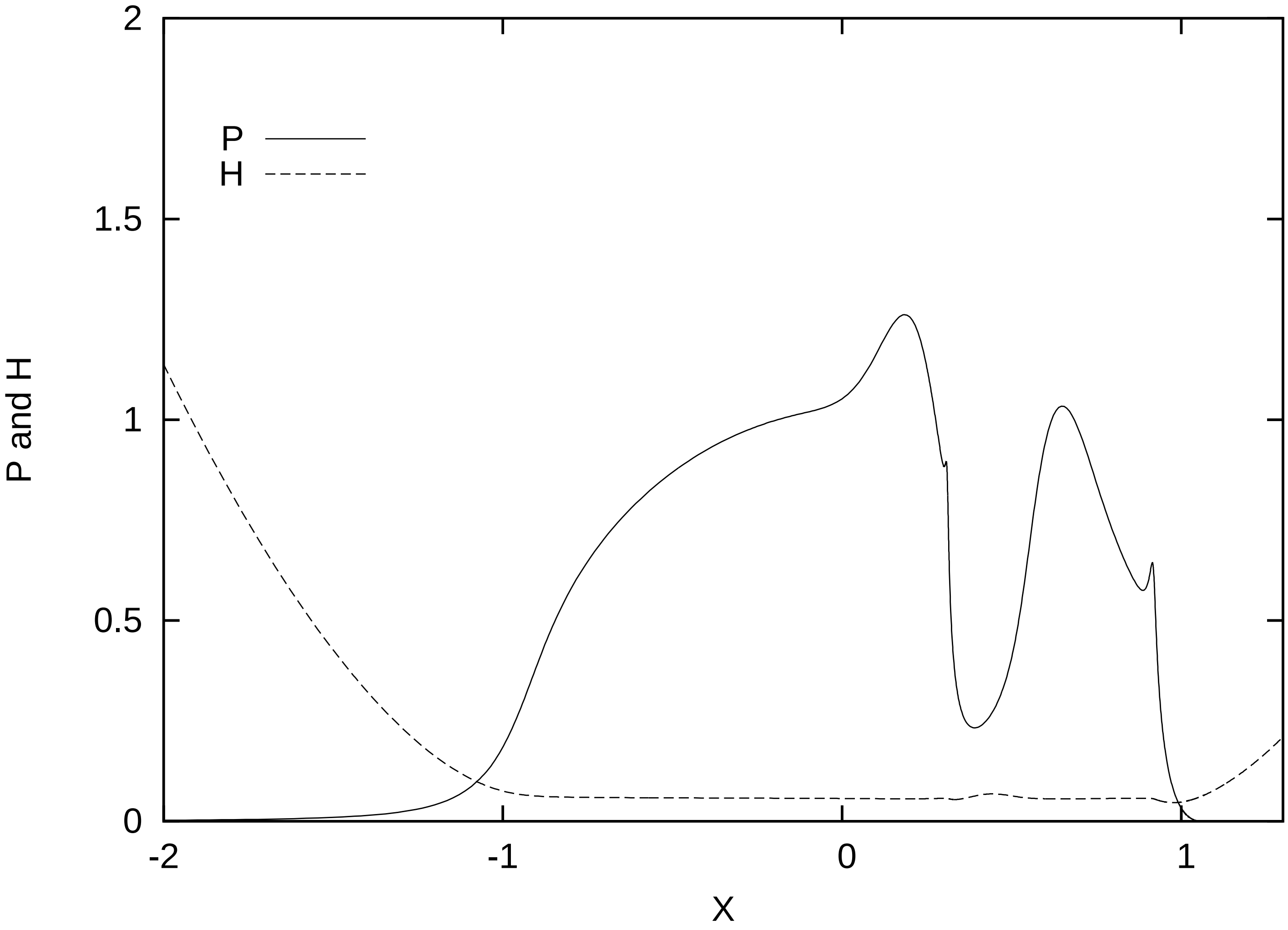

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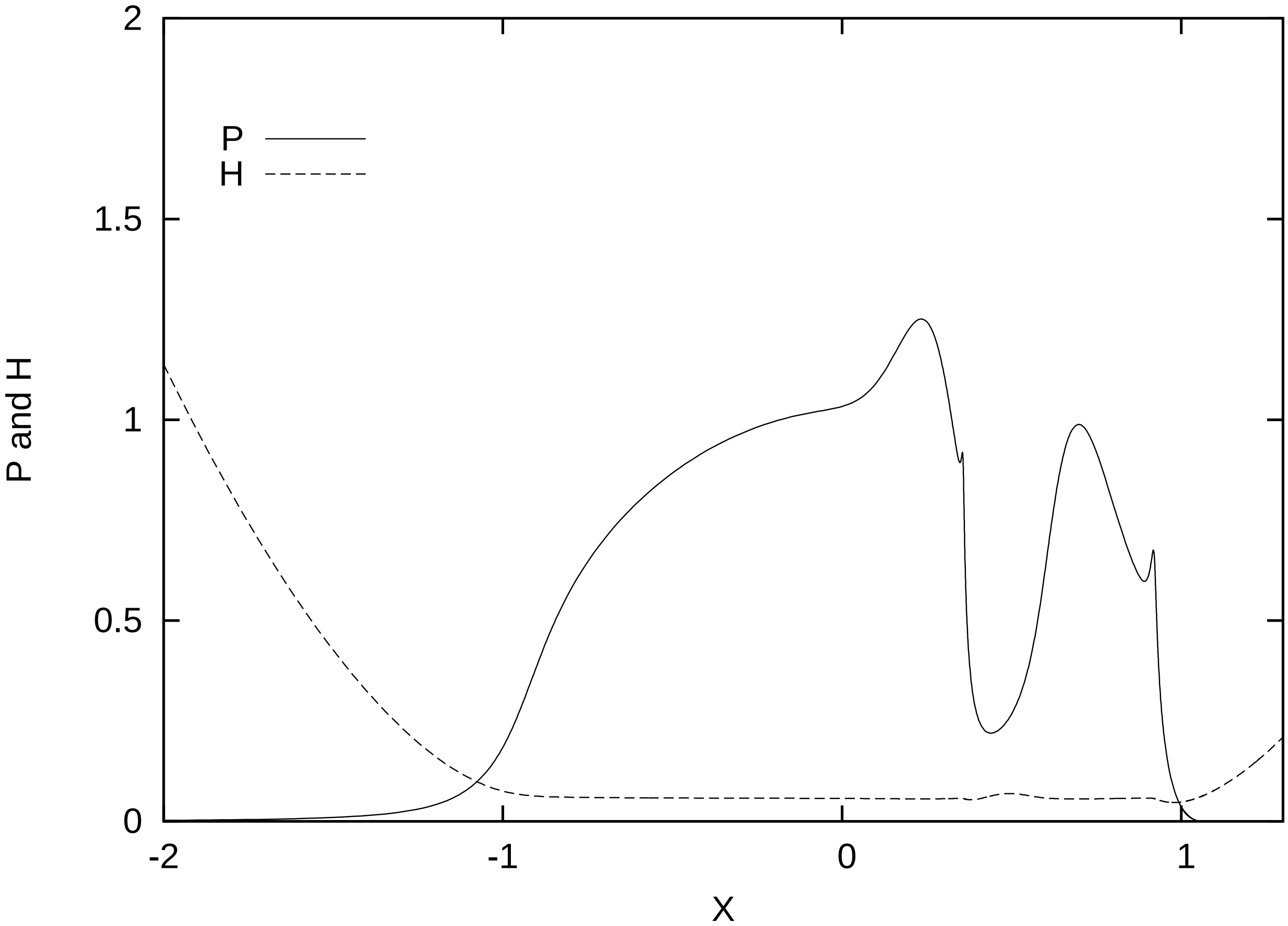

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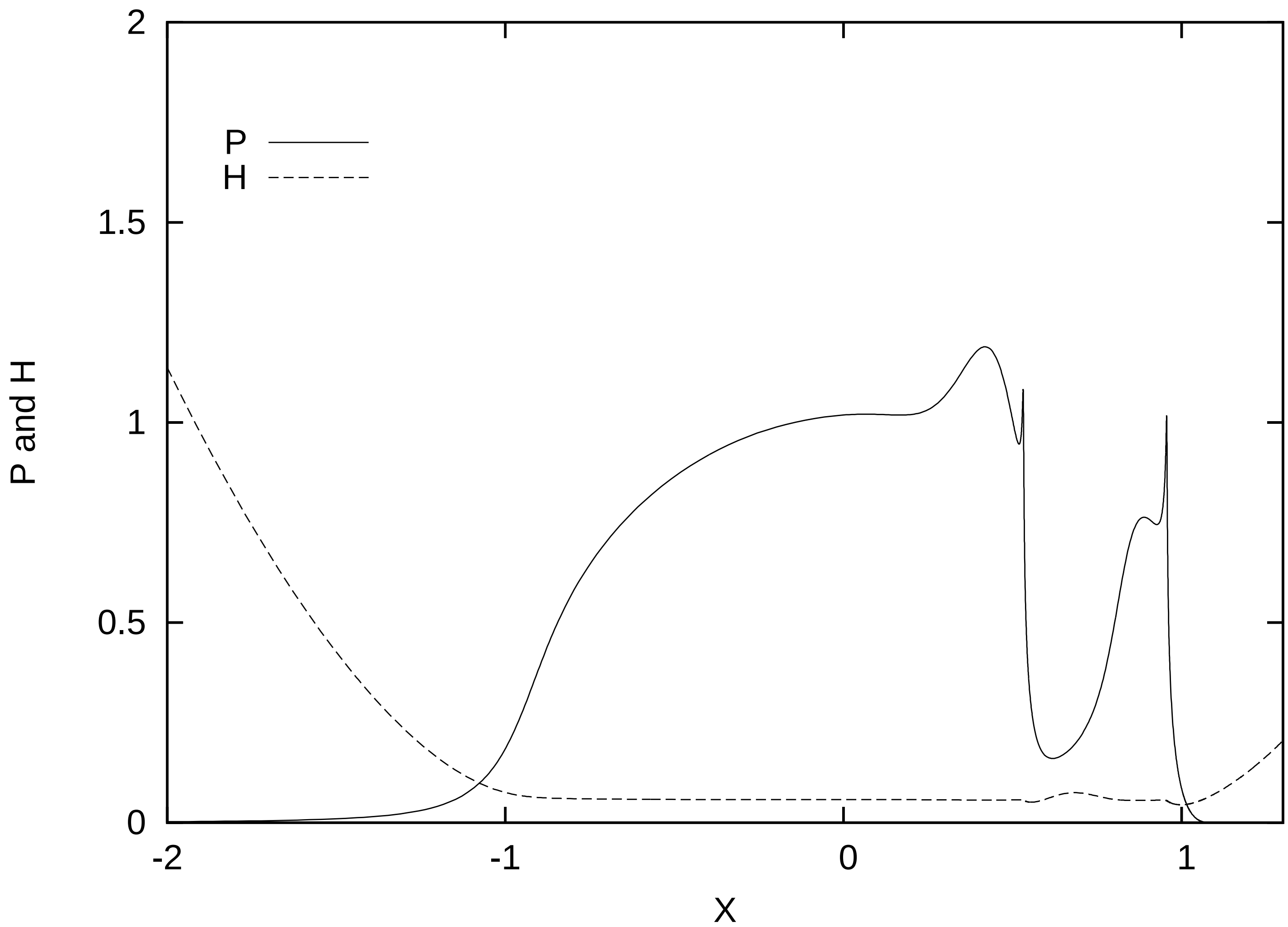

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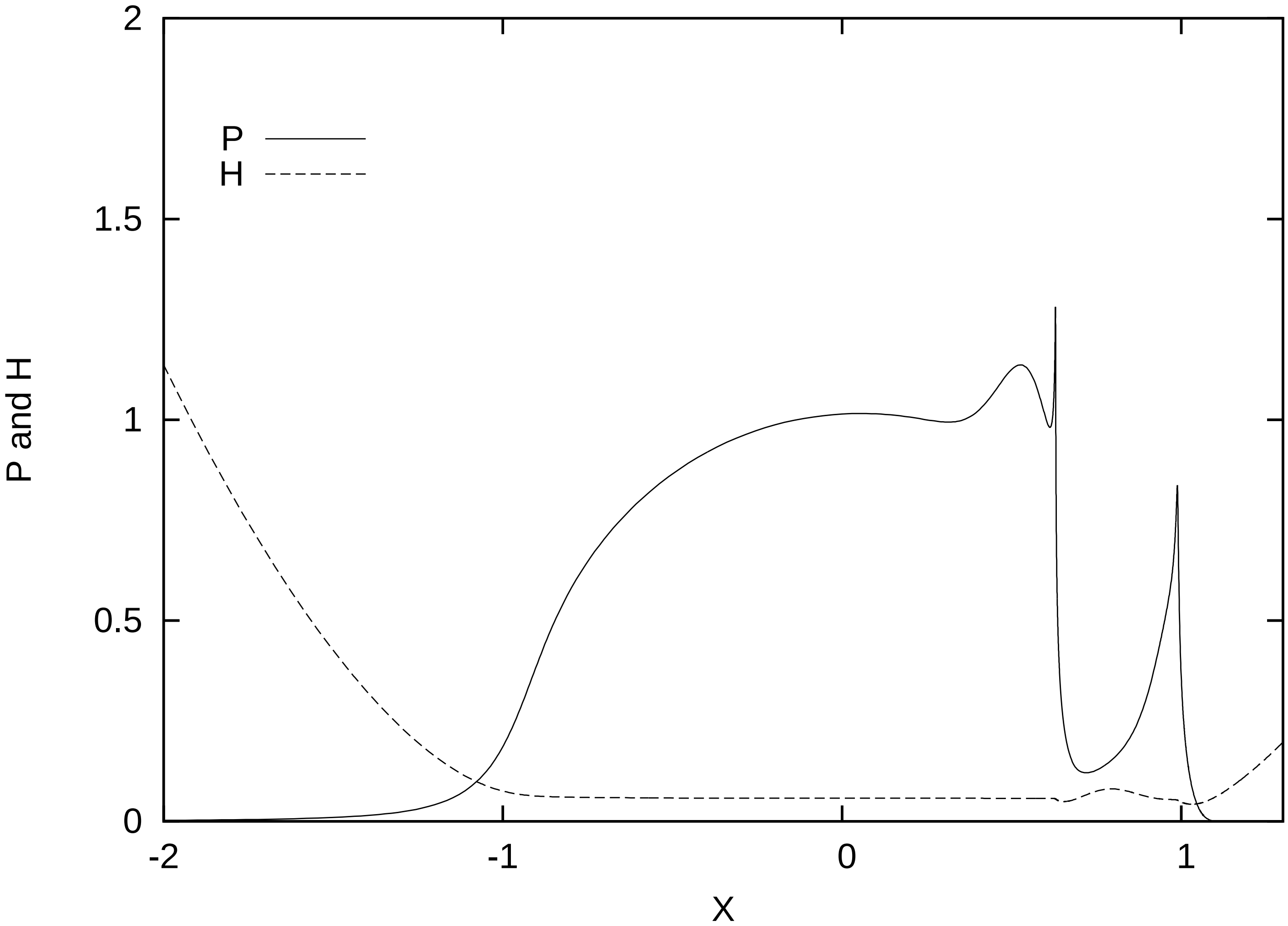

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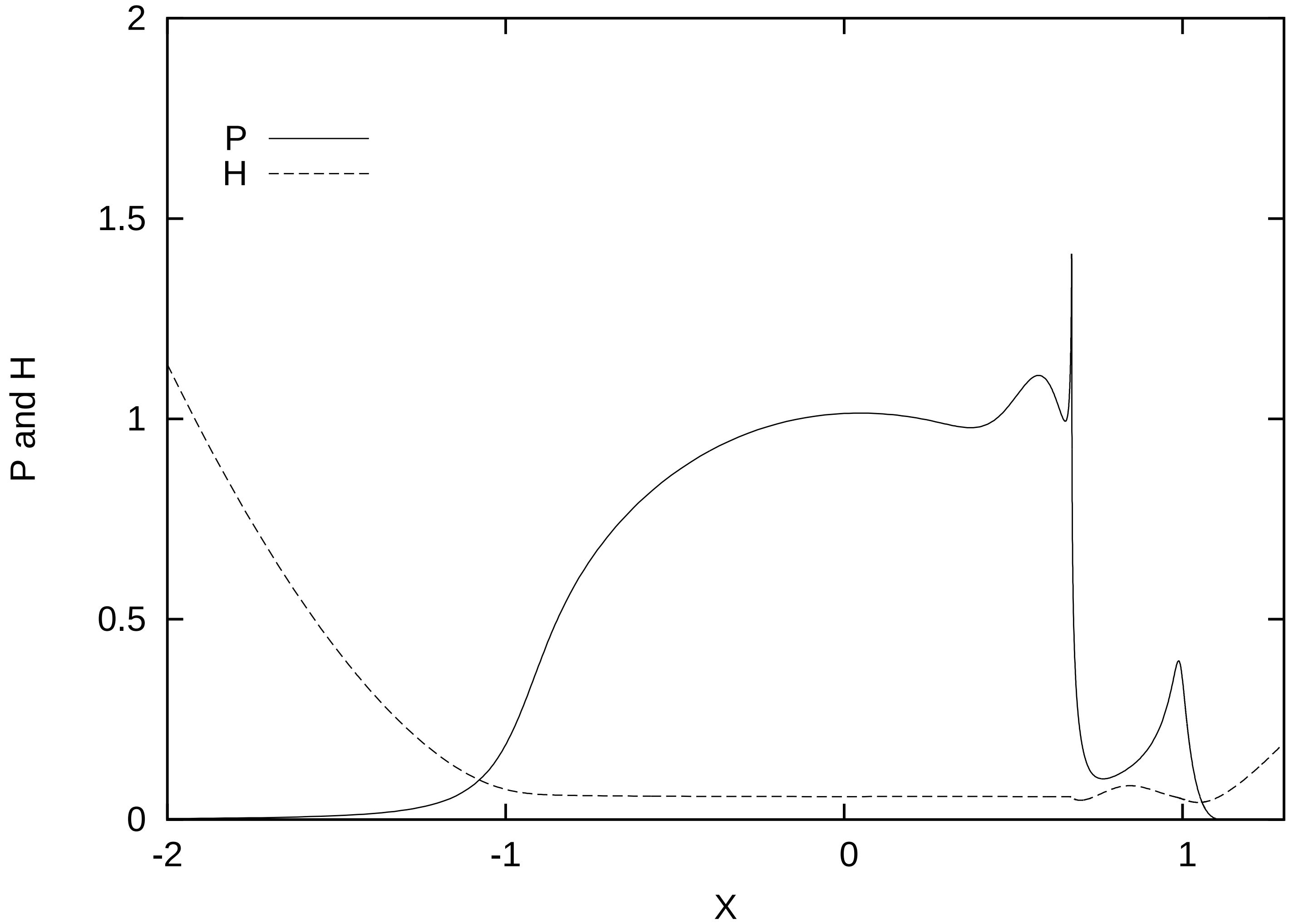

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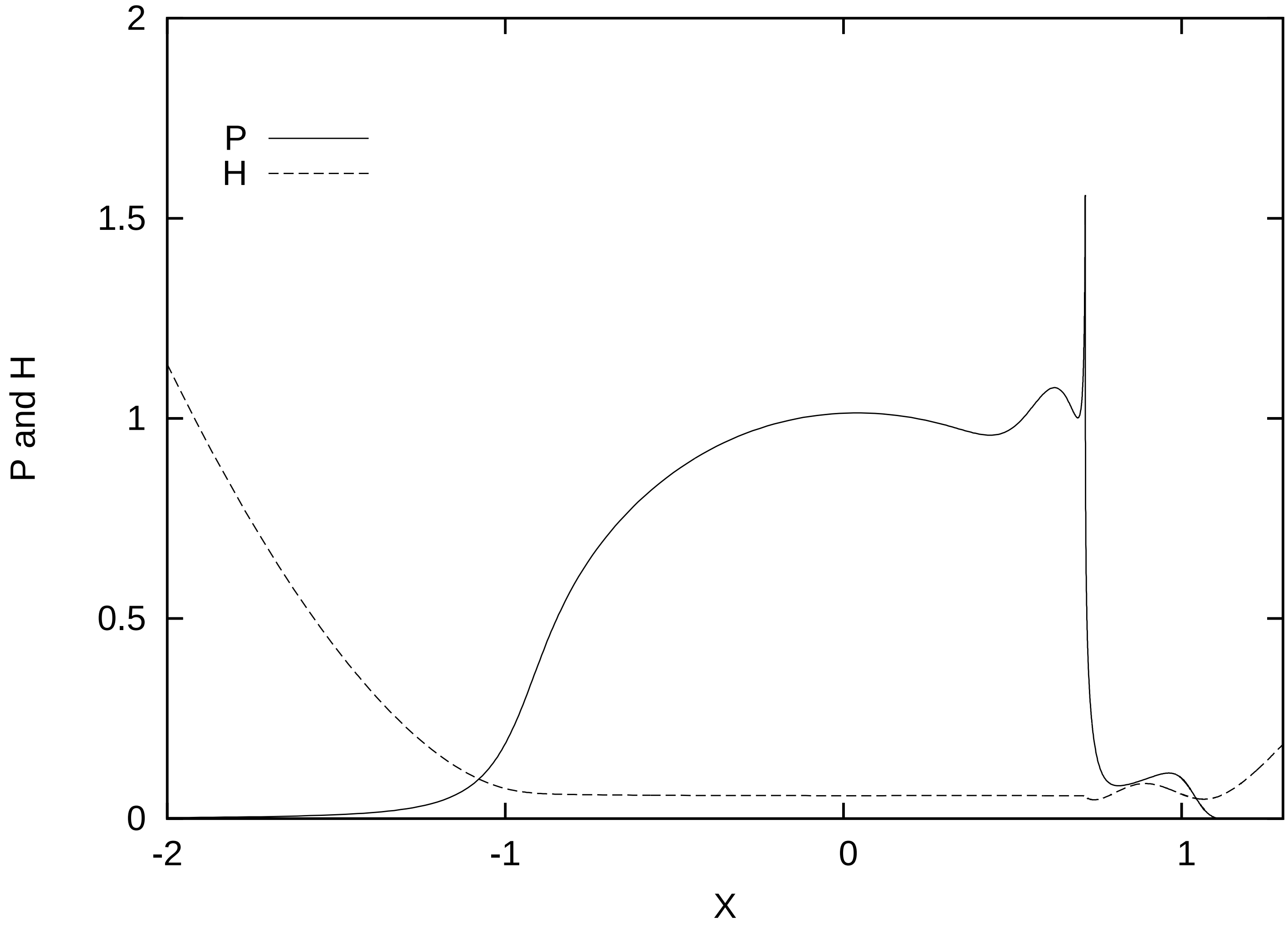

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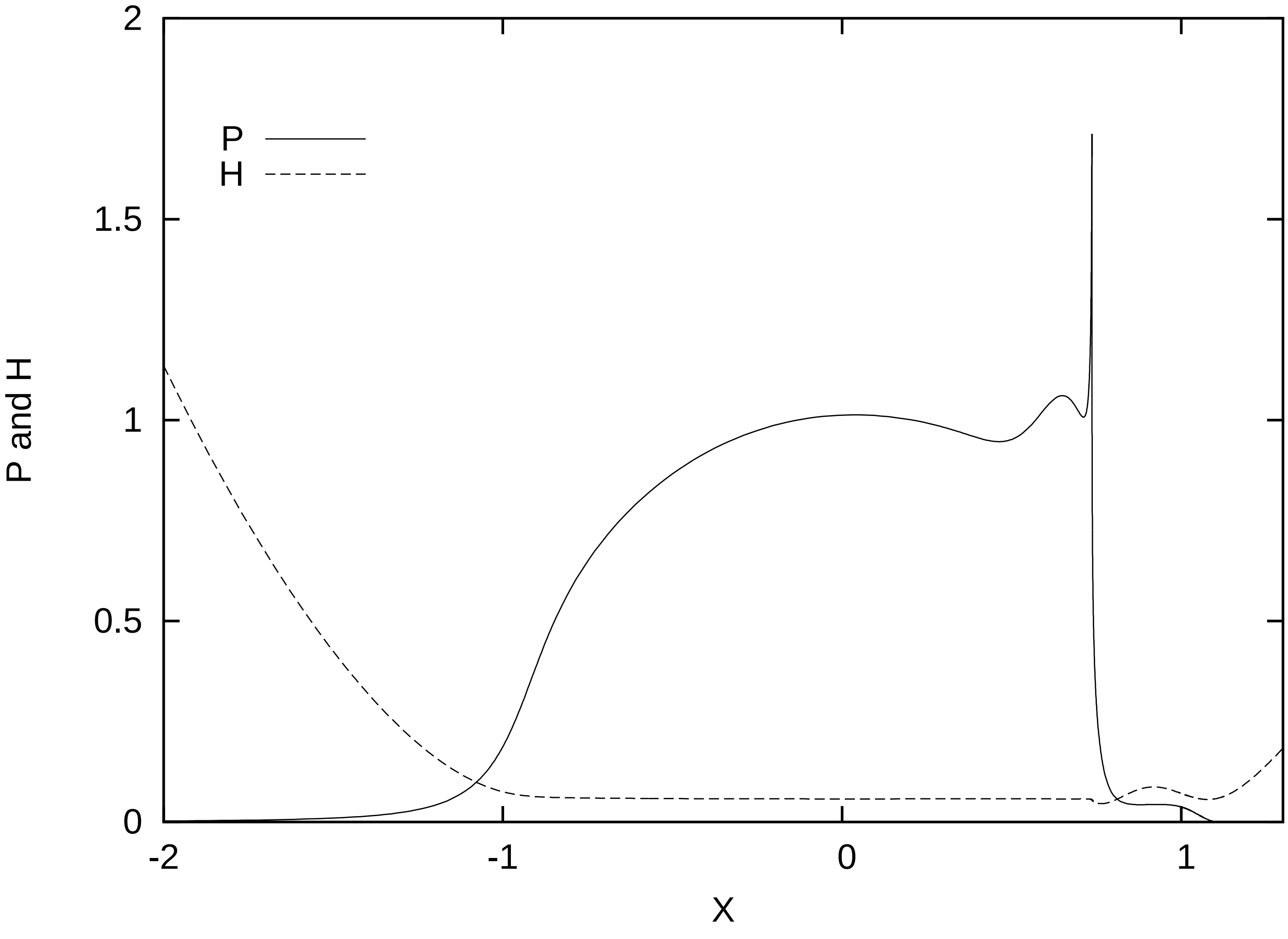

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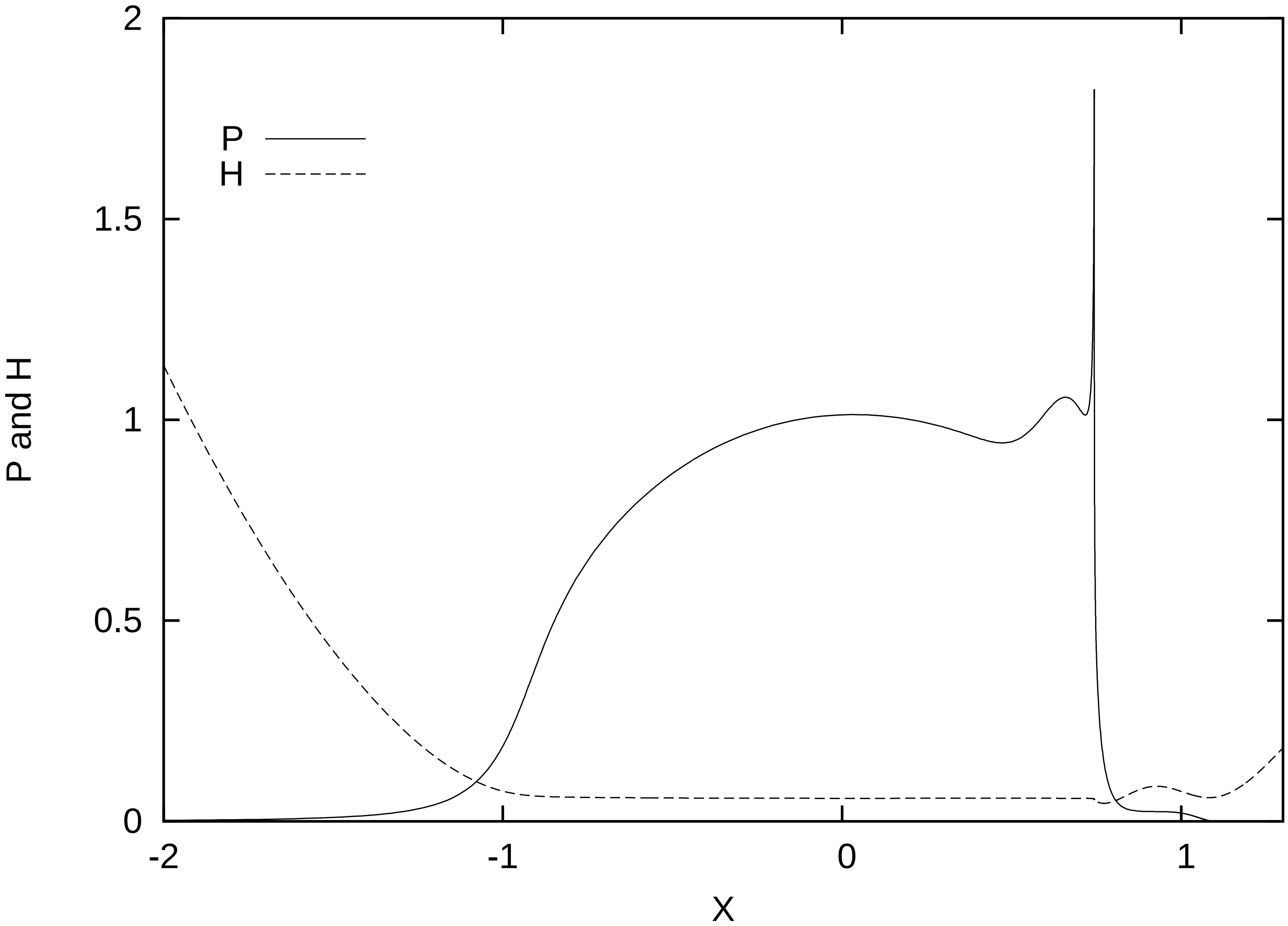


Steady Equations (2d)

Reynolds equation:

$$-\bigtriangledown \cdot (\varepsilon \bigtriangledown P) + \bigtriangledown \cdot (\beta \bar{\rho} H) = 0, \; (\beta = (1,0)^T)$$

Film thickness equation:

$$H(X,Y) = H_{00} + \frac{X^2}{2} + \frac{Y^2}{2} + \frac{2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{P(X',Y')}{\sqrt{(X-X')^2 + (Y-Y')^2}} \mathrm{d}X' \mathrm{d}Y'$$

Force balance equation:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(X,Y)\, dX' dY' - \frac{2\pi}{3} = 0$$

Cavitation boundary is now a unknown curve.

Discretization and Solution (2d)

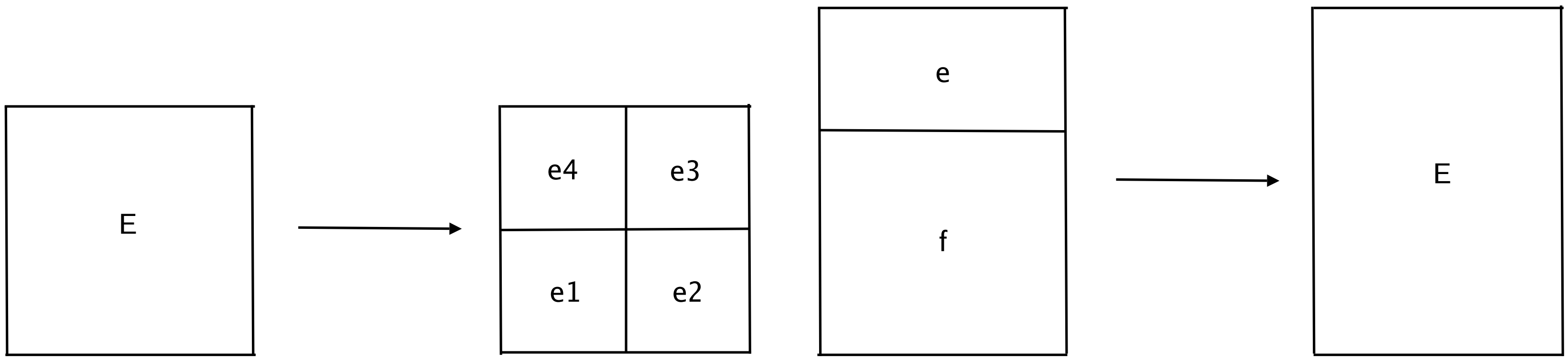
Generalization of the 1d approach...

- High order DG plus Penalty method for Reynolds equation and cavitation boundary.
- P-version of FAS (nonlinear) multigrid for solver.
- Smoother based upon sparse approximate Jacobian.
- Film thickness evaluated using pre-computed kernels.
- Adaptive h-refinement (see below).

H-Adaptivity

A simple error indicator is provided by considering

$$E^e = \left\| \sum_{i=n(p^e-1)}^{n(p^e)} u_i^e N_i^e(X) \right\|_2 = \sqrt{\int_{\Omega_e} \left(\sum_{i=n(p^e-1)}^{n(p^e)} u_i^e N_i^e(X) \right)^2 \mathrm{d}X}$$

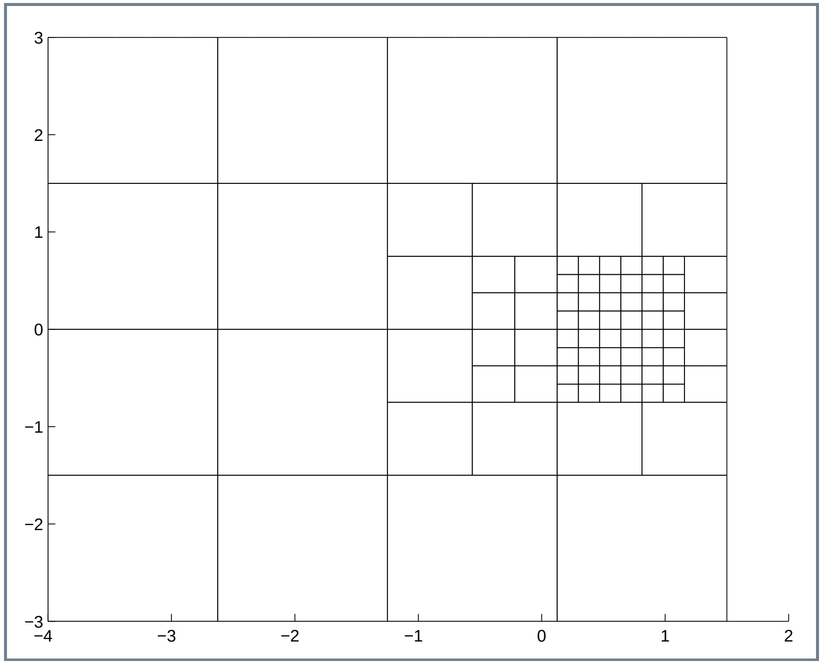


(a) Mesh refinement (b) Mesh Coarsening

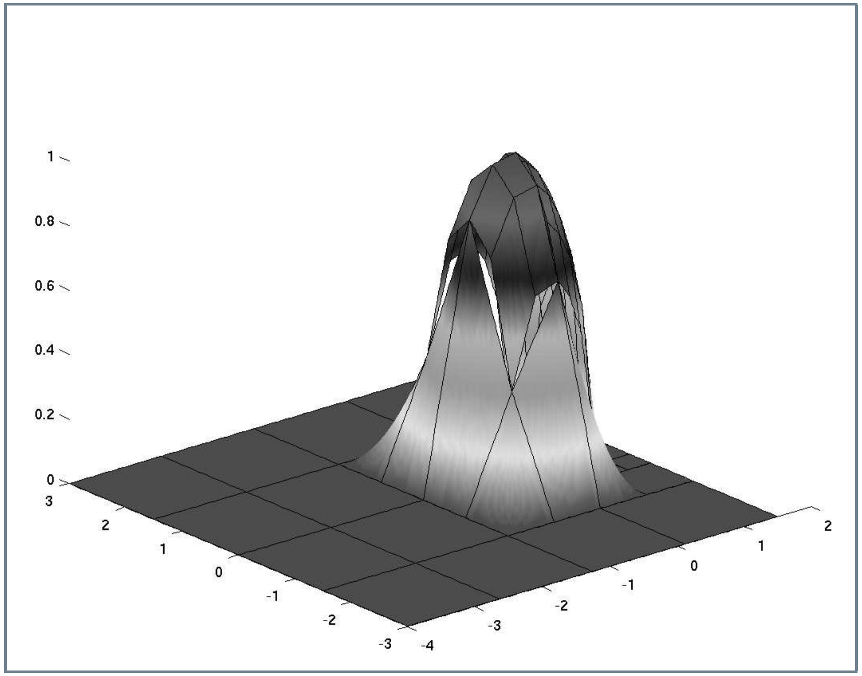
Overall Solution Procedure

1. Give an initial grid and ensure that this grid covers the pressurized domain.
2. Initialize the pressure on the given grid. Give an initial guess for H_{00}.
3. Calculate the kernels.
4. Perform 1 or 2 V-cycles on the current grid to update the solution. H_{00} is updated on the finest level.
5. Check if the grid needs to be adjusted.
6. Stop if the grid does not need to be adjusted and the residual is smaller than a given tolerance (10^{-10} say).
7. Adjust the grid if needed and transfer the current pressure profile from the old grid onto the new grid. Calculate the kernels related to those new elements. Go to 3.

Initial Grid and Pressure Profile

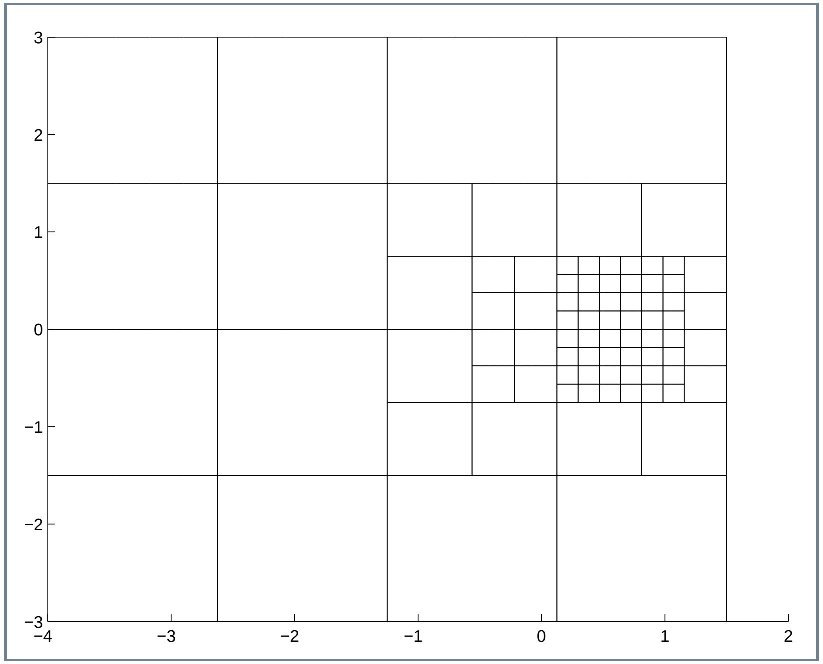


Initial grid

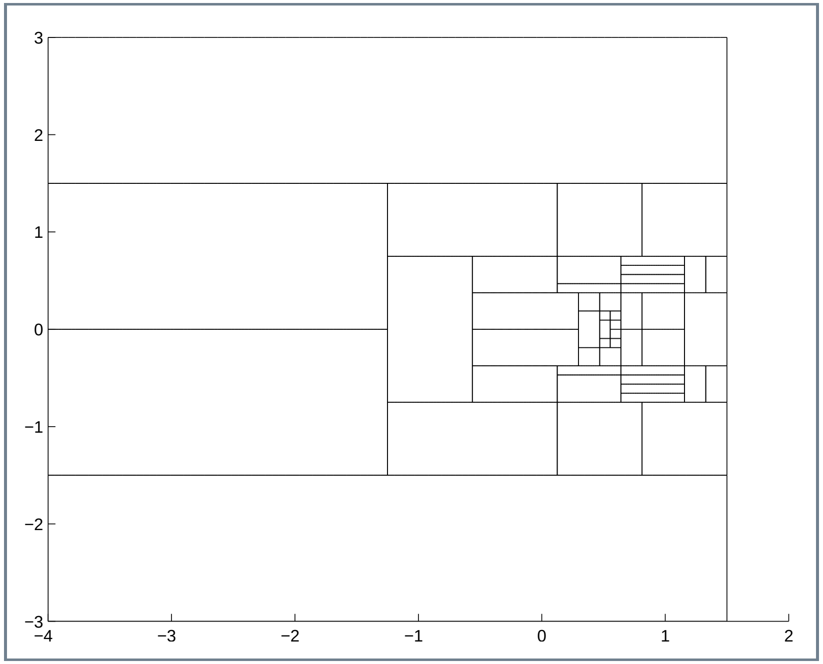


Initial pressure profile

Slightly Loaded Case

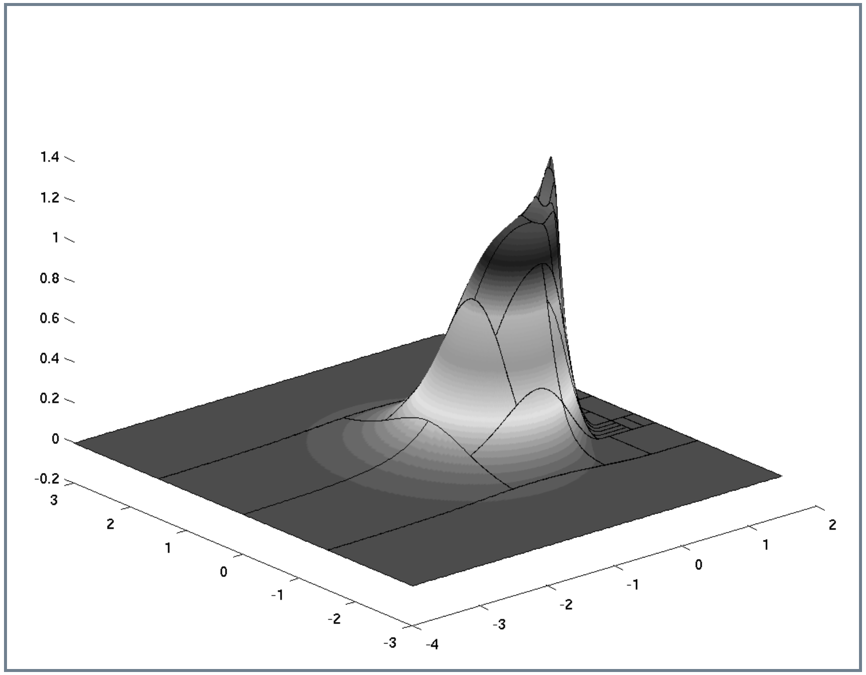


(a) Initial grid

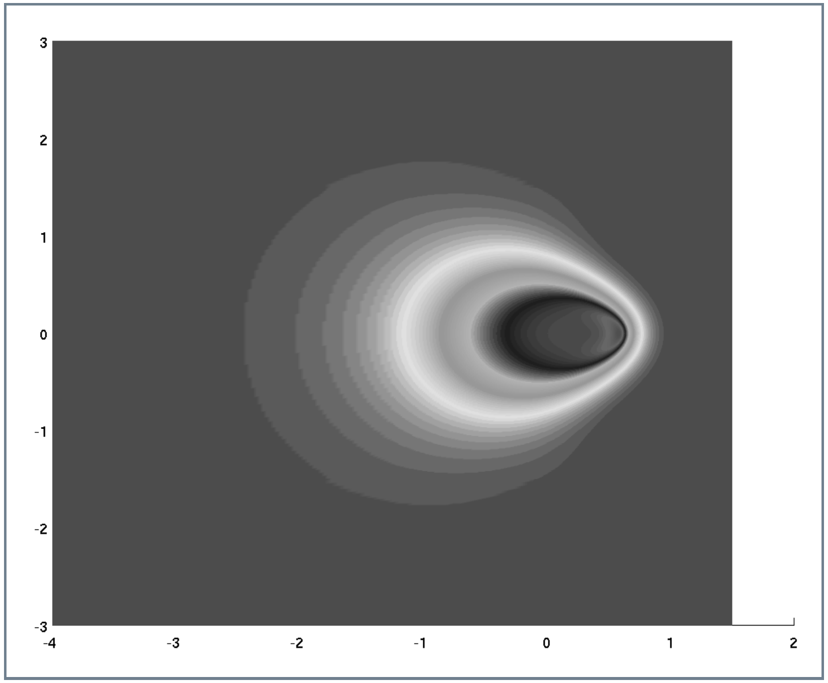


(b) Final grid

Slightly Loaded Case

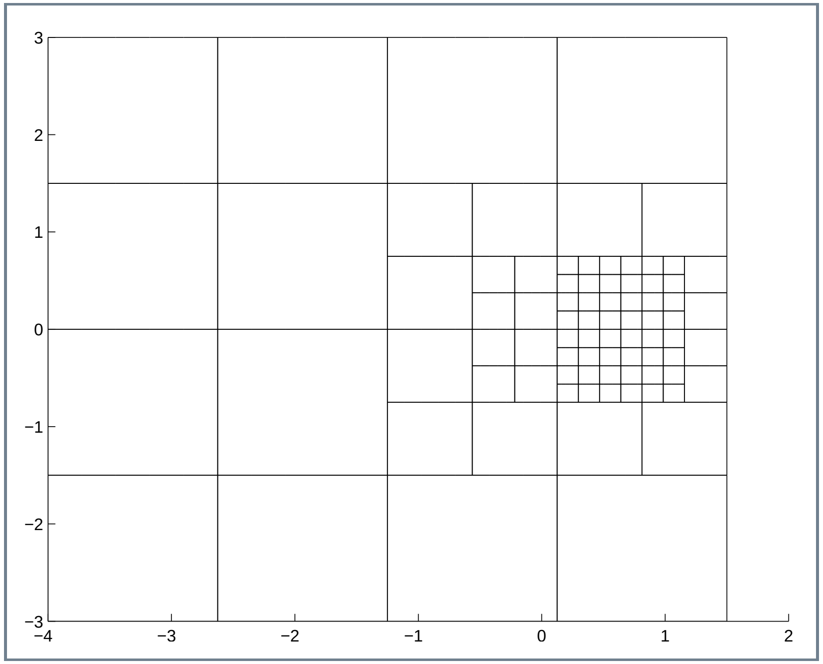


(a) Converged pressure profile

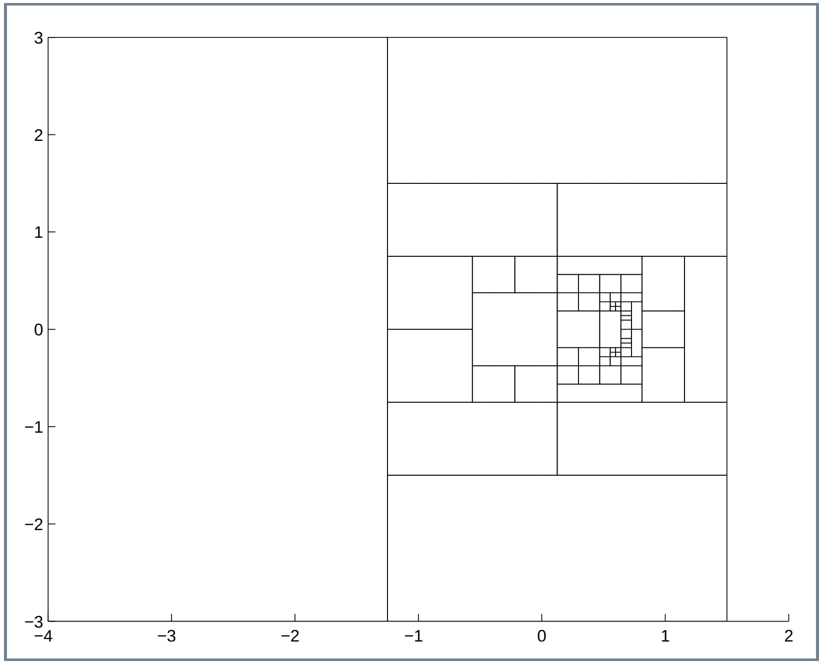


(b) Viewed from above

Moderately Loaded Case

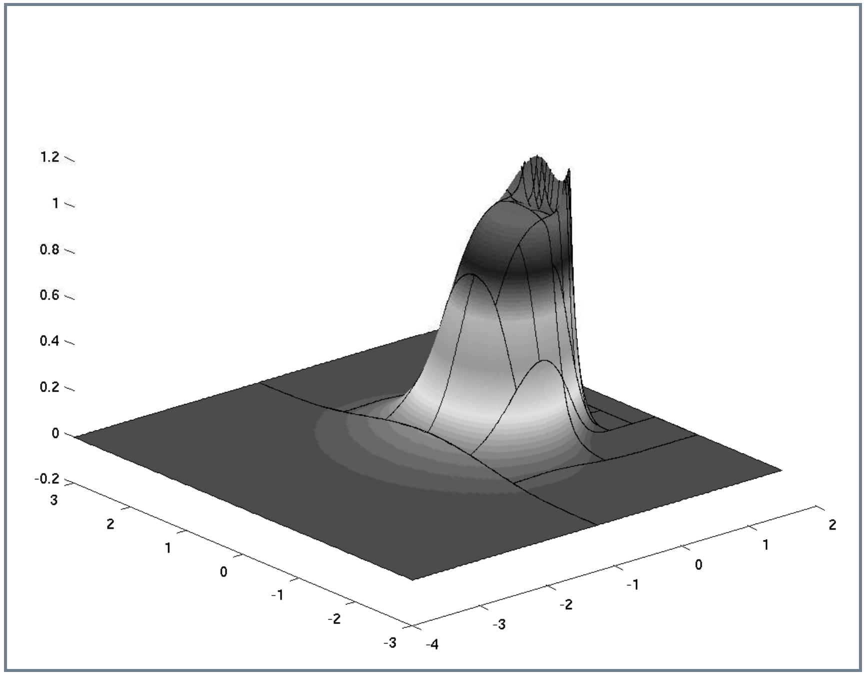


(a) Initial grid

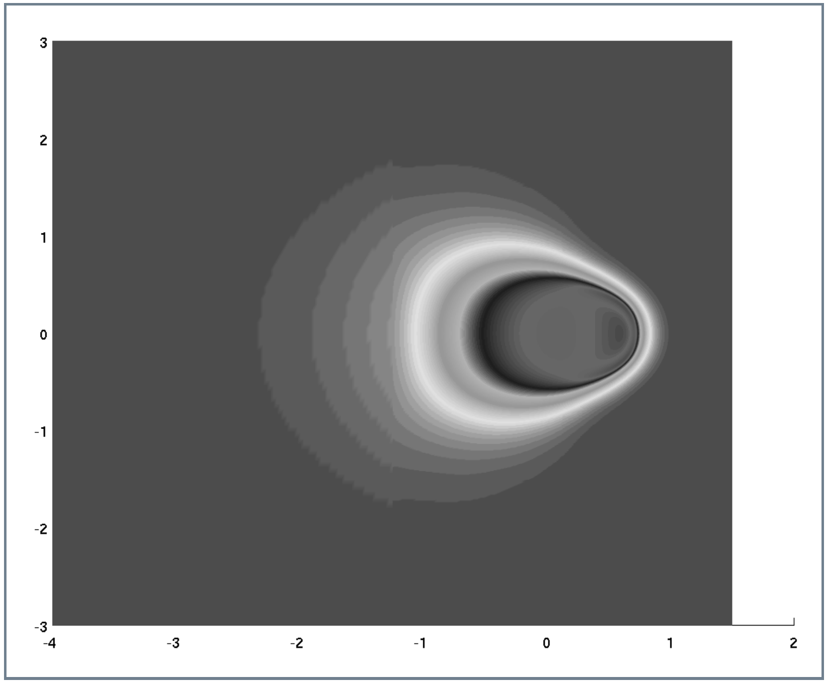


(b) Final grid

Moderately Loaded Case

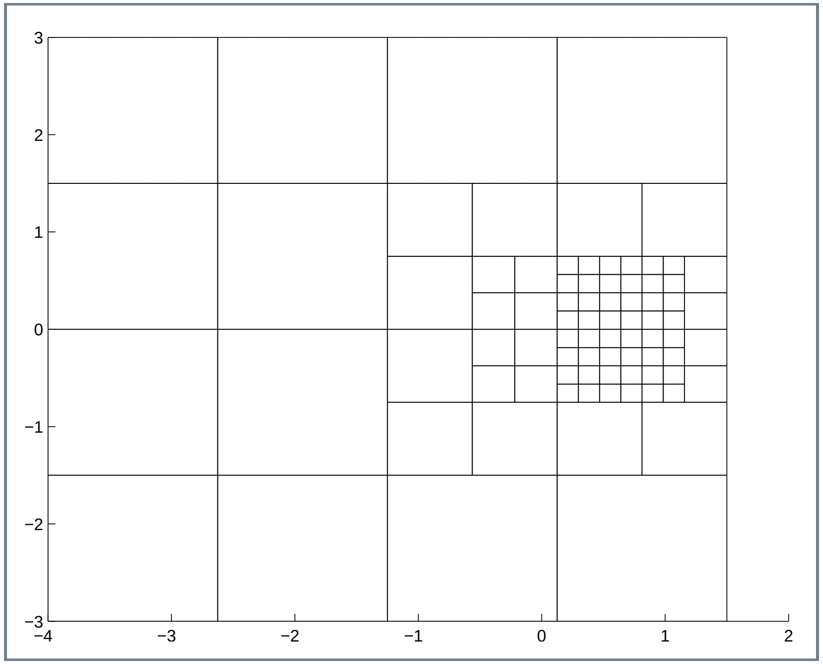


(a) Converged pressure profile

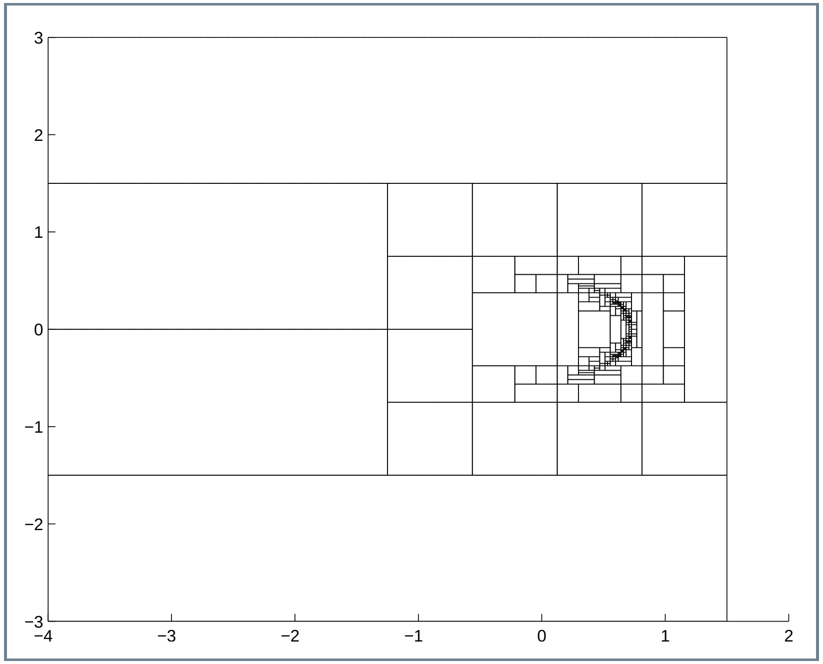


(b) Viewed from above

Highly Loaded Case

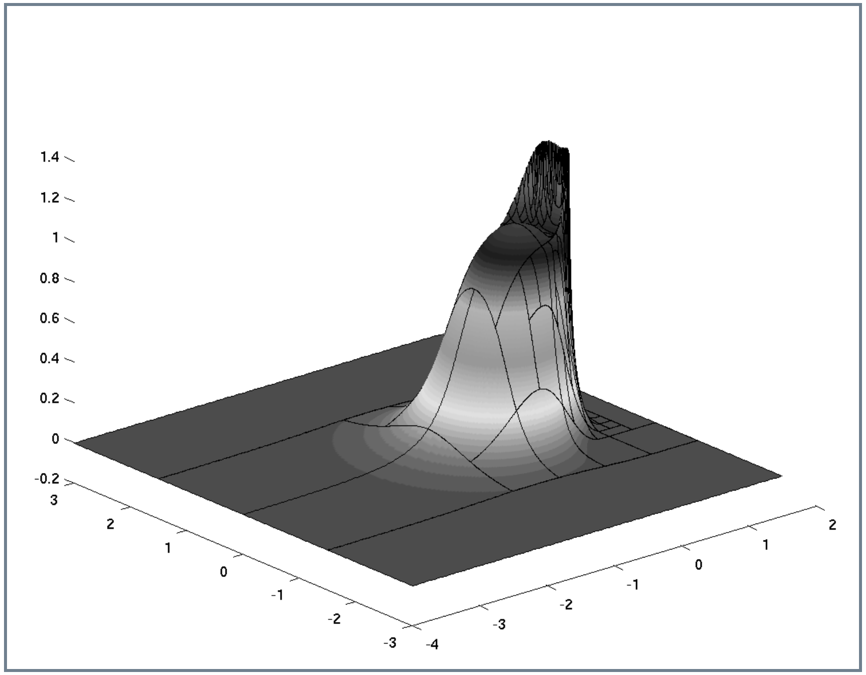


(a) Initial grid

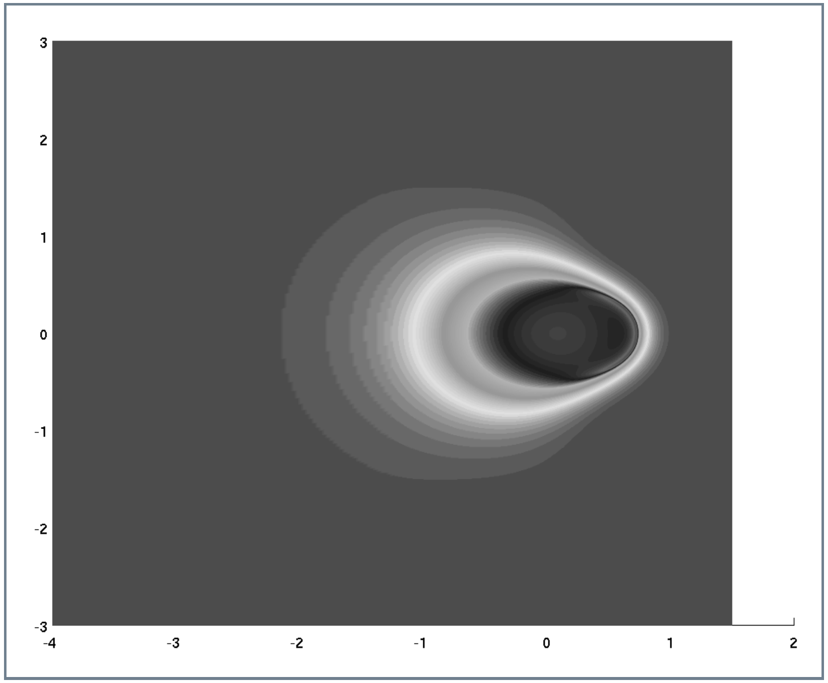


(b) Final grid

Highly Loaded Case



(a) Converged pressure profile



(b) Viewed from above

Discussion

Considerable research effort required...

1. The computation of the film thickness is still relatively expensive compared to the overall solution time.
2. Parallel implementation?
3. Transient point contact problems.
4. Improve the efficiency and accuracy of time-stepping for transient EHL.
5. Error estimates for h-adaptivity.
6. H-p adaptivity....

Discussion (cont.)

To be of "industrial strength" even more is required...

1. Intuitive user-interface for pre- and post-processing.
2. Minimal choice of parameters with robust default values.
3. Robust solver that always delivers a solution!
   - May need to use continuation
   - May need to use fewer degrees of freedom
   - Must not require modification of the source code for different problems!