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Preconditioning Saddle-Point Systems arising in a Stochastic Mixed Finite Element Problem

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Joint work with:

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Related work with:

- Elisabeth Ullmann, Oliver Ernst (U. of Freiberg), David Silvester (U. of Manchester)

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Outline

- Mixed SFEM on: $\mathcal{A}\left(\boldsymbol{x},\omega\right)^{-1}\boldsymbol{u}\left(\boldsymbol{x},\omega\right)-\nabla p\left(\boldsymbol{x},\omega\right) = 0, \; -\nabla\cdot\boldsymbol{u}\left(\boldsymbol{x},\omega\right) = f\left(\boldsymbol{x}\right)$

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- Mixed SFEM on: $\mathcal{A}\,(\boldsymbol{x},\omega)^{-1}\,\boldsymbol{u}\,(\boldsymbol{x},\omega) - \nabla p\,(\boldsymbol{x},\omega) \;=\; 0,\;\; -\nabla\cdot\boldsymbol{u}\,(\boldsymbol{x},\omega) \;=\; f\,(\boldsymbol{x})$

- Solving stochastic saddle-point systems

$$\begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix}$$

  ▷ Weak problem in $H(div;D)\otimes L^2(\Gamma)$ and $L^2(D)\otimes L^2(\Gamma)$

  ▷ Inf-sup stability

  ▷ Block-diagonal preconditioner

  ▷ Multigrid implementation

  ▷ Eigenvalue bounds

  ▷ Numerical results

Model Problem

Let $A(\boldsymbol{x}, \omega) : D \times \Omega \to \mathbb{R}$ be a random field.

For $\boldsymbol{x} \in D$, $A(\omega)$ is a random variable with finite variance; for $\omega \in \Omega$, $\mathcal{A}(\boldsymbol{x}) \in L^{\infty}(D)$.

We seek random fields $p(\boldsymbol{x}, \omega)$, $\boldsymbol{u}(\boldsymbol{x}, \omega)$ such that P-almost everywhere $\omega \in \Omega$:

$$\begin{aligned} \mathcal{A}(\boldsymbol{x}, \omega)^{-1} \boldsymbol{u}(\boldsymbol{x}, \omega) - \nabla p(\boldsymbol{x}, \omega) &= 0, \\ \nabla \cdot \boldsymbol{u}(\boldsymbol{x}, \omega) &= -f(\boldsymbol{x}) \quad \boldsymbol{x} \text{ in } D, \\ p(\boldsymbol{x}, \omega) &= g(\boldsymbol{x}) \quad \boldsymbol{x} \text{ on } \partial D_D, \\ \boldsymbol{u}(\boldsymbol{x}, \omega) \cdot \boldsymbol{n} &= 0 \quad \boldsymbol{x} \text{ on } \partial D_N. \end{aligned}$$

Finite Noise Assumption

We assume that the input random field can be represented by a finite number of random variables.

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Here, we consider a truncated Karhunen-Loève expansion :

$$A^{-1}(\boldsymbol{x},\omega) \approx A_M^{-1}(\boldsymbol{x},\boldsymbol{\xi}) = \mu(\boldsymbol{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i} c_i(\boldsymbol{x}) \xi_i,$$

where $\boldsymbol{\xi} = \{\xi_1(\omega), \ldots \xi_M(\omega)\}$ are independent random variables and $\{\lambda_i, c_i(\boldsymbol{x})\}$ are the eigenpairs of the correlation function $C_{A^{-1}}(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

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Note that:

$$\int_D Var\left(A^{-1} - A_M^{-1}\right) = \int_D \sigma^2(\boldsymbol{x}) dD - \sum_{i=1}^{M} \lambda_i$$

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Example

Consider the covariance function

$$C(\boldsymbol{x}, \boldsymbol{z}) = \sigma^2 \exp\left(-\frac{|x_1 - z_1|}{b_1} - \frac{|x_2 - z_2|}{b_2}\right),$$

$D = [0,1] \times [0,1]$ and Gaussian random variables.

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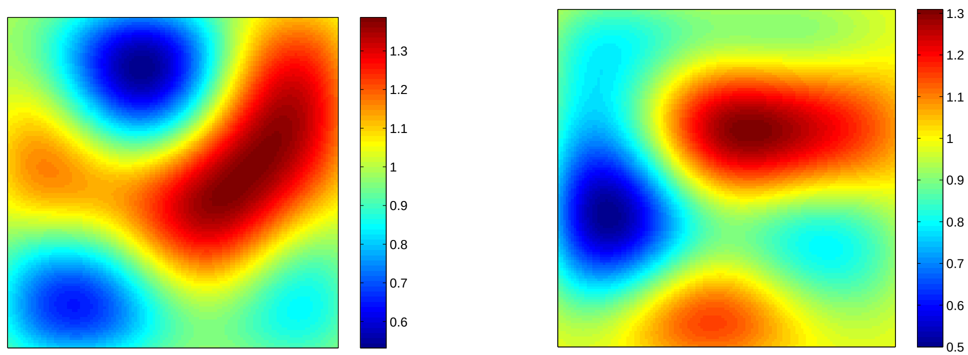
If $b_1 = 1 = b_2$ then 10 term KL expansion, yields relative error of 0.01

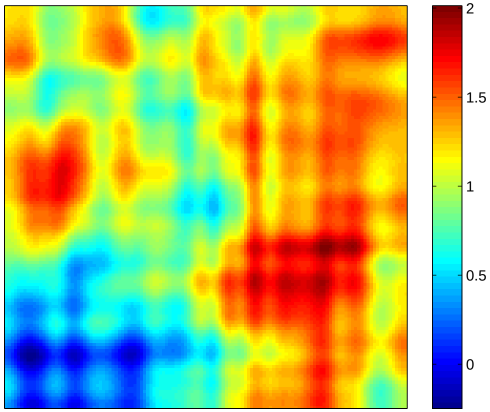
Two realisations of the resulting random field:

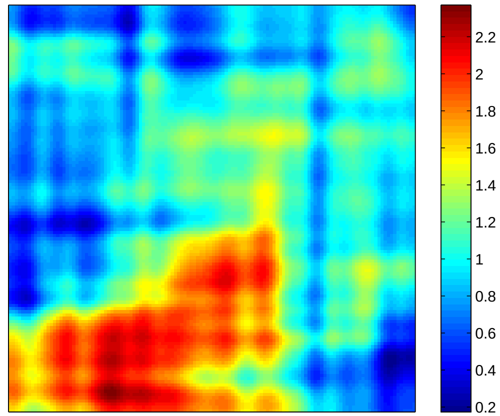


Example

If $b_1 = \frac{1}{4} = b_2$ then a 200 term KL expansion, yields relative error of 0.08

Two realisations of the resulting random field:





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Mixed Stochastic Galerkin Formulation

Let $y_i = \xi_i(\omega) \in \Gamma_i$, and write $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_M$.

If the random variables are independent then the joint density function has the form:

$$\rho(\boldsymbol{y}) = \prod_i \rho_i(y_i)$$

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We also define the space $L^2_\rho(\Gamma)$ of random functions which satisfy:

$$\int_\Gamma \rho(\boldsymbol{y}) g(\boldsymbol{y})^2 \, d\boldsymbol{y} < \infty.$$

Mixed Stochastic Galerkin Formulation

Consider the tensor product spaces

$$V = H_{0,N}(div; D) \otimes L^2_\rho(\Gamma) \qquad W = L^2(D) \otimes L^2_\rho(\Gamma)$$

We seek $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \in V$ and $p(\boldsymbol{x}, \boldsymbol{y}) \in W$ such that:

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$$\begin{aligned} \int_\Gamma \rho(\boldsymbol{y}) \left(\mathcal{A}_M^{-1} \boldsymbol{u}, \boldsymbol{v} \right) d\boldsymbol{y} + \int_\Gamma \rho(\boldsymbol{y}) (p, \nabla \cdot \boldsymbol{v}) \, d\boldsymbol{y} &= \int_\Gamma \rho(\boldsymbol{y}) (g, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial \Gamma_D} \, d\boldsymbol{y}, \\ \int_\Gamma \rho(\boldsymbol{y}) (w, \nabla \cdot \boldsymbol{u}) \, d\boldsymbol{y} &= -\int_\Gamma \rho(\boldsymbol{y}) (f, w) \, d\boldsymbol{y} \end{aligned}$$

$\forall \; \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \in V$ and $w(\boldsymbol{x}, \boldsymbol{y}) \in W$.

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$$\begin{aligned} \left\langle \left(\mathcal{A}_M^{-1} \boldsymbol{u}, \boldsymbol{v} \right) \right\rangle + \langle (p, \nabla \cdot \boldsymbol{v}) \rangle &= \left\langle (g, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial \Gamma_D} \right\rangle, \\ \langle (w, \nabla \cdot \boldsymbol{u}) \rangle &= -\langle (f, w) \rangle \end{aligned}$$

$\forall \ \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) \in V$ and $w(\boldsymbol{x}, \boldsymbol{y}) \in W$.

Finite-Dimensional Problem

Find $\boldsymbol{u}_{hd}(\boldsymbol{x}, \boldsymbol{y}) \in V_h \otimes S_d$ and $p_{hd}(\boldsymbol{x}, \boldsymbol{y}) \in W_h \otimes S_d$ satisfying:

$$\begin{aligned} \left\langle \left(\mathcal{A}_M^{-1} \boldsymbol{u}_{hd}, \boldsymbol{v} \right) \right\rangle + \langle (p_{hd}, \nabla \cdot \boldsymbol{v}) \rangle &= \left\langle (g, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial \Gamma_D} \right\rangle, \\ \langle (w, \nabla \cdot \boldsymbol{u}_{hd}) \rangle &= -\langle (f, w) \rangle \end{aligned}$$

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- $V_h \subset H(div; D)$, $W_h \subset L^2(D)$ are a deterministic inf-sup stable pairing e.g. $RT_0(D)$-$P_0(D)$.
- $S_d \subset L^2(\Gamma)$ is set of multivariate polynomials in M random variables. Choose from:
  1. total degree d (generalised polynomial chaos) of dimension $N_\xi = \frac{(M+d)!}{M!d!}$
  2. degree d in each random variable of dimension $N_\xi = (d+1)^M$

Abstract Saddle-Point Problem

We seek $\boldsymbol{u}_{hd}(\boldsymbol{x},\boldsymbol{y}) \in V_h \otimes S_d$, and $p_{hd}(\boldsymbol{x},\boldsymbol{y}) \in W_h \otimes S_d$ s.t.:

$$
\begin{aligned}
a\left(\boldsymbol{u}_{hd},\boldsymbol{v}\right) + b\left(p_{hd},\boldsymbol{v}\right) &= \left\langle \left(g, \boldsymbol{v}\cdot\boldsymbol{n}\right)_{\partial\Gamma_D} \right\rangle, \\
b\left(w,\boldsymbol{u}_{hd}\right) &= -\left\langle \left(f, w\right)\right\rangle
\end{aligned}
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$\forall\ \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}) \in V_h \otimes S_d$ and $w(\boldsymbol{x},\boldsymbol{y}) \in W_h \otimes S_d$

which leads to a symmetric indefinite system of the form:

$$\begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix}$$

of dimension $N_x \times N_\xi$ where $N_x = N_u + N_p$.

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Matrix structure

$$V_h = \mathsf{span}\,\{\boldsymbol{\varphi}_i(\boldsymbol{x})\}_{i=1}^{Nu}\,, \qquad W_h = \mathsf{span}\,\{\phi_j(\boldsymbol{x})\}_{j=1}^{Np}\,, \qquad S_d = \mathsf{span}\,\{\psi_k(\boldsymbol{y})\}_{k=1}^{N\xi}$$

with $\{\psi_k(\boldsymbol{y})\}$ orthonormal w.r.t $\langle \cdot, \cdot \rangle$, the saddle-point matrix has the structure:

$$\begin{pmatrix} I \otimes A_0 + \sum_{k=1}^{M} G_k \otimes A_k & I \otimes B^T \\ I \otimes B & 0 \end{pmatrix}$$

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where

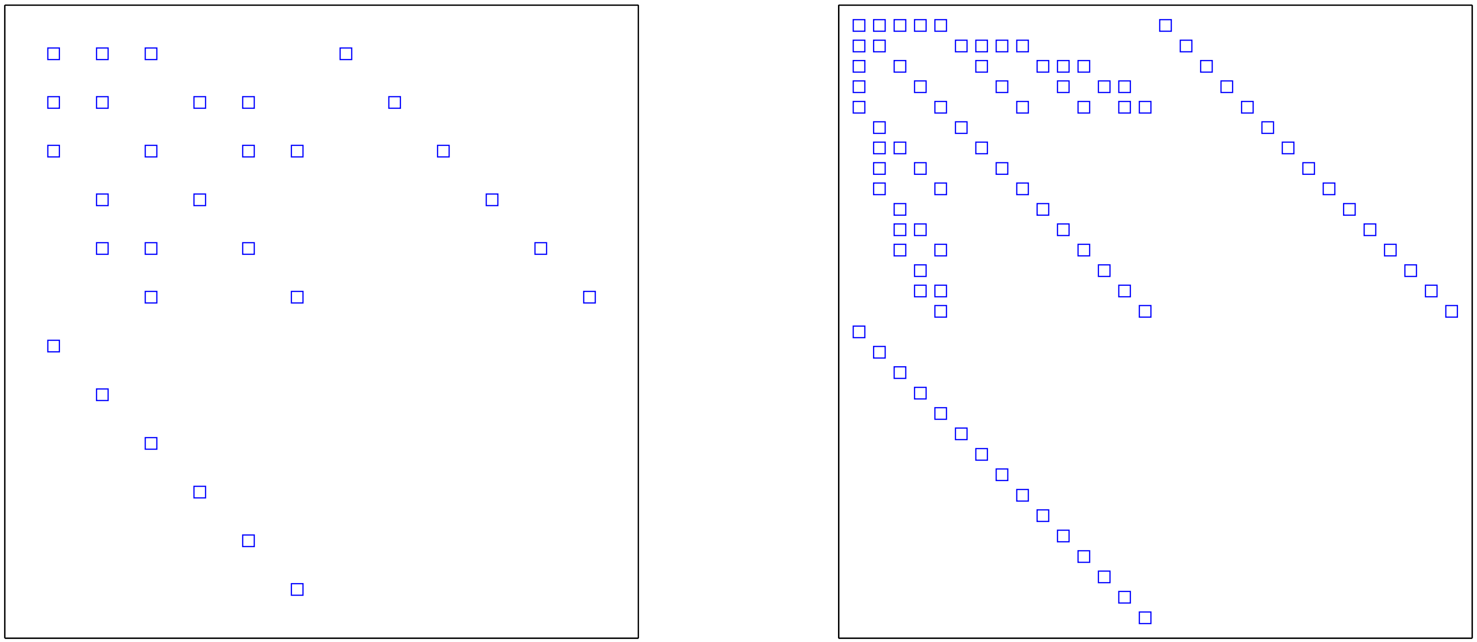
$$[A_0]_{ij} = \int_D \mu(\boldsymbol{x}) \boldsymbol{\varphi}_i(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_j(\boldsymbol{x}) \qquad [A_k]_{ij} = \sqrt{\lambda_k} \int_D c_k(\boldsymbol{x}) \boldsymbol{\varphi}_i(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_j(\boldsymbol{x})$$

and

$$[B]_{ij} = \int_D \nabla \cdot \boldsymbol{\varphi}_i(\boldsymbol{x})\, \phi_j(\boldsymbol{x}) \qquad [G_k]_{rs} = \langle\, y_k \psi_r(\boldsymbol{y}) \psi_s(\boldsymbol{y})\, \rangle$$

Examples

$M = 2, d = 2$ (left) and $M = 4, d = 2$ (right)



Well-posedness

The well-posedness of the stochastic saddle-point problem can be analysed using the standard Brezzi-Babuska stability criteria.

Define the following norms on the tensor product spaces:

$$\begin{aligned} \| \boldsymbol{q}_{hd} \|^2_{div \otimes L^2} &= \left\langle \| \boldsymbol{q}_{hd} \|^2_{div(D)} \right\rangle, \qquad \boldsymbol{q}_{hd} \in V_h \otimes S_d \\ \| w_{hd} \|^2_{L^2 \otimes L^2} &= \left\langle \| w_{hd} \|^2_{L^2(D)} \right\rangle, \qquad w_{hd} \in W_h \otimes S_d \end{aligned}$$

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If we choose:

- $V_h := RT_0(D)$ (lowest-order Raviart-Thomas elements)
- $W_h := P_0(D)$ (piecewise constants)

and assume that:

$$\boxed{0 < a_{min} \leq A_M^{-1}(\boldsymbol{x}, \boldsymbol{y}) \leq a_{max} < +\infty, \quad \text{a.e. in } D \times \Gamma}$$

then, the following results can be proved independently of the choice of $S_d \subset L^2_\rho(\Gamma)$.

Well-posedness

- $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms

- Ellipticity

$$a\left(\boldsymbol{v}_{hd},\boldsymbol{v}_{hd}\right) \geq a_{min} \parallel \boldsymbol{v}_{hd} \parallel^2_{div\otimes L^2} \qquad \forall\, \boldsymbol{v}_{hd} \in Z_{hd}$$

where

$$Z_{hd} = \{\boldsymbol{v}_h \in V_h \otimes S_d \text{ s.t. } b\left(\boldsymbol{v}_{hd}, w_{hd}\right) = 0, \quad \forall w_{hd} \in W_h \otimes S_d\}$$

- **Theorem** (inf-sup stability)

There exists a constant $\tilde{\beta} > 0$ depending only on the domain D and the Raviart-Thomas interpolation operator (and therefore independent of h, M and d) such that:

$$\sup_{\boldsymbol{v}_{hd}\in V_h\otimes S_d\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}_{hd}, w_{hd}\right)}{\parallel \boldsymbol{v}_{hd} \parallel_{div\otimes L^2}} \geq \tilde{\beta} \parallel w_{hd} \parallel_{L^2\otimes L^2} \qquad \forall\, w_{hd} \in W_h \otimes S_d.$$

Ideal ‘Hdiv’ Preconditioner

Define deterministic matrices $D \in \mathbb{R}^{N_u \times N_u}$, and $M \in \mathbb{R}^{N_p \times N_p}$ via:

$$[D]_{ij} = \int_D \nabla \cdot \boldsymbol{\varphi}_i \nabla \cdot \boldsymbol{\varphi}_j, \qquad [M]_{rs} = \int_D \phi_r \phi_s.$$

We then have matrix representations of the following stochastic norms:

$$\| \boldsymbol{v}_{hd} \|^2_{div, \mathcal{A}^{-1} \otimes L^2} = \underline{v}^T \left(\tilde{A} + \tilde{D} \right) \underline{v} \qquad \text{where } \tilde{D} = I \otimes D$$

$$\| w_h \|^2_{L^2 \otimes L^2} = \underline{w}^T \tilde{M} \underline{w}, \qquad \text{where } \tilde{M} = I \otimes M.$$

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$$\| w_h \|^2_{L^2 \otimes L^2} = \underline{w}^T \tilde{M} \underline{w}, \qquad \text{where } \tilde{M} = I \otimes M.$$

Note that the discrete inf-sup condition tells us that:

$$\tilde{\beta}^2 \min\left(1, \frac{1}{a_{max}}\right) \le \frac{\underline{w}^T \tilde{B} \left(\tilde{A} + \tilde{D}\right)^{-1} \tilde{B}^T \underline{w}}{\underline{w}^T \tilde{M} \underline{w}} \qquad \forall \underline{w} \in \mathbb{R}^{N_p N_\xi} \setminus \{\underline{0}\}$$

Eigenvalue bounds

Consider the 'ideal' preconditioner

$$P = \begin{pmatrix} \tilde{A} + \tilde{D} & 0 \\ 0 & \tilde{M} \end{pmatrix} = \begin{pmatrix} \tilde{A} + \tilde{B}^T \tilde{M}^{-1} \tilde{B} & 0 \\ 0 & \tilde{M} \end{pmatrix}$$

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Theorem

The eigenvalues of

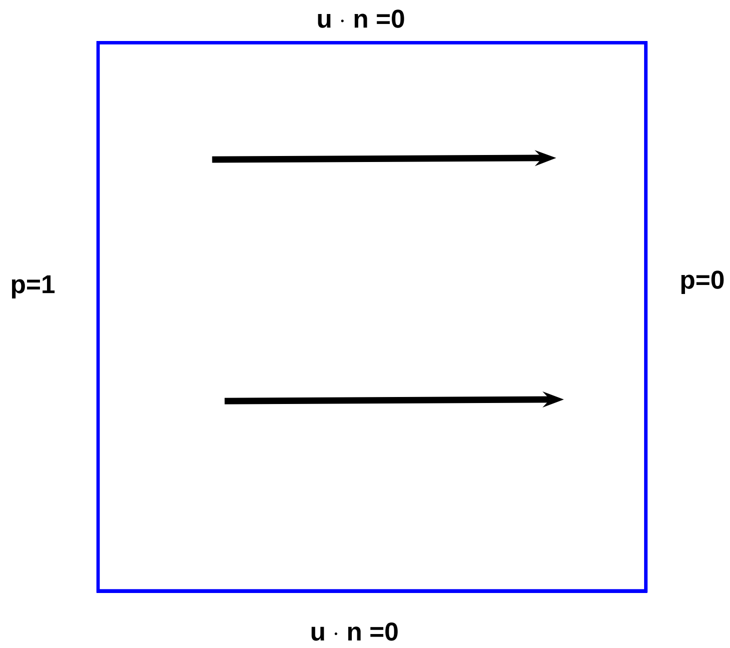
$$\begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{A} + \tilde{D} & 0 \\ 0 & \tilde{M} \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix}$$

are bounded and lie in the union of the intervals,

$$\left[-1, -\frac{\tilde{\beta}^2}{a_{max}} \right] \cup \{1\}$$

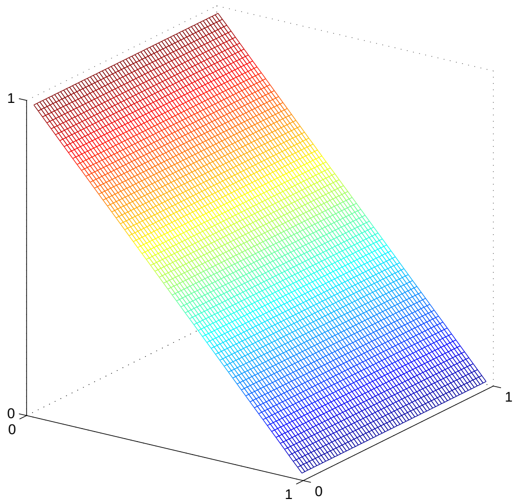
Example

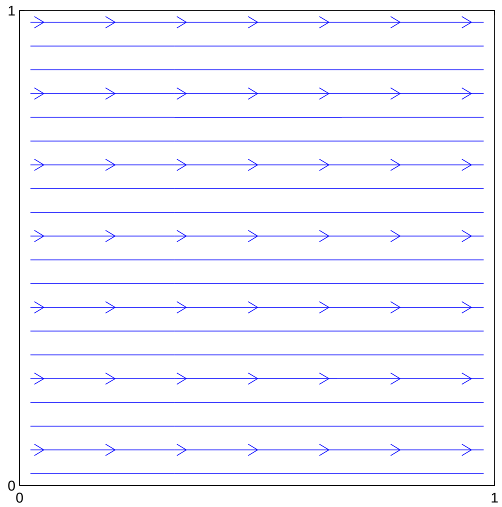
Let $D = [0,1] \times [0,1]$, with mixed bcs. We choose an exponential covariance function for the random input with $\mu(\boldsymbol{x}) = 1$ and $\sigma(\boldsymbol{x}) = 0.2$.



Mean of numerical solution

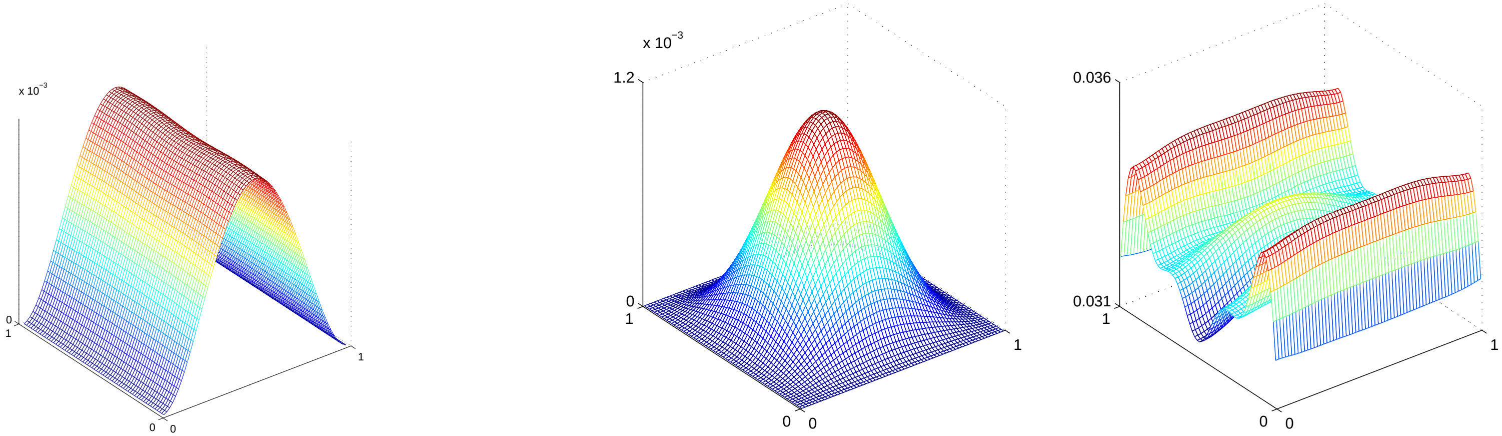
Pressure (left), Flux (right)





Variance of numerical solution

Pressure (left), y component (middle) and x component (right) of the Flux



(Exact) Preconditioned Minres iterations

In this example $a_{max} = O(1)$

			$h = \frac{1}{16}$		$h = \frac{1}{32}$	
	d	N_ξ	Iter	dimension	Iter	dimension
M=4	1	5	6	6,560	6	25,920
-	2	15	6	19,650	6	77,760
-	3	35	6	45,920	6	181,440
-	4	70	6	91,840	6	362,880
M=5	1	6	6	7,872	6	31,104
-	2	21	6	27,552	6	108,864
-	3	56	6	73,472	6	290,304
-	4	126	6	165,312	6	653,184
M=6	1	7	6	9,184	6	36,288
-	2	28	6	36,736	6	145,152
-	3	84	6	110,208	6	435,456
-	4	210	6	275,520	6	1,088,640

(Exact) Preconditioned Minres iterations

With $h = \frac{1}{32}$, $M = 4$ and $d = 2$ fixed and varying ratio $\frac{\sigma}{\mu}$

$\frac{\sigma}{\mu}$	0.1	0.2	0.4	0.8
Iter	**6**	**6**	**6**	**6**

With $h = \frac{1}{32}$, $M = 4$ and $d = 2$ and $\frac{\sigma}{\mu} = 0.1$ so that only a_{max} is varying

μ	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}
Iter	**4**	**4**	**6**	**9**	**22**

Practical Implementation

We need a fast solver for systems with the coefficient matrix:

$$\tilde{A} + \tilde{D} = I \otimes \left(A_0 + B^T M^{-1} B \right) + \left(\sum_{k=1}^{M} G_k \otimes A_k \right)$$

which represents a weighted stochastic $H(div; D) \otimes L^2_\rho(\Gamma)$ operator:

$$\tilde{\mathcal{H}}_{\mathcal{A}} : RT_0(D) \otimes S_d(\Gamma) \to RT_0(D) \otimes S_d(\Gamma)$$

defined via:

$$\boxed{\left(\tilde{\mathcal{H}}_{\mathcal{A}} \boldsymbol{v}_{hd}, \boldsymbol{v}_{hd} \right) = \int_\Gamma \rho(\boldsymbol{y}) \left(\int_D \mathcal{A}_M^{-1} \boldsymbol{v}_{hd} \cdot \boldsymbol{v}_{hd} + \nabla \cdot \boldsymbol{v}_{hd} \, \nabla \cdot \boldsymbol{v}_{hd} \, dD \right) \, d\,\boldsymbol{y}.}$$

Note that this is not an elliptic operator.

Eigenvalue bounds

Theorem: Suppose there exists a matrix $\tilde{V}$ satisfying

$$\theta \leq \frac{\underline{v}^T \left(\tilde{A} + \tilde{D}\right) \underline{v}}{\underline{v}^T \tilde{V} \underline{v}} \leq \Theta \leq 1$$

with positive constants θ and Θ. The eigenvalues of:

$$\begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{V} & 0 \\ 0 & \tilde{M} \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix}$$

lie in the union of the intervals,

$$\left[-1, -\frac{1}{2}\left(\theta\left(1-\alpha\right) - \sqrt{\theta^2\left(\alpha-1\right)^2 + 4\alpha\theta}\right)\right] \cup [\theta, 1]$$

where $\alpha = \frac{\tilde{\beta}^2}{a_{max}}$ is the corresponding bound for the ideal preconditioner.

Geometric H(div) Multigrid

We approximate the action of $\left(\tilde{A} + \tilde{D}\right)^{-1}$ via a specialised multigrid V-cycle.

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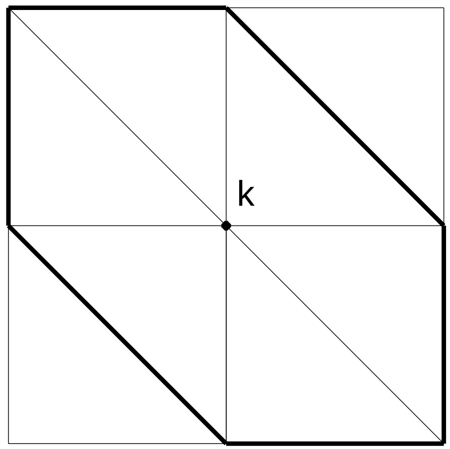
The main idea is to only vary the spatial discretisation from grid to grid whilst keeping the stochastic discretisation fixed.

Key ingredients:

- Prolongation: $\tilde{P} = I \otimes P_H^h$ where P_H^h is a standard spatial prolongation operator
- Restriction operator $\tilde{R} = \tilde{P}^T = I \otimes R_h^H$
- Smoother: additive Schwarz method (block Jacobi)

Additive Schwarz Smoothing

Let $\tilde{H}_h = \tilde{A} + \tilde{D}$ be the stochastic $H(div)$ matrix associated with a fixed spatial mesh T_h, decomposed into vertex-based patches:



The smoothing operator (in matrix form) is defined via:

$$\tilde{S}_h = \eta \sum_k \tilde{P}_h^k \tilde{H}_h^{-1}$$

Additive Schwarz Smoothing

where

$$\tilde{P}_h^k = (I \otimes R_k^T)\tilde{H}_{h,k}^{-1}(I \otimes R_k)\tilde{H}_h.$$

Then, for $\boldsymbol{v} \in \mathbb{R}^{N_u N_\xi}$ we have:

$$\boxed{\tilde{S}_h \boldsymbol{v} = \eta \sum_k (I \otimes R_k^T)\tilde{H}_{h,k}^{-1}(I \otimes R_k)\boldsymbol{v}}$$

where $\tilde{H}_{h,k}$ represents a local 'patch-version' of the matrix $\tilde{H}_h$.

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Smoothing requires multiple decoupled solves with $\tilde{H}_{h,k}$. In the stochastic problem:

$$\tilde{H}_{h,k} = I \otimes \left(A_{0,k} + D_{0,k}\right) + \sum_{i=1}^{M} G_i \otimes A_{i,k}$$

and so the dimension of each local matrix is $N_\xi N_k$.

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This is tractable for a few thousand stochastic degrees of freedom.

Multigrid Convergence

Theorem

Let $\tilde{V}$ denote the matrix corresponding to the inverse of the multigrid V-cycle operator described above. Then,

$$\boxed{\theta \leq \frac{\underline{v}^T \left(\tilde{A} + \tilde{D}\right) \underline{v}}{\underline{v}^T \tilde{V} \underline{v}} \leq 1}$$

where

$$\theta = 1 - \frac{C}{C + 2\nu}$$

depends only on the number of smoothing steps ν and a_{min} and a_{max}.

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Combining this result with eigenvalue bound for preconditioned saddle-point system, we have a solver that is optimal w.r.t all discretisation parameters.

Example 1

$$P = \begin{pmatrix} \tilde{V} & 0 \\ 0 & \tilde{M} \end{pmatrix}$$

- 1 multigrid V-cycle per minres iteration; 1 pre and 1 post smoothing step;
- Uniform random variables; $\mu(\boldsymbol{x}) = 1$, $\sigma = 0.1$ ($\Rightarrow a_{max} = O(1)$)

	d	$M = 1$	$M = 2$	$M = 3$	$M = 4$
$h = \frac{1}{32}$	1	**17**	**17**	**17**	**17**
-	2	**17**	**17**	**17**	**17**
-	3	**17**	**17**	**17**	**17**
-	4	**17**	**17**	**17**	**17**
$h = \frac{1}{64}$	1	**17**	**17**	**17**	**17**
-	2	**17**	**17**	**17**	**17**
-	3	**17**	**17**	**17**	**17**
-	4	**17**	**17**	**17**	**17**

Example 2

- Fixed discretisation parameters: $h = \frac{1}{16}$, $M = 4$, $p = 2$.
- Varying $\frac{\sigma}{\mu}$

Preconditioned minres iterations:

$\frac{\sigma}{\mu}$	0.1	0.2	0.3	0.4	0.5	0.6
Ideal	**6**	**6**	**6**	**6**	**6**	**6**
Multigrid version	**17**	**17**	**17**	**17**	**17**	**17**

Multigrid constants

$\frac{\sigma}{\mu}$	0.2	0.3	0.4	0.5	0.6
θ	0.4576	0.4564	0.4548	0.4527	0.4495
Θ	1.0000	1.0000	1.0000	1.0000	1.0000

Example 3

- Fixed discretisation parameters: $h = \frac{1}{16}$, $M = 4$, $p = 2$.
- Vary a_{max} by varying μ and setting $\sigma = \frac{\mu}{10}$.

Preconditioned minres iterations:

μ	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
Ideal	**3**	**3**	**4**	**5**	**8**	**16**	**46**
Multigrid version	**15**	**15**	**16**	**17**	**21**	**40**	**99**

Multigrid constants

μ	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
θ	0.4552	0.4550	0.4556	0.4587	0.4864	0.6453	0.9172
Θ	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Summary

- Solving well-posed stochastic saddle-point problem
- Stochastic inf-sup stability result leads to nice eigenvalue bounds for $H(div)$ preconditioners
- Practical implementation based on deterministic Arnold-Falk-Winther multigrid
- Analysis of extended multigrid method available
- Preconditioner for saddle-point system is optimal w.r.t spatial and stochastic discretisation parameters
- Overall performance does depend on a_{min} and a_{max}
- Experiments with modified (cheaper) smoothers look promising

Alternative Preconditioning Scheme

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The blocks of this matrix represent norms in which an alternative inf-sup condition can be established. In particular,

$$\underline{w}^T \tilde{B}\tilde{A}_{diag}^{-1}\tilde{B}^T \underline{w} = \left\langle \| \, w_{h,d} \, \|_{1,h,\mathcal{A}}^2 \right\rangle$$

represents the expectation of a (weighted) mesh-dependent $H_1(D)$ norm.

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represents the expectation of a (weighted) mesh-dependent $H_1(D)$ norm.

Pro: Standard multigrid methods can be used. Con: Obtaining $\tilde{A}_{diag}$ that yields robustness w.r.t PDE coefficients is difficult.

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References

$$P = \begin{pmatrix} \tilde{A} + \tilde{D} & 0 \\ 0 & \tilde{M} \end{pmatrix}$$

- H. Elman, D. Furnival, C.E. Powell, $H(div)$ preconditioning for a mixed finite element formulation of the stochastic diffusion equation. Submitted.

$$P = \begin{pmatrix} \tilde{A}_{diag} & 0 \\ 0 & \tilde{B}\tilde{A}_{diag}^{-1}\tilde{B}^T \end{pmatrix}$$

- O.G. Ernst, C.E. Powell, D. Silvester and E. Ullmann, Efficient solvers for a Linear Stochastic Galerkin Mixed Formulation of the Steady-State Diffusion Equation, Under revision, SISC 2008.