Difficulties in computing the fundamental distortion mode in Coriolis mass flow meters

CLAPDE 2008

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Focus

- The Coriolis effect
- Coriolis mass flow meters
- Collaboration and background
- Eigenvalue problem & results

Coriolis

Gaspard-Gustave de Coriolis (Gustave Coriolis, May 21, 1792 — September 19, 1843) published the paper that described the effect that now bears his name in 1835: *Sur les équations du mouvement relatif des systèmes de corps* (On the equations of relative motion of a system of bodies).

Source: `en.wikipedia.org/wiki/Gaspard-Gustave_Coriolis`

The Coriolis effect

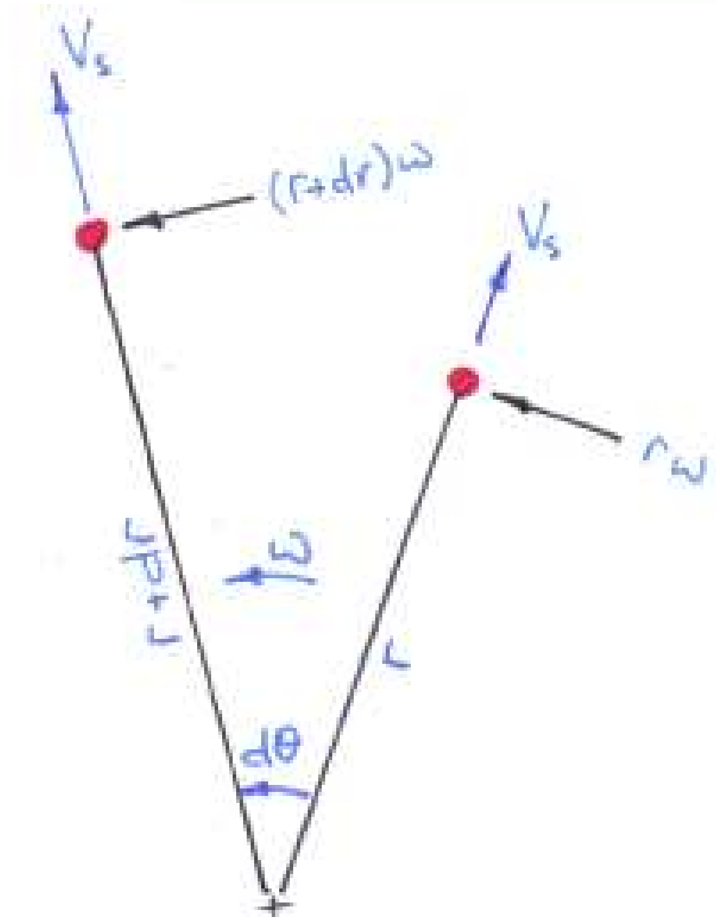
Imagine rolling a ball radially outward at constant velocity $V_s\,\mathrm{m/s}$ from the centre of a roundabout/carousel that is rotating at $\omega\,\mathrm{rad/sec}$.

The ball will experience:

- a radial centripetal acceleration
- a tangential Coriolis acceleration

Consider the change in velocity during dt...

Derivation I

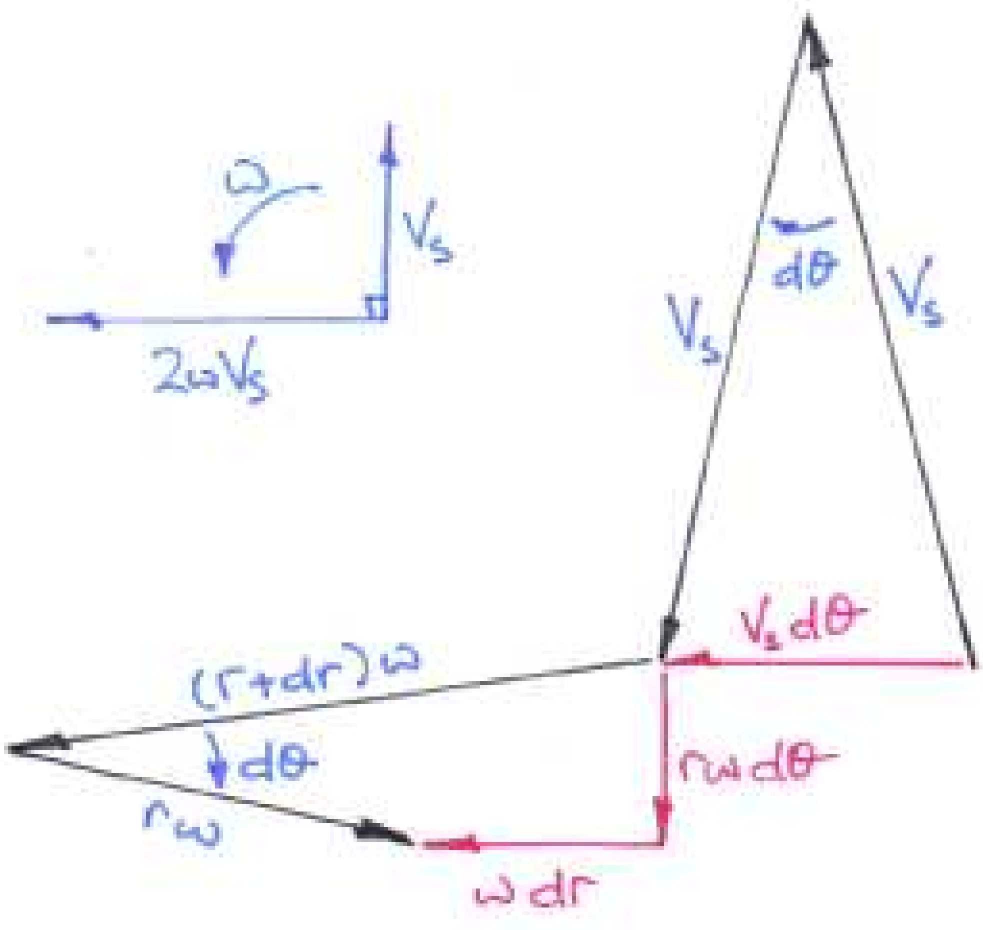


Sketch the velocity vectors at t

And those at $t + dt$

Hence obtain accelerations.

Derivation II



Change in tangential velocity

$$V_s\,d\theta + \omega\,dr$$

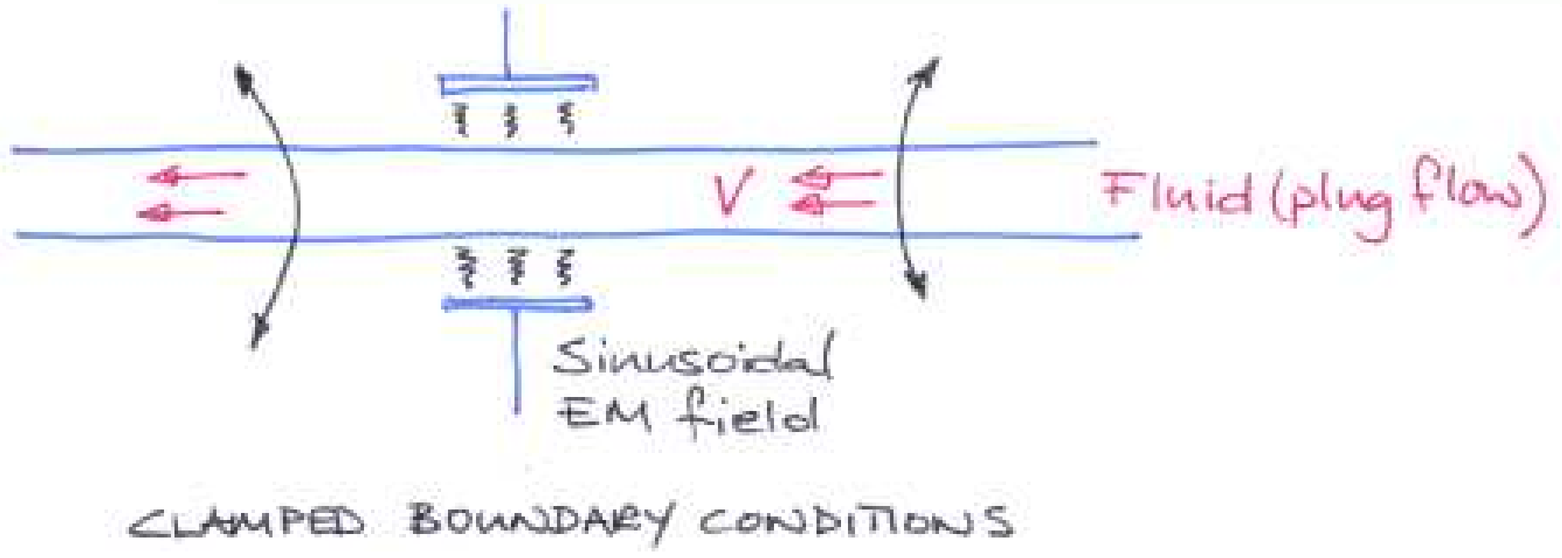
Change in radial velocity

$$r\omega\,d\theta$$

Divide by dt to get acceleration:

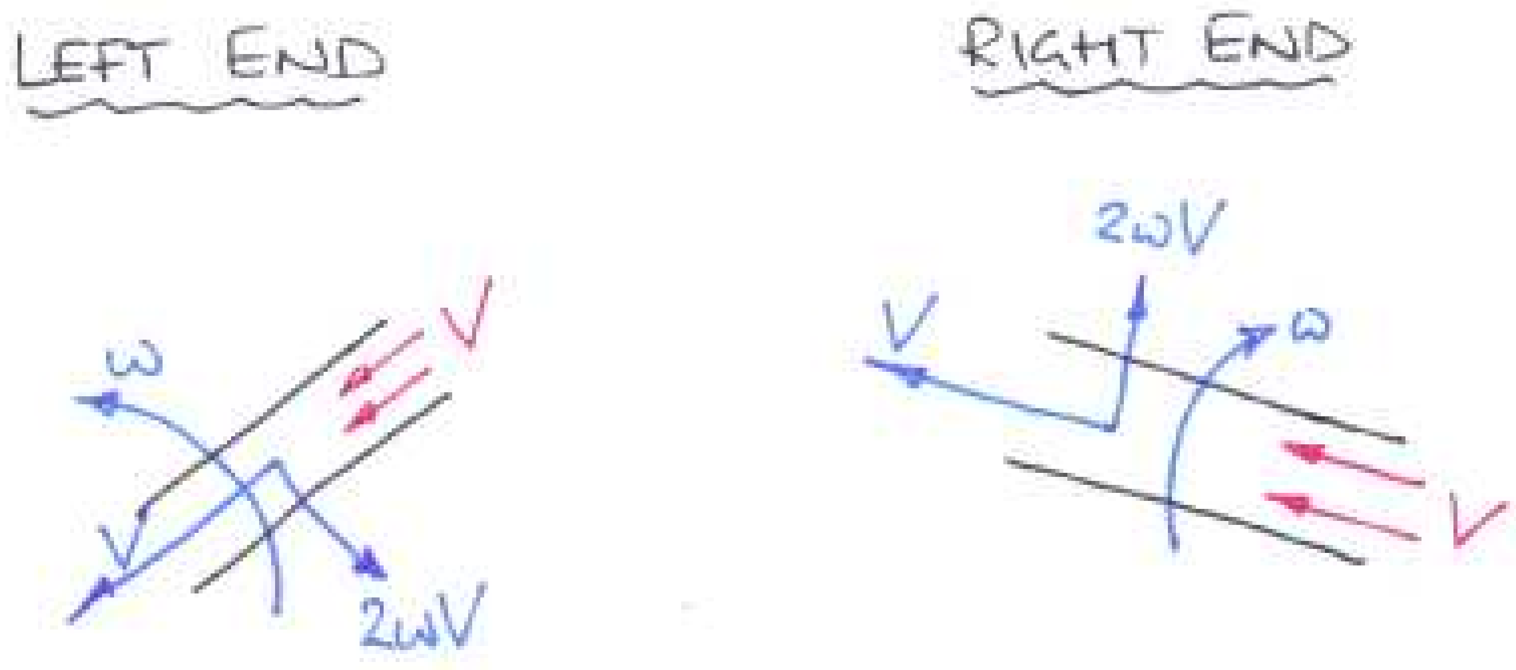
tangential:	$2\omega V_s$	Coriolis
radial:	$r\omega^2$	Centripetal

Pipe meter physics I



- Metal pipe carries a plug flow of fluid at velocity V.
- Pipe also vibrated by electromagnetic sine
- Bending theory applies

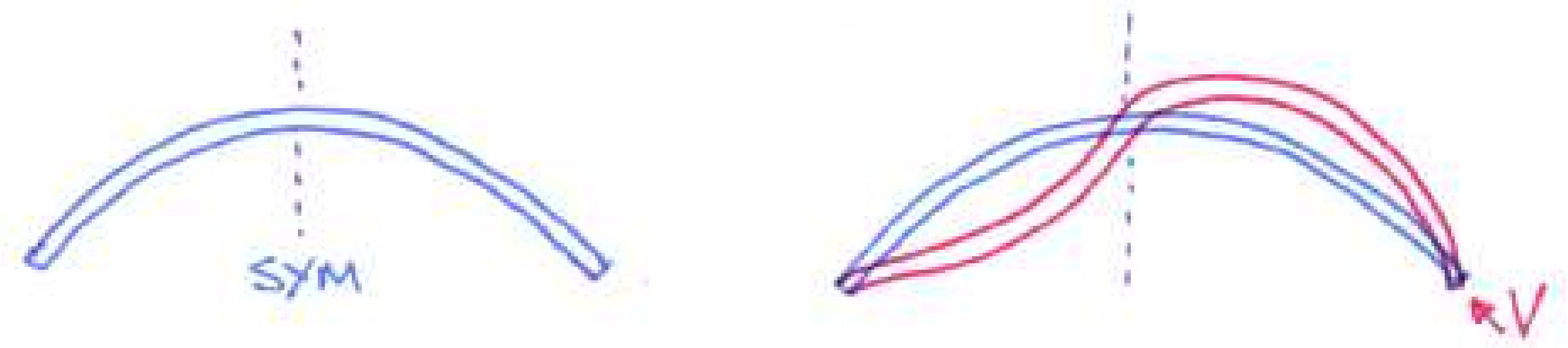
Pipe meter physics II



- Right end: fluid particle = ball on roundabout.
- Coriolis acceleration implies force implies deformation.
- Left end: Coriolis force is in opposite direction.

Pipe meter physics III

No flow on the left, flow at $V\ \mathrm{m/s}$ on the right:



- Overall effect: antisymetric Coriolis distortion superimposed on symmetric bending profile.
- Phase difference at quarter-points proportional to mass flow

So what?

Why does this matter?

- Coriolis distortion is an inertial effect and is proportional to the mass (not volume) flow rate.
- Mass flow is measured directly not by the indirect conversion of volume flow (e.g. bubbles).
- Important for accuracy:
  - Custodial transfers
  - medical drug dosing
- Meters range from over 1 metre diameter to micro-machined (fits on a thumb).

Example from wikipedia

An illustrative example:

- No flow
- With flow

Courtesy wikipedia.

The mathematical model

Incorporating the fluid flow (plug flow) into a Timoshenko beam model leads to,

$$(m_p + m_f)\frac{\partial^2 u}{\partial t^2} + m_f\left[2V\frac{\partial^2 u}{\partial x \partial t} + V^2\frac{\partial^2 u}{\partial x^2}\right]$$
$$-\kappa G A_p\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x}\right) = 0,$$
$$(\varrho_p I_p + \varrho_f I_f)\frac{\partial^2 \theta}{\partial t^2} - E I_p\frac{\partial^2 \theta}{\partial x^2} - \kappa G A_p\left(\frac{\partial u}{\partial x} - \theta\right) = 0,$$

where $V =$ fluid velocity, and the boundaries are clamped.

FEM

Finite element discretization leads to,

$$\boldsymbol{M}\frac{d^2\boldsymbol{U}}{dt^2} + \boldsymbol{E}\frac{d\boldsymbol{U}}{dt} + \boldsymbol{A}\boldsymbol{U} = \boldsymbol{0},$$

and setting $\boldsymbol{U} = \boldsymbol{V}e^{i\omega t}$ we get the complex eigenvalue problem,

$$(\boldsymbol{A} + i\omega\boldsymbol{E} - \omega^2\boldsymbol{M})\boldsymbol{V} = \boldsymbol{0}.$$

Here $\boldsymbol{E} = -\boldsymbol{E}^T$, $\boldsymbol{M} > 0$ and $\boldsymbol{A} > 0$ (if $m_f V^2$ is small enough).

It follows that: all $\omega \in \mathbb{R}$ and if

$(\omega, \boldsymbol{V})$ is an eigenpair then so is $(\omega, \overline{\boldsymbol{V}})$.

Three solution techniques

We are looking for the Coriolis distortion:

imaginary part of an eigenvector associated with the smallest-in-magnitude eigenvalue

Three methods are used:

- Matlab's `polyeig` routine.
- Matlab's `eig` routine.
- Inverse iteration.

All with and without shift.

polyeig for $(\boldsymbol{A} + i\omega \boldsymbol{E} - \omega^2 \boldsymbol{M})\boldsymbol{V} = \boldsymbol{0}$

The MATLAB fragment

```
[X, e] = polyeig(A, i*E, -M);
```

solves the quadratic eigenvalue problem in terms of a column of eigenvalues, e, and a matrix of eigenvectors, X. (Recall that $\boldsymbol{A}$ and $\boldsymbol{M}$ are invertible.)

eig for $(\boldsymbol{A} + i\omega\boldsymbol{E} - \omega^2\boldsymbol{M})\boldsymbol{V} = \boldsymbol{0}$

Set $\boldsymbol{W} = \omega\boldsymbol{V}$ so that $-\omega^2\boldsymbol{M}\boldsymbol{V} = -\omega\boldsymbol{M}\boldsymbol{W}$. Then:

$$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{M}^{-1}\boldsymbol{A} & i\boldsymbol{M}^{-1}\boldsymbol{E} \end{pmatrix} \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{W} \end{pmatrix} = \omega \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{W} \end{pmatrix}.$$

Hence: $\boldsymbol{B}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{L}$, with $\boldsymbol{L} =$ diagonal of eigenvalues and $\boldsymbol{X} =$ eigenvectors. Solve in MATLAB via the fragment,

```
B = [ zeros(N,N) eye(N) ;
          M\A       i*M\E ];
[X L] = eig(B);
```

(where $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{M}$ are $N \times N$).

Inverse iteration

Given $\boldsymbol{D} > 0$ and $\boldsymbol{x}_0$ the iteration:

- $\boldsymbol{z}_{n+1} = \boldsymbol{D}^{-1}\boldsymbol{x}_n$ for $n = 0, 1, 2, \ldots$
- $\boldsymbol{x}_{n+1} = \boldsymbol{z}_{n+1}\|\boldsymbol{z}_{n+1}\|_\infty^{-1}$

converges to the eigenvector of the eigenvalue of least modulus of $\boldsymbol{D}$ (if this is well-defined).

For our system this is,

$$\boldsymbol{W}_{n+1} = \boldsymbol{V}_n$$

$$\boldsymbol{V}_{n+1} = \boldsymbol{A}^{-1}\left(\boldsymbol{M}\boldsymbol{W}_n - i\boldsymbol{E}\boldsymbol{W}_{n+1}\right)$$

and we take $(1 + 10^{-5}i, 1 + 10^{-5}i, \ldots)$ as the initial guess.

Summary

These methods were used with and without shift.

The physical constants in the PDEs are ‘real life’ and correspond to a real straight-tube meter.

Most meters are far more complicated in terms of design and geometry.

Before the numerics here is the background.

Collaboration and background

Robert Cheesewright (Engineering, Brunel) was getting incorrect eigenvectors from ANSYS.

He asked for my help and advice in terms of FEM and ‘locking’.

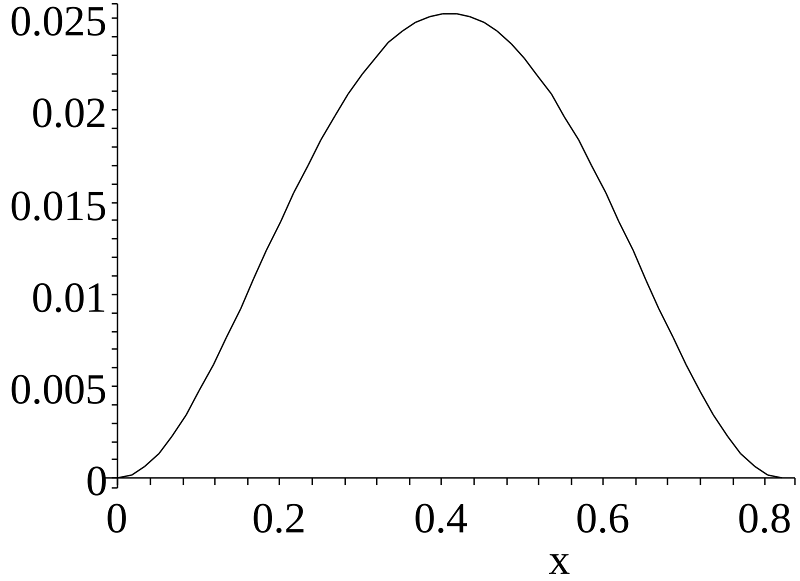
My independent C++ and matlab computations still gave incorrect results. . .

The Timoshenko beam results are shown

The Euler-Bernoulli results are essentially the same

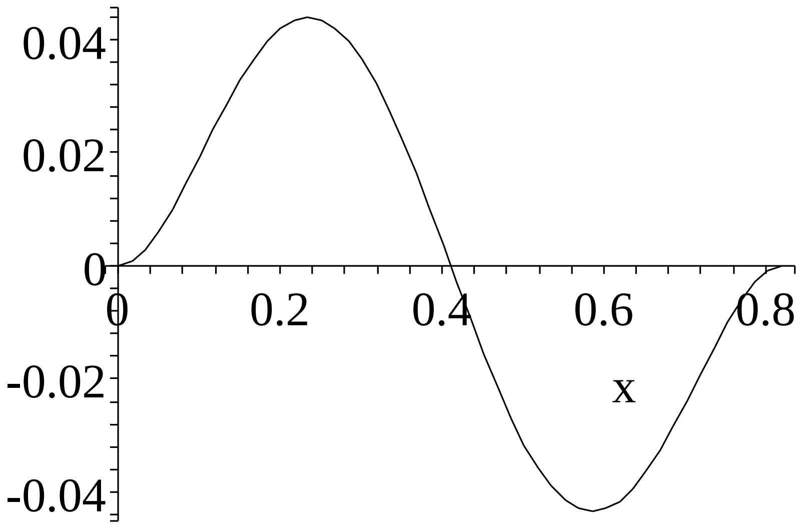
The results for a straight tube meter are. . .

Centrifugal distortion (by analysis)



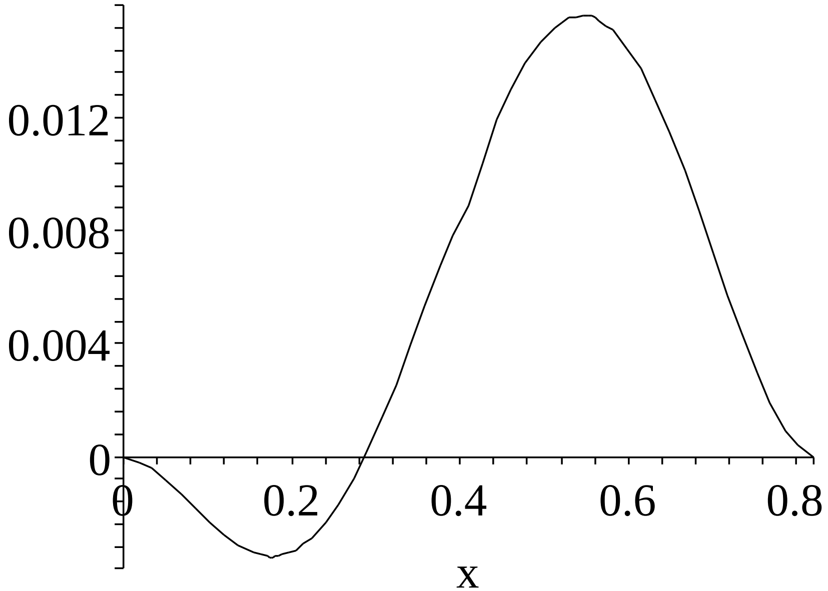
Distortion (in millimetres) due to centrifugal forces as a function of distance (in meters) along the tube, as predicted by analysis.

Coriolis distortion (by analysis)



Distortion (in millimetres) due to Coriolis forces as a function of distance (in meters) along the tube, as predicted by analysis.

Centrifugal distortion (by ANSYS)



Distortion due to Coriolis forces as a function of distance along the tube, as predicted by FE modelling using ANSYS.

My efforts. . .

The coriolis distortion is wrong and not physical.

My efforts:

- C++ code to generate FE matrices
- Data read by matlab for eigen-computations
- The results were. . .

Eigenvalues (no shift)

	$\|\omega\|_{\min}$ by technique…		
N_e	1	2	3
16	940.1753	940.1753	$\approx 1 \leftrightarrow 13000$
32	938.9266	938.9266	$\approx 1 \leftrightarrow 13000$
64	938.8359	938.8359	$\approx 1 \leftrightarrow 13000$
128	938.8298	938.8300	$\approx 1 \leftrightarrow 13000$
256	938.8296	938.8296	$\approx 1 \leftrightarrow 13000$

Computed $|\omega|_{\min}$ for the Timoshenko beam (quadratic elements) with no shift, $p = 0$.

Eigenvalues (with shift)

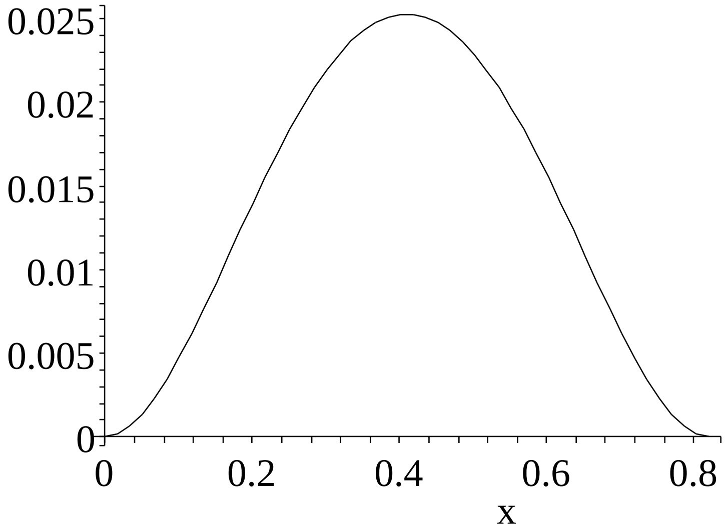
	$\|\omega\|_{\min}$ by technique…		
N_e	1	2	3
16	940.1753	940.1753	940.1753
32	938.9266	938.9266	938.9266
64	938.8359	938.8359	938.8359
128	938.8300	938.8300	938.8300
256	938.8296	938.8296	938.8296

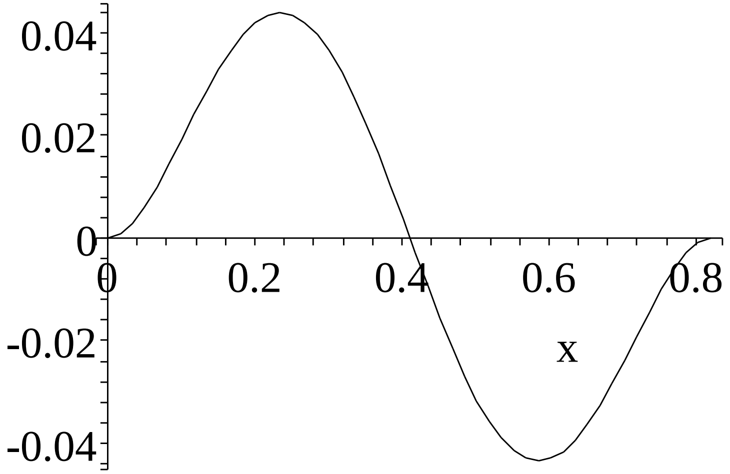
Computed $|\omega|_{\min}$ for the Timoshenko beam (quadratic elements) with shift $p = 900$.

Eigenvectors

What about the eigenvectors?

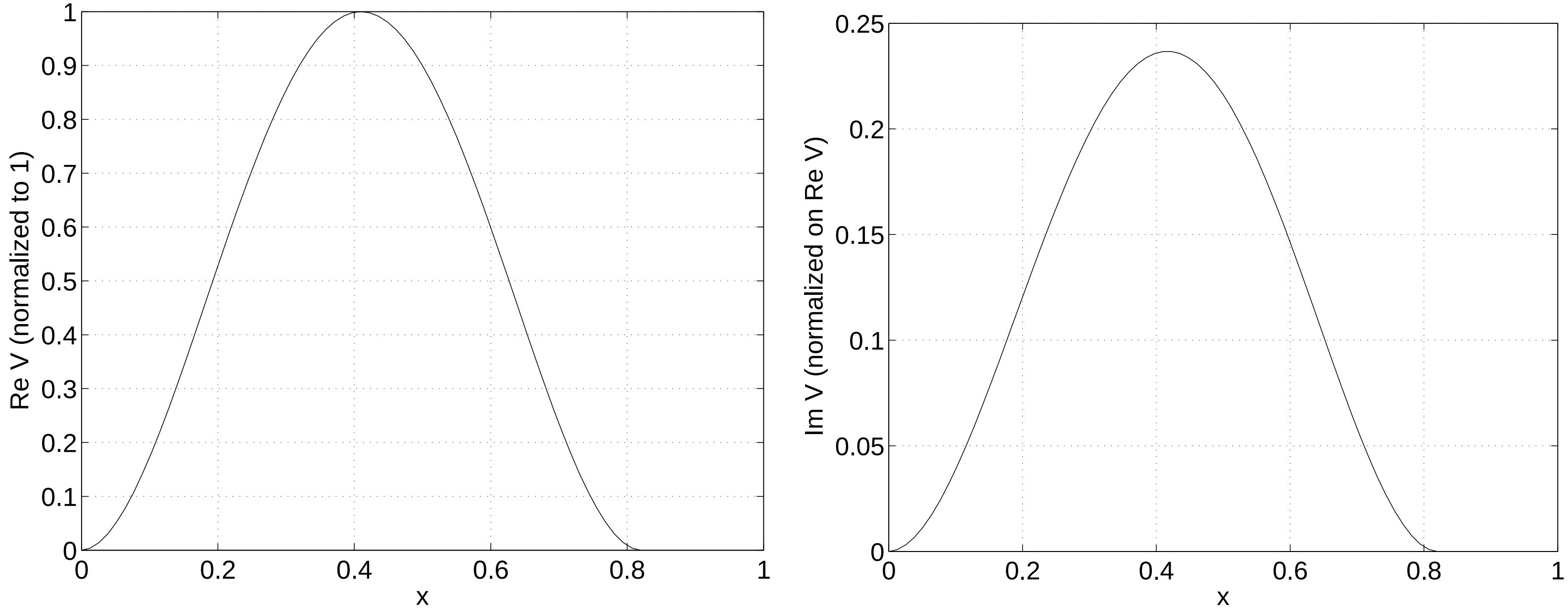
We are expecting to see. . .





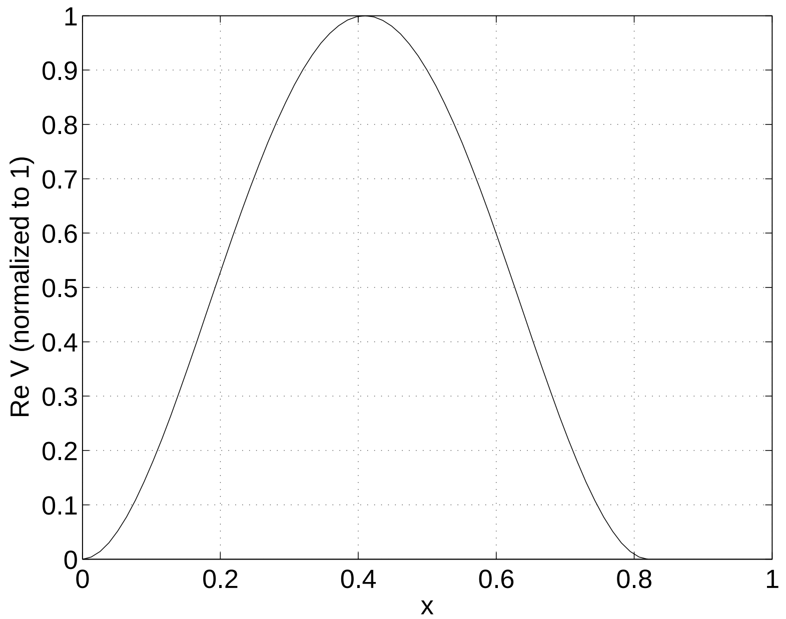
(centrifugal = real part, coriolis = imaginary part.)

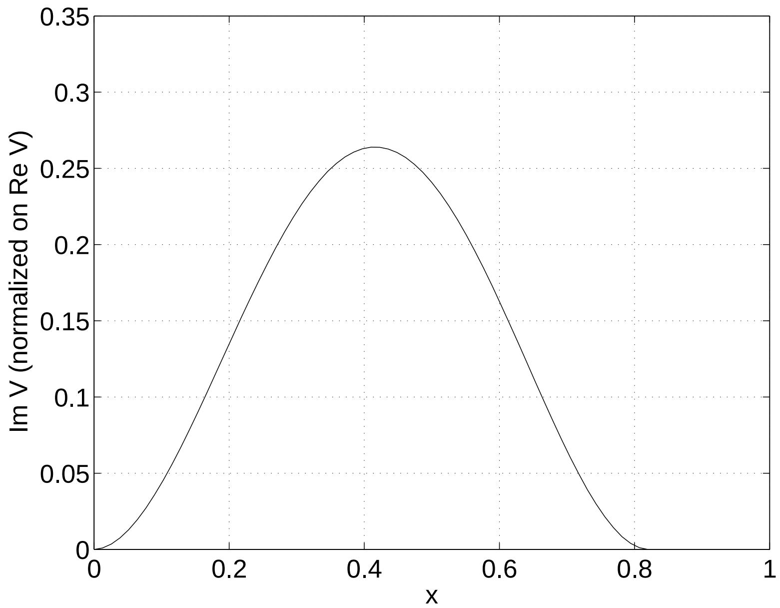
`polyeig` *without shift*



$\mathrm{Re}\, V$ and $\mathrm{Im}\, V$ from Matlab's `eig` routine. No shift (32 elements).

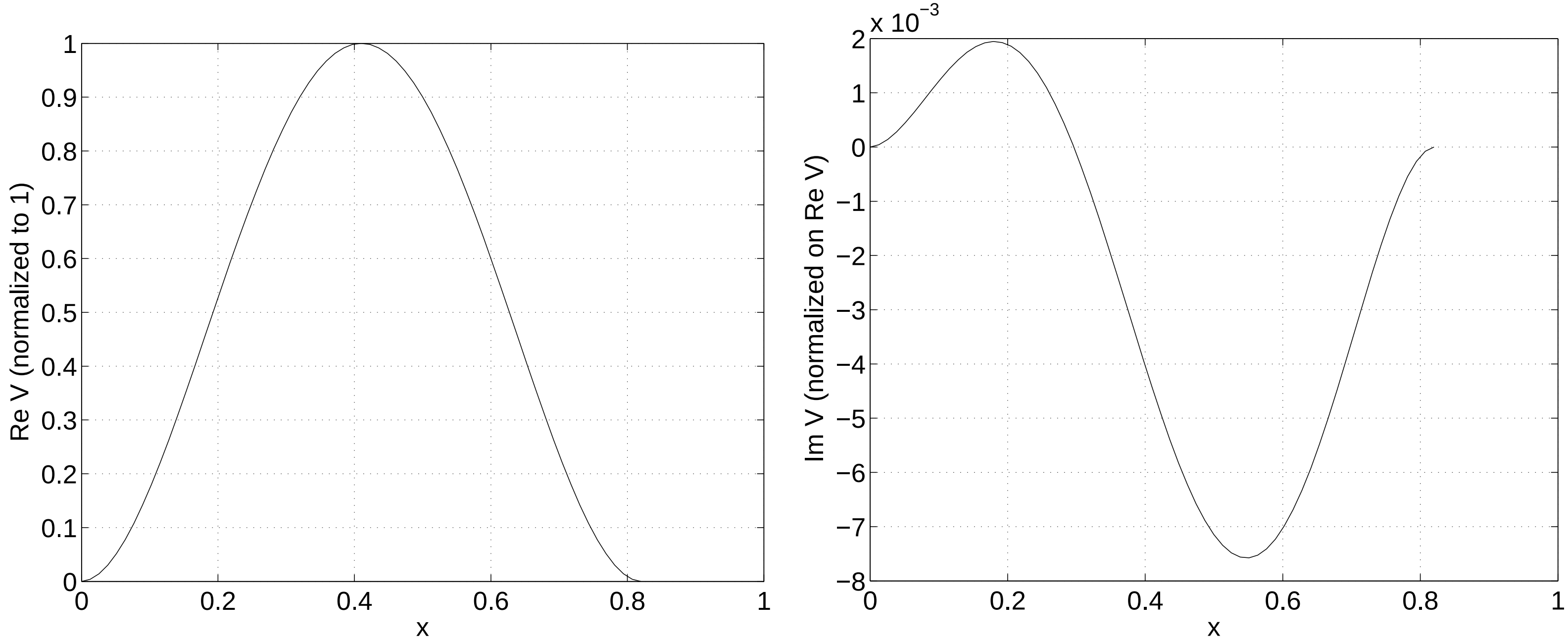
polyeig with shift





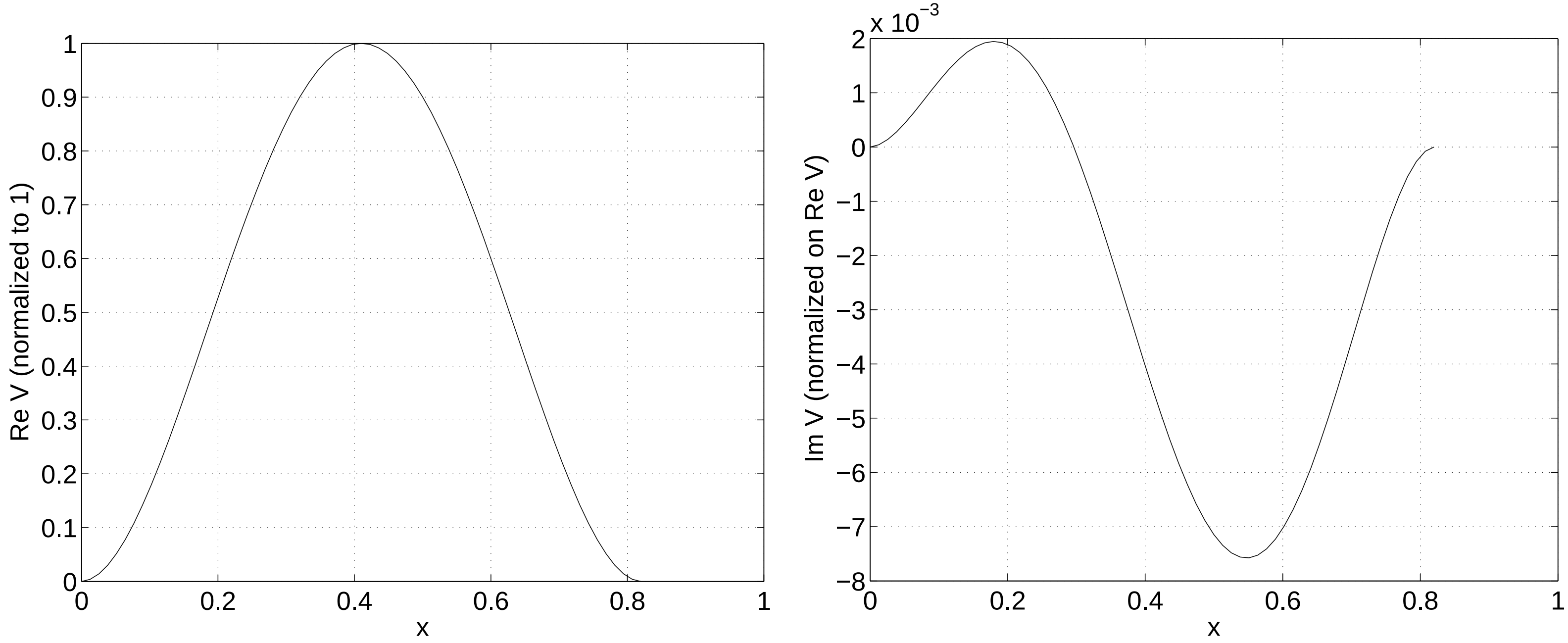
$\mathrm{Re}\,V$ and $\mathrm{Im}\,V$ from Matlab's `eig` routine. Shift $= 900$ (32 elements).

eig without shift



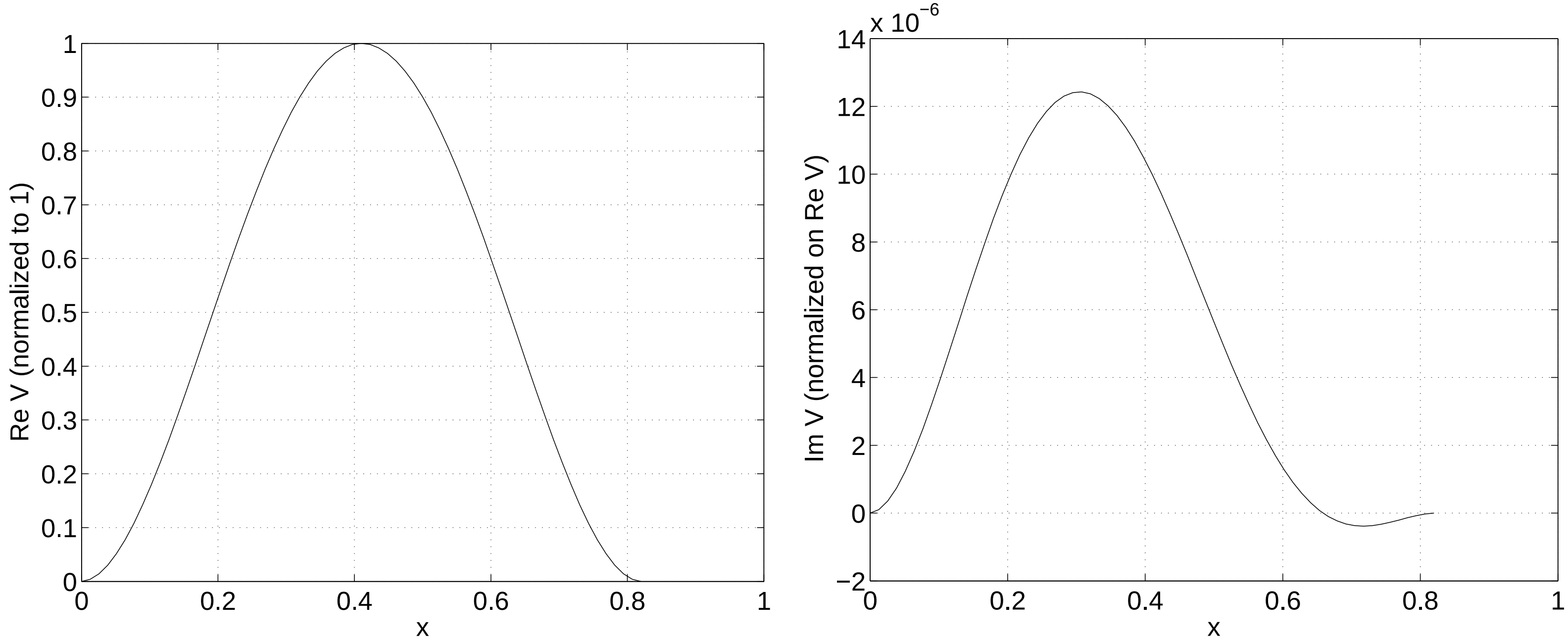
$\mathrm{Re}\, V$ and $\mathrm{Im}\, V$ from Matlab's `eig` routine. No shift (32 elements).

`eig` with shift



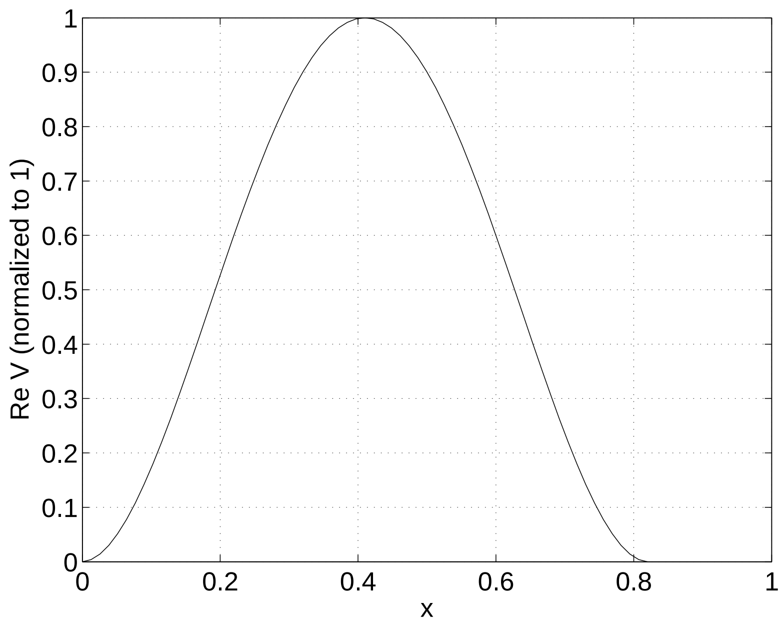
$\mathrm{Re}\, V$ and $\mathrm{Im}\, V$ from Matlab's `eig` routine. Shift $= 900$ (32 elements).

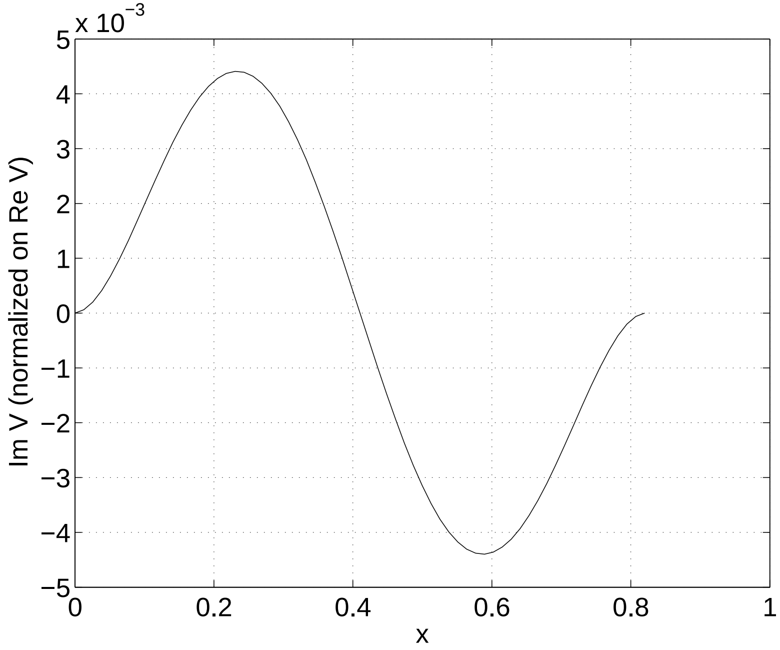
Inverse iteration without shift



$\mathrm{Re}\, V$ and $\mathrm{Im}\, V$ from inverse iteration. No shift (32 elements).

Inverse iteration with shift





$\mathrm{Re}\, V$ and $\mathrm{Im}\, V$ from inverse iteration. Shift $= 900$ (32 elements).

Conclusion

- Shifted inverse iteration is most robust (given a good initial guess).
- ANSYS and matlab seem to struggle.
- Problem is due to rounding error (a hunch!)
  - $\omega^2 r$ and $2\omega V$ are different orders of magnitude.
- A challenge for eigen-solvers?
- . . . or is there an ‘easy’ remedy?

FIAM

I'll leave you with the fundamental inequality of applied mathematics:

$$\|T - P\|_T \ll \|T - P\|_P$$

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The difference between theory and practice in theory
is less than
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Anon, circa 20th Century

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The End. . .