Lecture 1: Stability and Bifurcations for the Discretised Incompressible Navier-Stokes Equations

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Outline

The 3 lectures

- Lecture 1: Basic ideas of bifurcation/stability in time dependent PDEs; The Taylor-Couette problem - a comparison of experimental results with numerics; Numerical linear algebra of bordered matrices
- Lecture 2: Hopf bifurcations and periodic orbits in large systems; some open questions; The Taylor problem again
- Lecture 3: Inexact Inverse Iteration and Jacobi-Davidson with preconditioning; numerical results from Navier-Stokes and other problems
- "Cliffe, Spence & Tavener", review in Acta Numerica (2000)
- "Spence & Graham", introductory notes from 1998 Leicester Summer School

Stability and Bifurcation: the basics

- The Taylor-Couette Problem: Benjamin & Mullin experiments (1978,1981,...)
- (Linearised) Stability for time dependent discretised PDEs
- Bordered matrices
- Numerical continuation and bifurcations
- The Taylor problem again: comparison of numerics with experiments
- Conclusions

Outline

The Taylor Problem (Benjamin & Mullin)

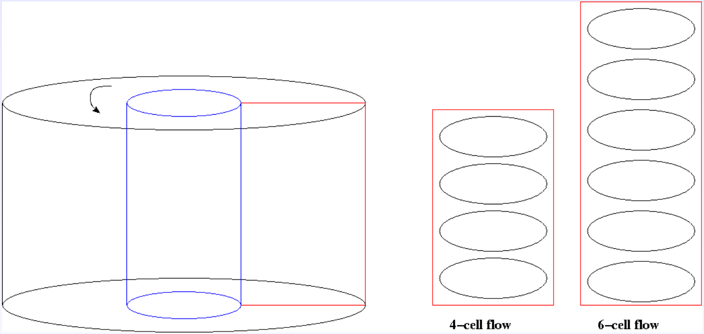
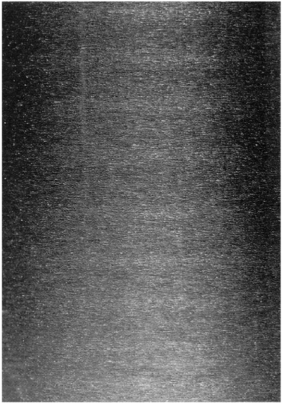
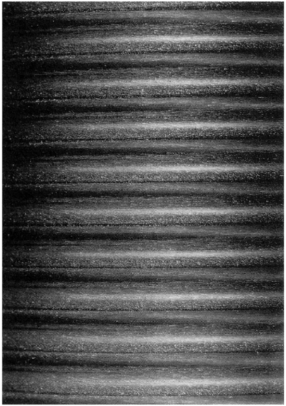


Figure: The Taylor problem showing 4-cell and 6-cell flows

- Two parameters: R, Reynold's number (speed of inner cylinder) and α, the aspect ratio (height/gap)
- Experiment:
  1. Fix α
  2. Increase R slowly from zero, or start up suddenly with large R

Taylor problem: photos





The Taylor Problem: Schematic of experimental results

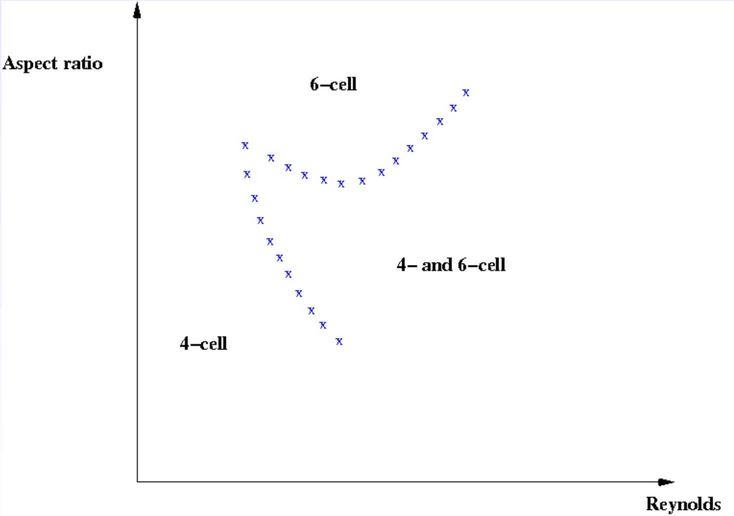


Figure: Parameter space plot showing loss of stability of 4 and 6 cell flows

The Taylor Problem: Anomalous modes (Benjamin & Mullin)

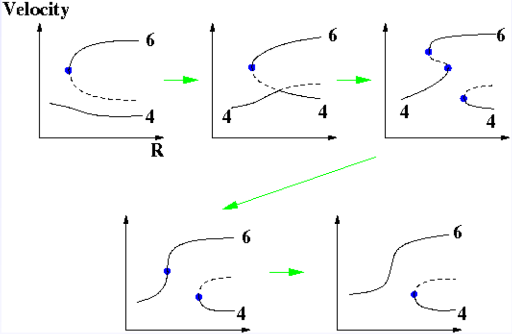


Figure: 4 and 6 cell anomalous modes: sequence of bifurcation diagrams as aspect ratio varies

Question

Can we reproduce these experimental results using numerical methods?

Outline

Linearised Stability

- $\dot{x} = F(x, \lambda), \ x(t) \in \mathbb{R}^n$
- Bifurcation Theory: change of stability of solutions (steady, periodic, homoclinic,...) as λ varies
- Steady solution: $0 = F(x, \lambda)$

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- Linearised Stability
  1. Perturbation: $x \to x + \delta$
  2. $\boxed{\dot{\delta} = A(\lambda)\delta}$ $\quad A(\lambda) = F_x(x, \lambda)$, Jacobian
  3. $\delta = e^{\mu\, t}\phi$
  4. $\boxed{A(\lambda)\phi = \mu\phi}$

Linearised Stability

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- As λ varies, μ varies in $\mathbb{C}$. Loss of stability arises:
  1. μ passes through 0, so F_x is singular
  2. a complex pair crosses imaginary axis: in this case F_x is non-singular (Lecture 2 on Hopf bifurcation.)
- left-half plane is "stable"; right-half plane is "unstable"
- Pseudo-eigenvalues?

Incompressible Navier-Stokes

Discretisation of linearised equations using mixed finite elements leads to the following eigenvalue problem:

- $$\begin{bmatrix} K(\lambda) & C \\ C^T & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \mu \begin{bmatrix} M & O \\ O & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}$$
- $$\boxed{A(\lambda)\phi = \mu B\phi}$$
- "Saddle point" $A(\lambda)$, but $K(\lambda)$ nonsymmetric
- μ could be complex
- B positive semidefinite: $\mu =$"∞"
- $\begin{bmatrix} K(\lambda) & \gamma C \\ \gamma C^T & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \mu \begin{bmatrix} M & C \\ C^T & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}$ "∞" mapped to γ

Strategy for Stability Analysis

- Compute steady state diagram: $F(x, \lambda) = 0$ Task 1
- Detect existence of bifurcation points (i.e. where real or complex eigenvalues of $F_x = A(\lambda)$ cross imaginary axis), and then locate them accurately Task 2
- In two parameter problems (e.g. Reynold's number and aspect ratio): Compute paths of bifurcation points Task 3
- Key tool: Bordered matrices

Outline

Background on Bordered matrices

- $A \in \mathbb{R}^{n\times n}, \quad b, c \in \mathbb{R}^n$
- $M = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$

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- $M = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$
- If $\text{Rank}(A) = n$ and $(d - c^T A^{-1} b) \neq 0$, then M is nonsingular
- If $\text{Rank}(A) < n - 1$ then M is singular
- If $\text{Rank}(A) = n - 1$ with $A\phi = 0, \quad \psi^T A = 0^T$ then

$$\psi^T b \neq 0, \quad c^T \phi \neq 0 \iff M \text{ nonsingular}$$

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- Bordering is important
- Example: Assume A has singular values $\sigma_1 \geq \cdots \geq \sigma_{n-1} > 0$. Then

$$M = \begin{bmatrix} A & \psi \\ \phi^T & 0 \end{bmatrix}$$

has singular values $\sigma_1 \geq \cdots \geq \sigma_{n-1}, 1, 1$

Solving bordered systems: A nearly singular

- Assume A has structure
- Consider

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- Doolittle (D)

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} = \begin{bmatrix} I & 0 \\ w^T & 1 \end{bmatrix} \begin{bmatrix} A & b \\ 0 & \delta \end{bmatrix}$$

  Forward/back substitutions use 1 solve with A^T, ($A^T w = c$), and 1 solve with A

- Crout (C)

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} = \begin{bmatrix} A & 0 \\ c^T & \delta \end{bmatrix} \begin{bmatrix} I & v \\ 0 & 1 \end{bmatrix}$$

  Forward/back substitutions use 2 solves with A

Block Elimination Algorithm for A nearly singular: Govaerts&Pryce

Consider

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- Crout (C) and Doolittle (D) both fail when A is nearly singular
- BUT:
  1. (D) computes y well
  2. If y is known accurately, (C) computes x well

Block Elimination Algorithm for A nearly singular: Govaerts&Pryce

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- Crout (C) and Doolittle (D) both fail when A is nearly singular
- BUT:
  1. (D) computes y well
  2. If y is known accurately, (C) computes x well
- Method: Use (D) to get $\tilde{y}$. Apply iterative refinement on (C) with starting guess $(0, \tilde{y})$
- Govaerts&Pryce: Backward stable
- Cost: 1 solve with A^T, 2 solves with A

Bordered matrices and Iterative solvers

- Calvetti&Reichel (2000)
- A symmetric
- monitor eigenvalues of F_x along $F(x, \lambda) = 0$ using Implicitly Restarted Block Lanczos
- solve bordered systems using FOM with basis from Block Lanczos
- No preconditioning?
- Extension to nonsymmetric problems -OK for real eigenvalues but complex eigenvalues?
- LOCA "Library of Continuation Algorithms", Sandia

Bordered Matrices

We shall see that bordered matrices arise naturally in the following 3 tasks:

1. Numerical Continuation (i.e. computing $F(x, \lambda) = 0$)
2. (i) Detecting when $\mathrm{Det}(F_x)$ changes sign, and
   (ii) accurate calculation of singular points
3. Numerical continuation of paths of singular points in 2-parameter problems
4. Requirement: Efficient algorithms for bordered matrices with structure

Outline

To compute $F(x, \lambda) = 0$; Pseudo-arclength continuation (Keller)

- Implicit Function Theorem (IFT):

$$F(x_0, \lambda_0) = 0, \text{ and } F_x(x_0, \lambda_0) \text{ nonsingular} \Rightarrow,$$

$$F(x(\lambda), \lambda) = 0 \quad \text{near } \lambda = \lambda_0$$

- (x_0, λ_0) is regular. IFT$\Rightarrow \exists$ path of regular points near (x_0, λ_0)
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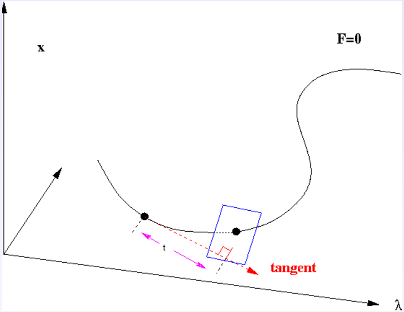
- To "pass over" singular points add an extra normalisation:

$$G(y, t) = \begin{bmatrix} F(x, \lambda) \\ c^T(x - x_0) + d(\lambda - \lambda_0) - t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

- $$G_y(y, t) = \begin{bmatrix} F_x & F_\lambda \\ c^T & d \end{bmatrix}$$

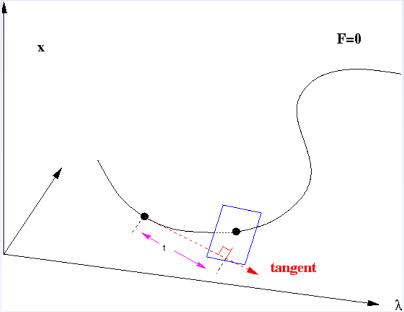
- c^T, d?
- Key tool: Efficient treatment of bordered matrices near points where F_x is nearly singular

Pseudo-arclength continuation



- The "normalisation"= equation of the plane ⊥ tangent
- t is the "length along the tangent" ("pseudo-arclength")
- $G(y, t) = 0$ represents the point where curve intersects the plane
- Method: compute tangent; form $G(y, t) = 0$; solve using Newton

Pseudo-arclength continuation



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- Method: compute tangent; form $G(y,t) = 0$; solve using Newton
- Aside: DAETS - $F(x,\lambda) = 0, \quad \dot{x}^T\dot{x} + \dot{\lambda}^2 = 1$

Generic bifurcations in 1-parameter problems

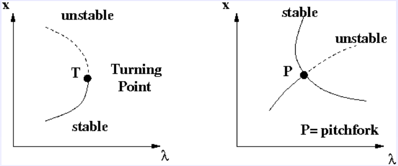

Figure: Generic behaviour for singular points in 1-parameter problems

- Lecture 2: Complex pair crosses Imaginary axis
- Two cases: (a) Turning Point
  (b) If a symmetry is broken (i.e eigenvector ϕ 'breaks' the symmetry) then Symmetric Pitchfork
- Taylor problem has a reflectional symmetry
- In both cases: $F(x(t), \lambda(t)) = 0$: $\mu(t)$ is eigenvalue of $F_x(x(t), \lambda(t))$ then

$$\mu(t) = 0, \ \frac{d}{dt}\mu(t) \neq 0 \quad \text{at the singular point}$$

  That is, an eigenvalue of F_x passes through zero "smoothly"
- loss of stability at the singular points

Detection then accurate calculation

- Seek (x, λ) such that $F_x(x, \lambda)$ is singular
- Consider
$$\begin{bmatrix} F_x(x,\lambda) & F_\lambda(x,\lambda) \\ c^T & d \end{bmatrix} \begin{bmatrix} * \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- $g = g(x, \lambda)$
- Cramer's Rule: $\mathrm{Det}(F_x) = 0 \iff g = 0$.

Detection then accurate calculation

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- $g = g(x,\lambda)$
- Cramer's Rule: $\mathrm{Det}(F_x) = 0 \iff g = 0$.
- Accurate calculation: Consider the pair
$$F(x,\lambda) = 0, \quad g(x,\lambda) = 0$$
- Newton's Method:
$$\begin{bmatrix} F_x(x,\lambda) & F_\lambda(x,\lambda) \\ g_x(x,\lambda)^T & g_\lambda(x,\lambda) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} F \\ g \end{bmatrix}$$
- System nonsingular if $\frac{d}{dt}\mu \neq 0$ at singular point

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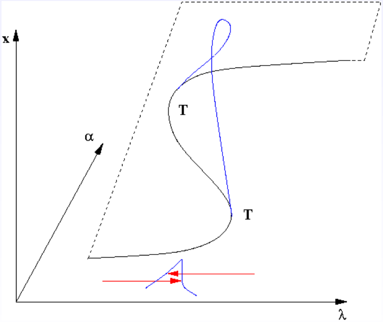
- System nonsingular if $\frac{d}{dt}\mu \neq 0$ at singular point
- Extended Systems:
$$F(x,\lambda) = 0, \quad F_x(x,\lambda)\phi = 0, \quad c^T\phi = 1$$
  Also reduces to solving 1-bordered systems (numerics for Taylor problem)
- Adapt for symmetry-breaking

2-parameter problems (e.g. The Taylor Problem)

- Use system that is nonsingular at a bifurcation point (F_x singular)
- Use pseudo-arclength to follow paths bifurcation points. For example:

$$F(x,\lambda,\alpha)=0, \quad g(x,\lambda,\alpha)=0, \quad n(x,\lambda,\alpha,t)=0$$

  where $n(x,\lambda,\alpha,t)=0$ is the "normalisation" (2-bordered systems)
- detect singular points on path of bifurcations?



Transcritical bifurcation

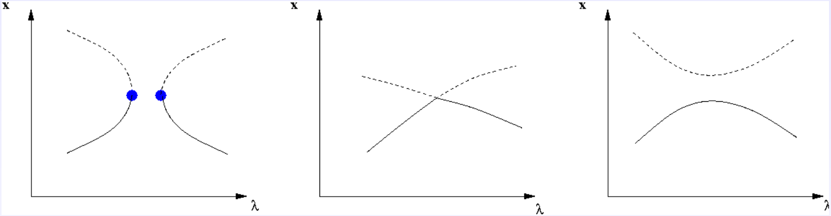


Figure: The sequence to a transcritical bifurcation for $F(x, \lambda, \alpha) = 0$

- solid lines represent stable solutions
- Transcritical bifurcations should not occur in 1-parameter problems
- A Transcritical bifurcation, and a Cusp are Turning points in a path of Turning points
- Transcritical and Cusp bifurcations are "codimension 1"
- Multi-parameter problems?
- High codimension points are called Organising Centres

Outline

Recall the Taylor Problem: Schematic of experimental results

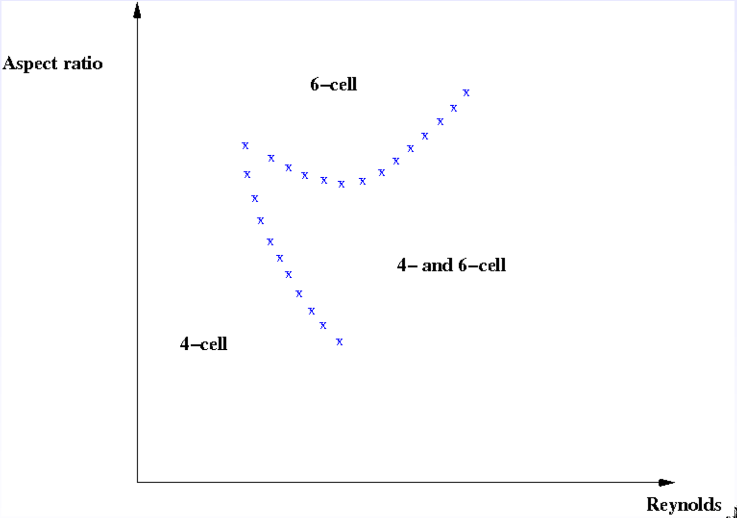


Figure: Parameter space plot showing loss of stability of 4 and 6 cell flows

Recall the Taylor Problem

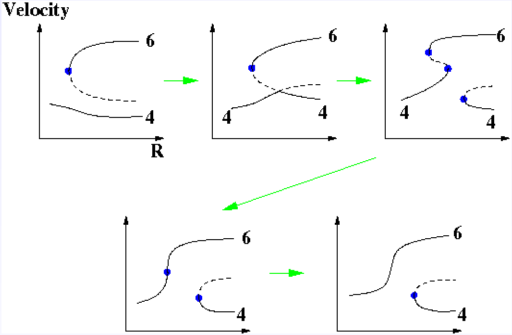
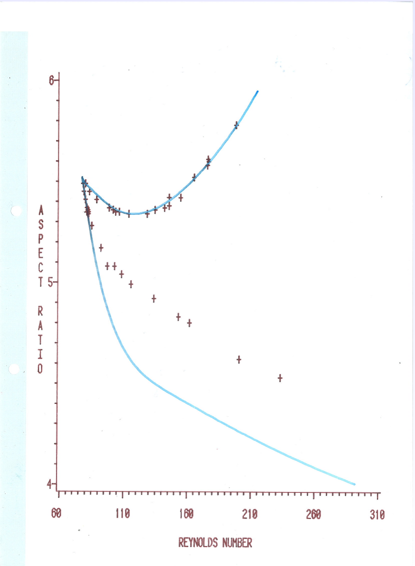
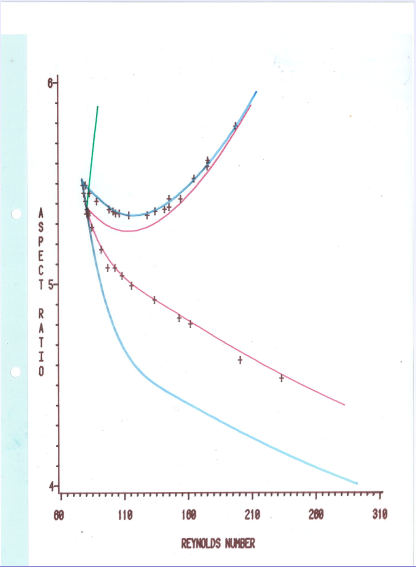


Figure: 4 and 6 cell anomalous modes: sequence of bifurcation diagrams as aspect ratio varies

Numerical results for the 4-6 cell interchange (Cliffe)



The 4-6 cell interchange including symmetry-breaking (Cliffe)



Outline

Conclusions

- Numerical methods work!
- Excellent agreement between numerics and experiment
- Eigenvalues work! (Problem isn't very "non-normal")
- The numerical methods gave extra insight via symmetry-breaking
- Efficient methods for bordered systems are crucial
- Iterative methods for bordered sytems in continuation and bifurcation analysis?
- Lecture 2: Hopf bifurcations and periodic orbits