Accurate and stable recovery of functions from spectral data

Ben Adcock & Anders C. Hansen

DAMTP, University of Cambridge

UNIVERSITY OF CAMBRIDGE

Abstract

We consider the problem of reconstructing a function (defined on some bounded domain) to high accuracy from a finite number of its coefficients with respect to some orthogonal basis. Straightforward expansion in this basis may converge slowly. Yet it is always possible to reconstruct the function in another basis. Such reconstruction technique is stable, and the resultant approximation near-optimal.

A simple example of this approach is the accurate reconstruction of an analytic, nonperiodic function from its Fourier coefficients, with numerous applications including image and signal processing. The Fourier series of such a function converges slowly. Nonetheless, by reconstructing in a polynomial basis, we obtain exponential convergence in terms of n (the polynomial degree), or root exponential convergence in m (the number of Fourier coefficients). The procedure can be implemented in $\mathcal{O}(mn)$ operations.

A general reconstruction theorem

Let $\{\psi_j\}_{j=1}^{\infty}$ be an orthonormal basis for the Hilbert space $\mathrm{H}(\Omega)$ of real-valued functions (the sampling basis). Suppose further that the first m coefficients of a function $f \in \mathrm{H}(\Omega)$ with respect to this basis are known:

$$\hat{f}_j = \langle f, \psi_j \rangle, \quad j = 1, \ldots, m.$$

Direct approximation of f via the orthogonal projection $\mathcal{P}_m : \mathrm{H}(\Omega) \to S_m = \mathrm{span}\{\psi_1, \ldots, \psi_m\}$ may converge slowly. Instead, we seek to recover f in the reconstruction basis $\{\phi_j\}_{j=1}^{\infty}$.

Let $T_n = \mathrm{span}\{\phi_1, \ldots, \phi_n\}$ and $\mathcal{Q}_n : \mathrm{H}(\Omega) \to T_n$ be the orthogonal projection. Our approximation $f_n \in T_n$ to f is defined by the equations

$$a_m(f_n, g) = a_m(f, g), \quad \forall g \in T_n,$$

where $a_m : T_n \times T_n \to \mathbb{R}$ is the bilinear form

$$a_m(g, h) = \langle \mathcal{P}_m g, \mathcal{P}_m h \rangle, \quad \forall g, h \in \mathrm{H}(\Omega).$$

The main result is as follows:

Theorem 1. *For every $n \in \mathbb{N}$ there exists an M such that the approximation f_n exists and is unique for all $m \geq M$, and satisfies the stability estimate $\|f_n\| \leq (1 - C_{n,m}^2)^{-1}\|f\|$. Furthermore,*

$$\|f - f_n\| \leq K_{n,m}\|f - \mathcal{Q}_n f\|, \quad K_{n,m} = \sqrt{1 + \frac{C_{n,m}^2}{(1 - C_{n,m}^2)^2}},$$

where, for fixed n, the constant $C_{n,m} \to 0$ as $m \to \infty$, and is given explicitly by $C_{n,m} = \|(\mathcal{I} - \mathcal{P}_m)|_{\mathrm{T}_n}\|$. In particular, the parameter M is the least value of m such that $C_{n,m} < 1$.

We conclude:

- Reconstruction is always possible, regardless of the two bases.
- For sufficiently large m, f_n is quasi-optimal: the error $\|f - f_n\|$ is bounded by a constant multiple of $\inf_{g \in T_n} \|f - g\|$.
- As $m \to \infty$, $f_n \to \mathcal{Q}_n f = \mathrm{argmin}_{g \in T_n}\|f - g\|$. Hence, f_n is asymptotically optimal.

Computing f_n

The equations for f_n can be interpreted as the normal equations of a least squares problem. If $f_n = \sum_{j=1}^{n} \alpha_j \phi_j$, $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$ and $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_m)^\top$, then $U^\top U \alpha = \hat{f}$, where U is the $m \times n$ matrix with $(j,k)^{\text{th}}$ entry $\langle \phi_k, \psi_j \rangle$.

These equations are well conditioned:

Lemma 1. *For $n \in \mathbb{N}$ and $m \geq M$ the condition number $\kappa(U^*U) \leq (1 - C_{n,m}^2)^{-1}$. In particular, for fixed n, $\kappa(U^*U) \to 1$ as $m \to \infty$.*

As a direct consequence, f_n can be computed in only $\mathcal{O}(mn)$ operations using, for example, conjugate gradients.

Guaranteed recovery

Naturally, to implement this method, we require conditions for guaranteed recovery. In other words, we must estimate the quantity:

$$\Phi(n;\theta) = \min\{m \in \mathbb{N} : C_{n,m} \leq \theta\}, \quad \theta \in (0,1).$$

By definition, $\Phi(n;\theta)$ is the least m such that $\|f - f_n\| \leq c(\theta) \inf_{g \in T_n} \|f - g\|$, where $c(\theta) = \sqrt{1 + \theta(1-\theta)^{-2}}$. In other words, this is the least m required for quasi-optimal recovery with constant $c(\theta)$.

The function $\Phi(n;\theta)$ depends only on the sampling and reconstruction bases. Estimates (analytical and numerical) are determined on a case-by-case basis.

Reconstruction from Fourier coefficients

Suppose that $f : [-1,1] \to \mathbb{R}$ is analytic and nonperiodic. In many applications we may only know the first m Fourier coefficients of f

$$\hat{f}_j = \int_{-1}^{1} f(x) \mathrm{e}^{-\mathrm{i}j\pi x}\,\mathrm{d}x, \quad j = -\frac{m}{2}, \ldots, \frac{m}{2}.$$

We wish to recover f to high accuracy. The truncated Fourier sum converges slowly (in the L^2 norm), and suffers from the well-known Gibbs phenomenon. We now seek to reconstruct f in the Legendre polynomial basis $\phi_j(x) = \sqrt{j + \frac{1}{2}} P_j(x)$, $j = 0, \ldots, n-1$.

By Theorem 1, the approximation f_n converges exponentially fast in n, provided $m \geq \Phi(n;\theta)$. It is therefore vital to estimate $\Phi(n;\theta)$.

Estimates for $\Phi(n;\theta)$

For these bases, we have the following analytical bounds for $\Phi(n;\theta)$:

Theorem 2. *The function $\Phi(n;\theta)$ satisfies*

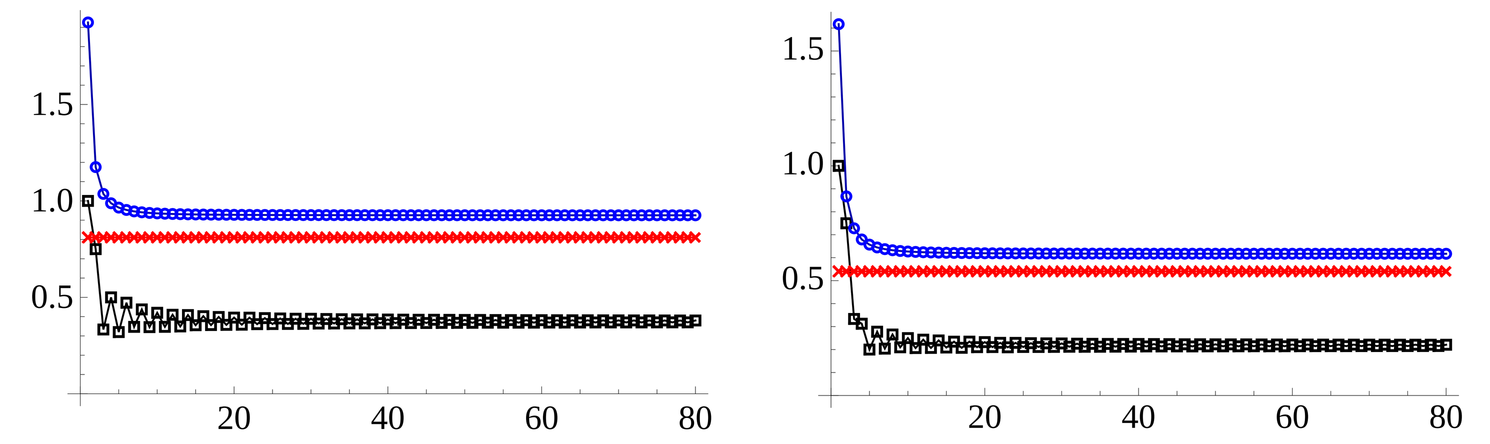
$$\Phi(n;\theta) \leq 1 + \frac{4(\pi - 2)}{\pi^2 \theta} n^2, \quad \forall n \in \mathbb{N}.$$

In addition,

$$n^{-2}\Phi(n;\theta) \leq \frac{4}{\pi^2 \theta} + \mathcal{O}\left(n^{-2}\right), \quad n \to \infty.$$

We conclude that $m = \mathcal{O}\left(n^2\right)$ for quasi-optimal recovery. Hence, root exponential convergence of f_n in terms of m.

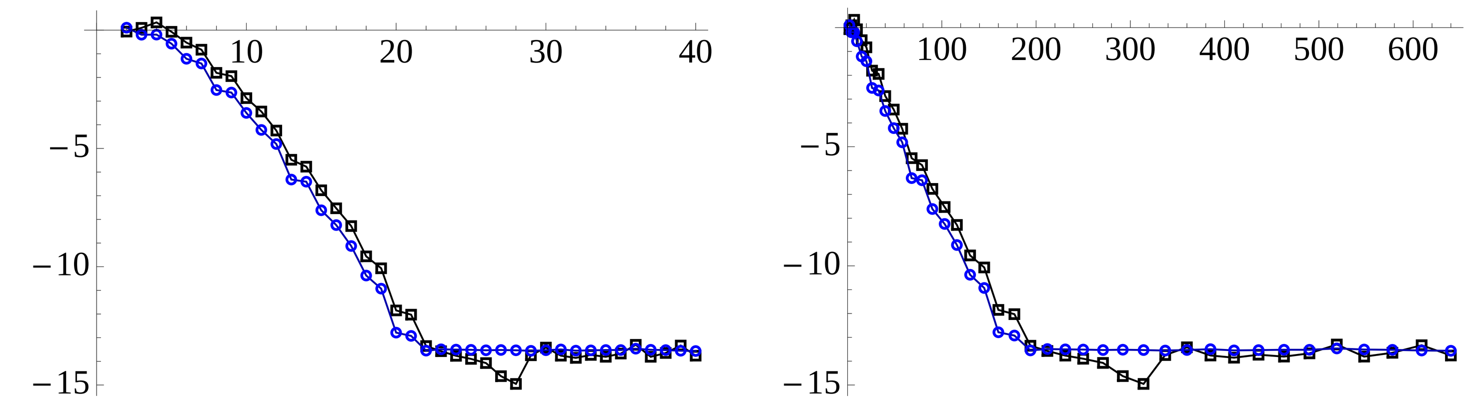
In fact, the bounds in this theorem are reasonably accurate:



The function $n^{-2}\Phi(n;\theta)$ (black), the global bound (blue) and the asymptotic bound (red), for $n = 1, \ldots, 80$ and $\theta = \frac{1}{2}$ (left), $\theta = \frac{3}{4}$ (right).

Numerical examples

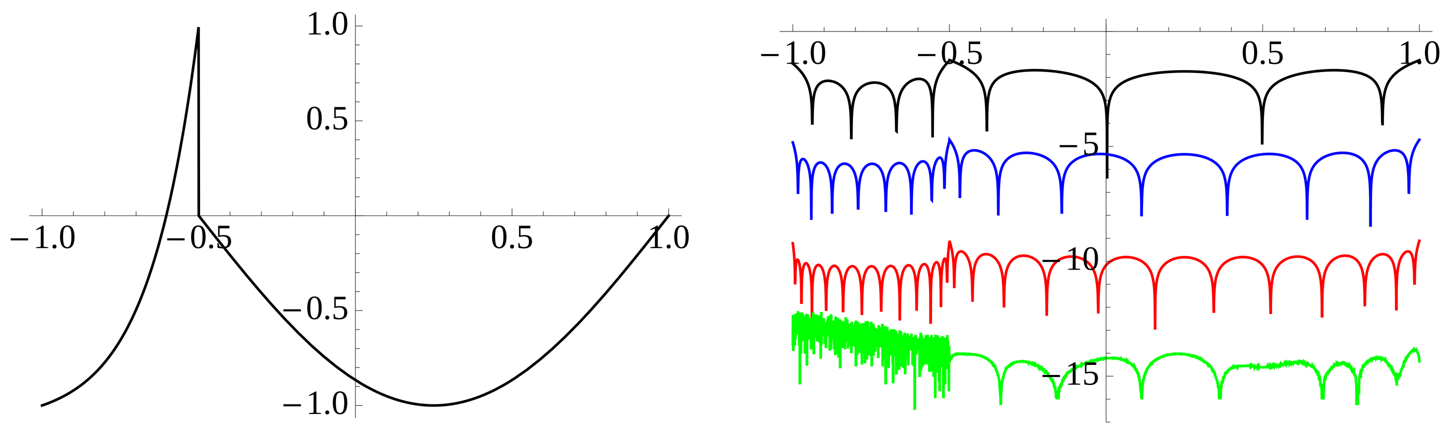
As a result, we typically use $m = 0.4n^2$ in computations:



Error in approximating $f(x) = \mathrm{e}^{-x}\cos 4x$ by $f_n(x)$. Left: log error $\log_{10}\|f - f_n\|_\infty$ (black) and $\log_{10}\|f - f_n\|$ (blue) against n. Right: log error against $m = 0.4n^2$.

Using only $n = 20$ and $m = 200$, we obtain 14 significant digits for this function.

Naturally, this technique can also be extended to piecewise analytic functions by reconstructing in a piecewise polynomial basis:



Error in approximating f (left) by f_n. Right: pointwise error $\log_{10}|f(x) - f_n(x)|$ for $-1 \leq x \leq 1$ and $n = 8, 16, 24, 32$.

Here, despite the sharp peak at $x = -\frac{1}{2}$, the parameters $n = 32$, $m = 410$ give 14 digits of accuracy.

Open problems

- **Chebyshev polynomials:** these are more convenient than Legendre polynomials. However, since they are not orthogonal on $\mathrm{L}^2(-1,1)$, a different implementation is needed.
- **Higher dimensions:** Theorem 1 holds for any two bases and any domain Ω. An obvious, and simple, extension is to the cube $[-1,1]^d$, but there are many other possibilities.