Mixed Multiscale Methods for Heterogeneous Elliptic Problems

Part 1: Introduction and Background
Part 2: Mixed Multiscale Numerics
Part 3: Mixed Multiscale Mortar Methods

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We consider the following four approaches:

1. Homogenization and Upscaling: Part 1
2. Multiscale Finite Elements: Parts 1–2
3. Variational Multiscale Method: Parts 1–2
4. Domain Decomposition and Mortar Methods: Part 3

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Part 1: Introduction and Background

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Outline

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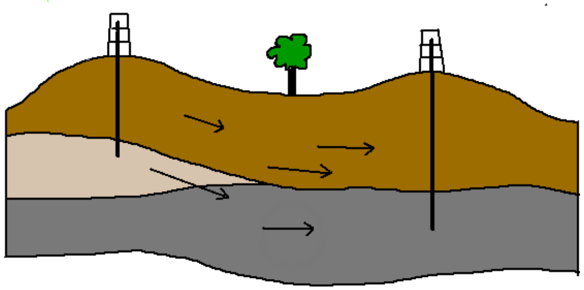
2. Homogenization and Upscaling
   - Simple Averaging
   - Mathematical Homogenization
3. Multiscale Numerics
   - The Nonmixed System: Multiscale Finite Elements
   - The Nonmixed System: Variational Multiscale Method
4. Some Numerical Examples
5. Summary and Conclusions

Elliptic Systems with a Heterogeneous Coefficient

A Second Order Elliptic PDE

Incompressible, single phase flow in a porous medium:

$$\begin{cases} \mathbf{u} = -a_\epsilon \nabla p & \text{in } \Omega \subset \mathbb{R}^d \quad \text{(Darcy's law)} \\ \nabla \cdot \mathbf{u} = f & \text{in } \Omega \quad \text{(Conservation)} \\ \mathbf{u} \cdot \nu = 0 & \text{on } \partial\Omega \quad \text{(BC for simplicity)} \end{cases}$$



p is the fluid pressure

$\mathbf{u}$ is the (Darcy) velocity of the fluid

a_ϵ is the medium permeability, heterogeneous on a scale ϵ

f is the source/sink term (i.e., the wells).

Objective: Given a_ϵ and f:

- Find an *accurate* approximation of $\mathbf{u}$ and p
- Respect the principle of mass *conservation*

Both properties are critical in many applications.

Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ or $(L^2(\Omega))^d$ inner product.

Find $p \in W = L^2(\Omega)/\mathbb{R}$ and $\mathbf{u} \in \mathbf{V} = H_0(\text{div};\Omega)$ such that

$$(a_\epsilon^{-1}\mathbf{u},\mathbf{v}) = -(\nabla p,\mathbf{v}) = (p,\nabla\cdot\mathbf{v}) \quad \forall\ \mathbf{v}\in\mathbf{V} \quad \text{(Darcy's law)}$$

$$(\nabla\cdot\mathbf{u},w) = (f,w) \quad \forall\ w\in W \quad \text{(Conservation)}$$

where

$$H_0(\text{div};\Omega) = \{\mathbf{v}\in(L^2(\Omega))^d : \nabla\cdot\mathbf{v}\in L^2(\Omega) \text{ and } \mathbf{v}\cdot\nu = 0 \text{ on } \partial\Omega\}$$

Remark: The mixed form preserves the conservation equation, and so allows locally conservative approximations.

Find $p \in W$ and $\mathbf{u} \in \mathbf{V}$ such that

$$A(\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = G(\mathbf{v}) \qquad \forall\ \mathbf{v} \in \mathbf{V}$$

$$(w, \nabla \cdot \mathbf{u}) = F(w) \qquad \forall\ w \in W$$

Theorem (Babuška 1973; Brezzi 1974). Suppose A is a continuous, symmetric bilinear form, coercive on $\mathbf{V} \cap \ker(\nabla \cdot)$, and $\exists \gamma > 0$ such that

$$\inf_{w \in W} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(w, \nabla \cdot \mathbf{v})}{\|w\|_W \, \|\mathbf{v}\|_{\mathbf{V}}} \geq \gamma$$

Then $\exists!$ solution $(p, \mathbf{u}) \in W \times \mathbf{V}$, and

$$\|p\|_W + \|\mathbf{u}\|_{\mathbf{V}} \leq C\{\|F\|_{W^*} + \|G\|_{\mathbf{V}^*}\}$$

Mixed Finite Element Approximation

Define

$\mathcal{T}_h$ a reasonable finite element partition of Ω

h the maximal element diameter

$W_h \times \mathbf{V}_h$ any reasonable mixed finite element spaces in $W \times \mathbf{V}$

Find $p \in W_h \subset W$ and $\mathbf{u} \in \mathbf{V}_h \subset \mathbf{V}$ such that

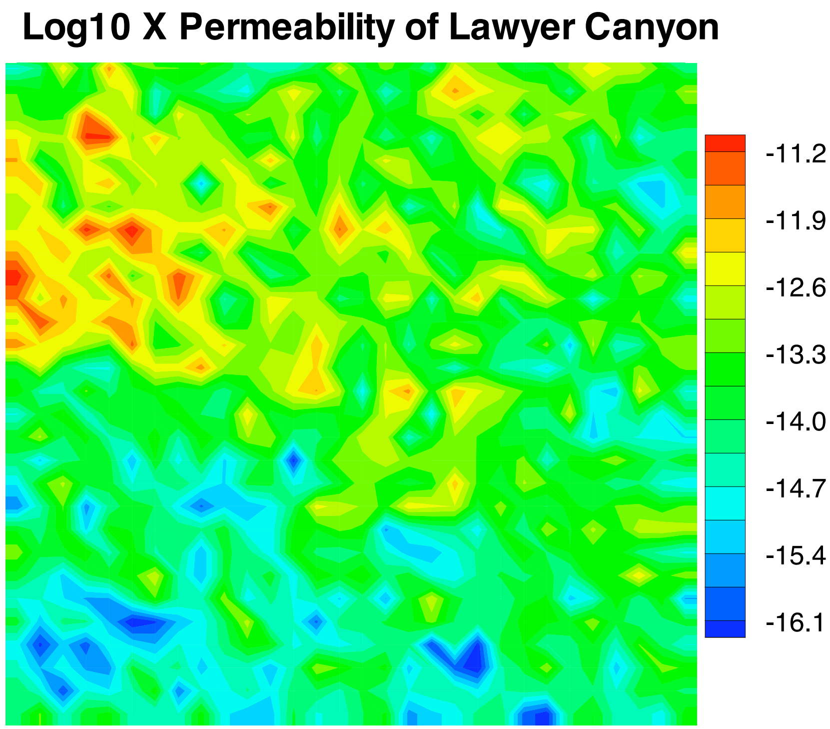
$$(a_\epsilon^{-1}\mathbf{u}_h, \mathbf{v}) = (p_h, \nabla \cdot \mathbf{v}) \quad \forall\ \mathbf{v} \in \mathbf{V}_h \quad \text{(Darcy's law)}$$

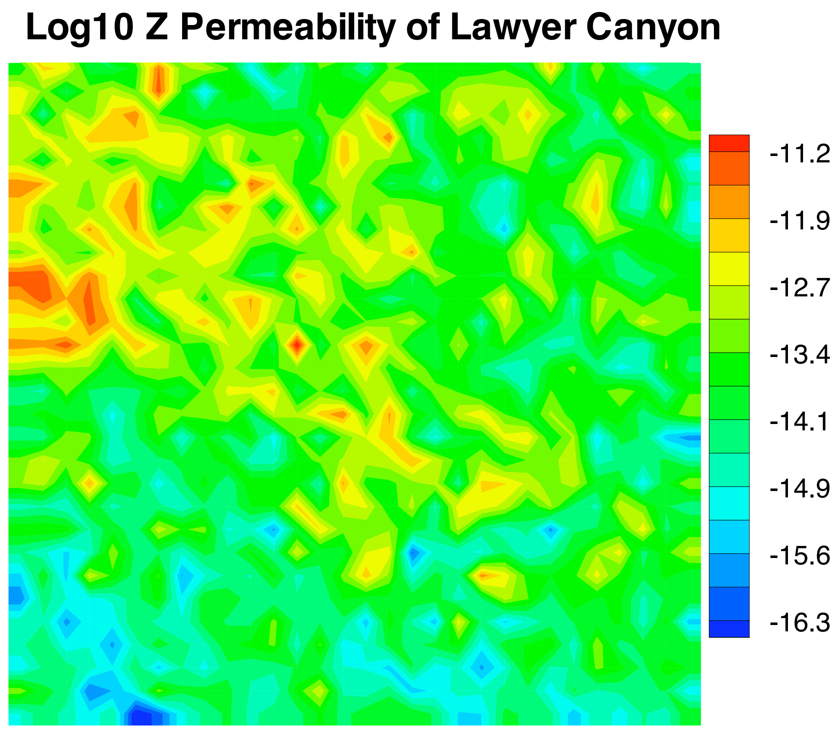
$$(\nabla \cdot \mathbf{u}_h, w) = (f, w) \quad \forall\ w \in W_h \quad \text{(conservation)}$$

Theorem: For mixed velocity spaces containing $\mathbb{P}_{k-1}$ on each element,

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_0 &\le C\|\mathbf{u}\|_k h^k &= \mathcal{O}(h^k) \\
\|p - p_h\|_0 &\le C\|p\|_{k+1} h^k &= \mathcal{O}(h^k) \\
\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_0 &\le C\|\nabla \cdot \mathbf{u}\|_k h^k &= \mathcal{O}(h^k)
\end{aligned}$$

where $\|\cdot\|_k$ is the norm in $H^k(\Omega)$.





Lawyer Canyon data, meter scale
(permeability ranges by a factor of 10^6)

Suppose a_ϵ varies on the the spatial scale ϵ. Then

$$|\mathbf{u}| = \mathcal{O}(\epsilon^{-1}) \quad \text{and} \quad |D^k \mathbf{u}| = \mathcal{O}(\epsilon^{-k-1})$$

Theorem: For mixed velocity spaces containing $\mathbb{P}_{k-1}$ on each element,

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C \|\mathbf{u}\|_k h^k = \epsilon^{-1} \mathcal{O}\left(\frac{h}{\epsilon}\right)^k$$

- If $h > \epsilon$, this is *not* small!
- To resolve p and $\mathbf{u}$, we need $h < \epsilon$. That is, we must resolve a_ϵ.

Problem: A direct computation is not feasible!

- $\Omega \sim 10^4 \times 10^4 \times 10^2$ m^3
- $h \sim 10^{-1}$ m

$\Longrightarrow$ a grid of size $10^5 \times 10^5 \times 10^3 = 10^{13}$ elements.

Currently, perhaps the largest supercomputers can handle 10^7 elements.

We consider the following four approaches:

1. *Homogenization and Upscaling:* (Bensoussan, Lions & Papanicolaou 1978; Sanchez-Palencia 1980)

   Replace the coefficient a_ϵ in the differential equation by one that is easier to resolve.

2. *Multiscale Finite Elements:* (Babuška & Osborn 1983; Babuška, Caloz & Osborn 1994; Hou & Wu 1997; Chen & Hou 2003)

   Define the finite element space to better capture fine scales.

3. *Variational Multiscale Method:* (Hughes 1995; Arbogast, Minkoff & Keenan 1998; Arbogast & Boyd 2006)

   Modify the variational form to better captures fine scales.

4. *Domain Decomposition and Mortar Methods:* (Schwartz 1870; Arbogast, Pencheva, Wheeler & Yotov 2007)

   Divide the problem into weakly coupled small subdomains that can be resolved.

Homogenization and Upscaling

Volume Averaging for Effective Properties

We want to solve the problem on a coarse grid.

Upscaling: The system is represented on a coarser scale by defining average or effective macroscopic parameters in place of the true parameters (in our case, a_ϵ).

Simple Averaging

A Naive Example

Consider 1-D. Select $\eta > 0$ as an averaging *window* and define the average

$$\bar{\psi}(x) = \frac{1}{\eta} \int_{x-\eta/2}^{x+\eta/2} \psi(\xi)\, d\xi$$

Upscale the micromodel to the macromodel

$$\begin{cases} \mathbf{u} = -a_\epsilon \nabla p \\ \nabla \cdot \mathbf{u} = f \end{cases} \quad \Longrightarrow \quad \begin{cases} \bar{\mathbf{u}} = -\overline{a_\epsilon \nabla p} \stackrel{?}{=} -\bar{a} \nabla \bar{p} \\ \nabla \cdot \bar{\mathbf{u}} \stackrel{?}{=} \overline{\nabla \cdot \mathbf{u}} = \bar{f} \end{cases}$$

Fundamental problem in upscaling: Nonlinearities!

$$\text{average of } F(x) \neq F(\text{average of } x)$$

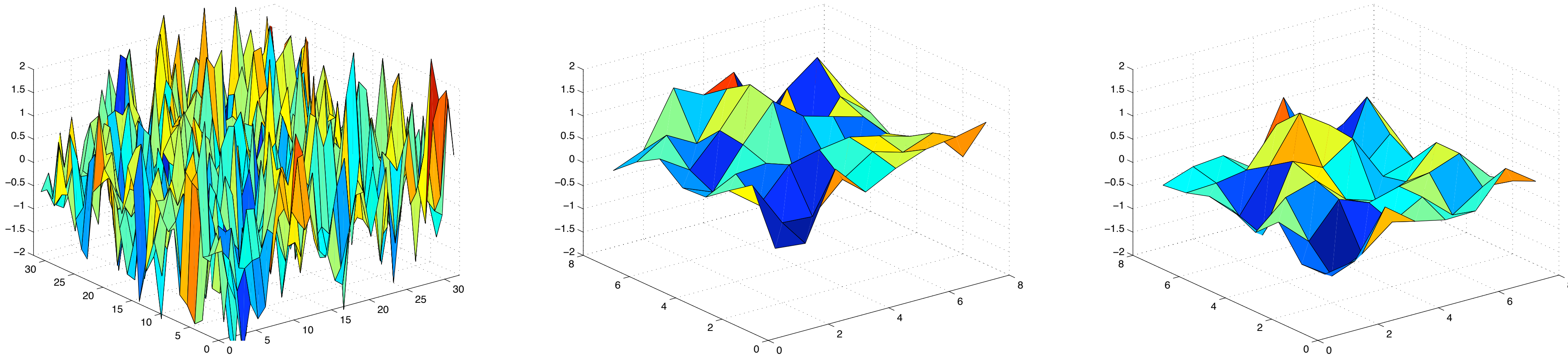
Suppose upscaling works. What average should we take?

- Arithmetic average: $\bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i$.
- Harmonic average: $\bar{a} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right)^{-1}$.

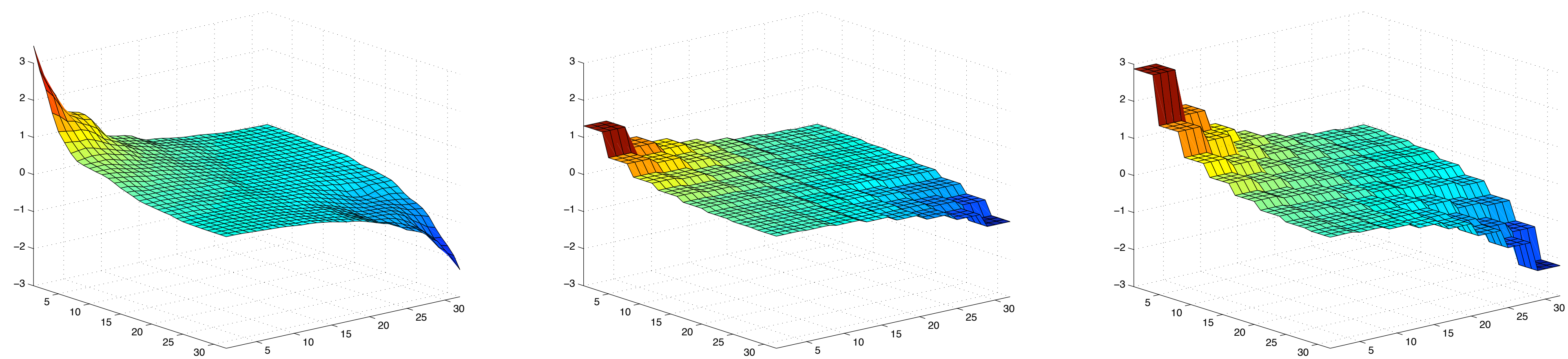
  The reciprocal of the average of the reciprocals. Emphasizes the small values.
- Something else?

Some Numerical Results using Averaging

Consider a small 2-D problem. Log-permeability and local averages:



Computed pressure:



32×32 $\quad$ 8×8 arithmetic average $\quad$ 8×8 harmonic average

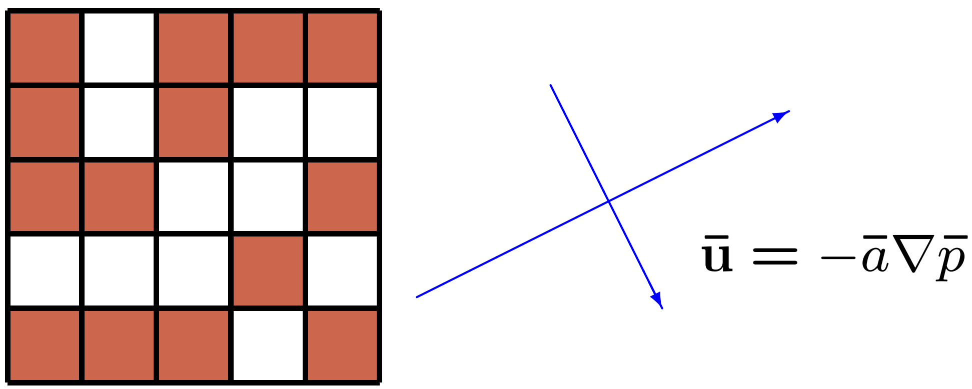
Relative errors: Arithmetic 0.43, Harmonic 0.40

Locally the medium is isotropic (i.e., the same in all directions).

However, $\bar{a}$ should be a full tensor!

$$\bar{a} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

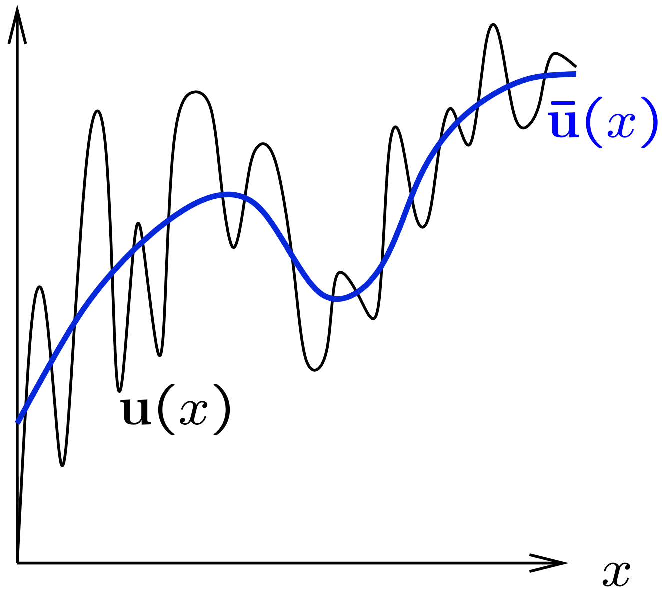
That is, $\bar{a}$ is anisotropic.



Remark: It is not so easy to quantify this anisotropy.

Mathematical Homogenization

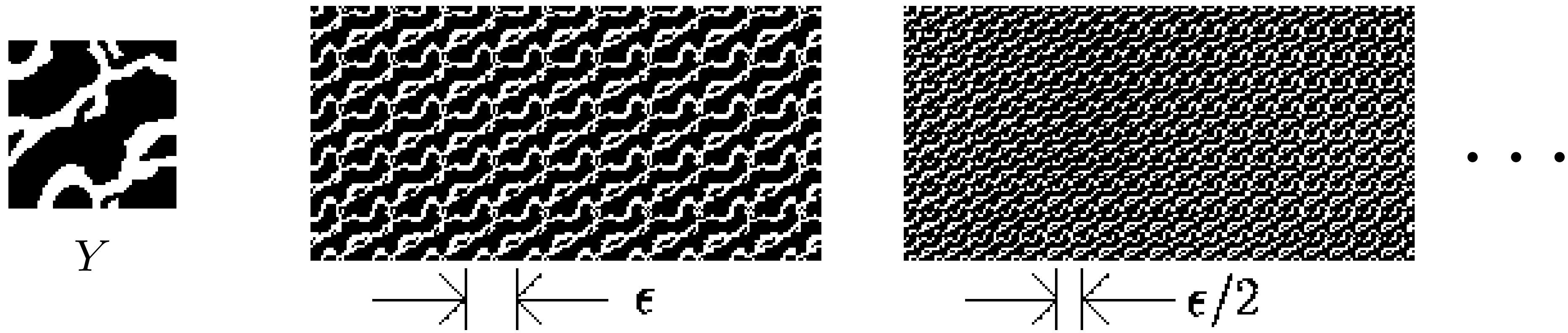
The solution $\mathbf{u}$ has high frequency wiggles due to the heterogeneity of a_ϵ.



Can we find $\bar{\mathbf{u}}(x)$ without knowing $\mathbf{u}(x)$? The wiggles are irregular, so they are hard to deal with.

- Assume that the heterogeneity is periodic, so that the wiggles are regular, and thus easily identified.
  [This is basically our closure assumption.]
- Let the period of oscillation be ϵ, and let $\epsilon \to 0$. This should remove the wiggles (at least in some weak sense).

Suppose that the domain Ω has a periodic structure with period ϵY. As $\epsilon \longrightarrow 0$, we obtain our macro-scale model for the average flow.



Homogenization is very mathematical, and involves deep analysis of partial differential equations.

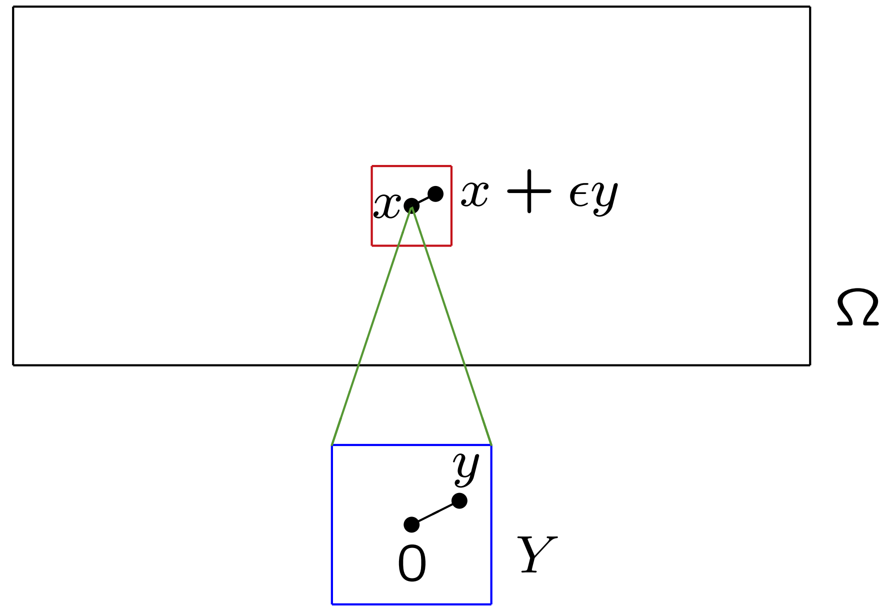
Fortunately there is a simpler, more physical view of homogenization.

Scaling. We assume that the space variable has both a slow (x) and fast (y) component.

$$x \sim x + \epsilon y$$

At any point x, y allows us to "see" the local details, which may affect larger scales.

The details disappear as $\epsilon \to 0$, but not necessarily their coarse-scale affects.



Local periodicity. We can assume that a_ϵ is locally periodic:

$$a_\epsilon(x) = a(x, y)$$

where $a(x, y)$ is periodic in y but varies slowly in x.

Formal Homogenization—1

Formal assumption: We *assume* without proof that we can expand the true solution $p(x)$ into a power series involving ϵ:

$$p(x) \sim p_0(x,y) + \epsilon\, p_1(x,y) + \epsilon^2 p_2(x,y) + \cdots$$

wherein $y = x/\epsilon$ and each p_k is periodic in y.

Gradient scaling: Then

$$\nabla \sim \nabla_x + \epsilon^{-1}\nabla_y$$

Procedure: We expect that

$$p_\epsilon \to p_0 \quad \text{as } \epsilon \to 0$$

Substitute the formal expansion into the equations

$$\begin{cases} \mathbf{u}_\epsilon = -a_\epsilon \nabla p_\epsilon & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_\epsilon = f & \text{in } \Omega \\ \mathbf{u}_\epsilon \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

Equating terms with like powers of ϵ leads to

1. $p_0(x,y) = p_0(x)$ only [i.e., homogenization removes y!]

2. Closure operator:

$$p_1(x,y) = \sum_j \omega_j(x,y)\,\partial_j p_0(x)$$

where the ω_j solve the local cell problems:

$$\begin{cases} -\nabla_y \cdot \Big[a(x,y)\nabla_y \omega_j(x,y)\Big] = \nabla_y \cdot \Big[a(x,y)\mathbf{e}_j\Big] & \text{in } \Omega \times Y \\ \omega_j(x,y) \text{ is periodic in } y & \end{cases}$$

3. By local averaging over the cell Y,

$$\begin{cases} \mathbf{u}_0 = -a_0 \nabla p_0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_0 = f & \text{in } \Omega \\ \mathbf{u}_0 \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

wherein $a_0(x)$ can be computed as the tensor

$$a_{0,ij}(x) = \frac{1}{|Y|}\int_Y a(x,y)\Big(\partial_i^y \omega_j(x,y) + \delta_{ij}\Big)\,dy$$

We have the homogenized permeability $a_0(x)$ and we can compute $p_0(x)$.

Lemma: a_0 is symmetric and positive definite:

$$\xi^T a_0 \xi = \sum_{i,j} \xi_i \, a_{0,ij} \, \xi_j > 0 \quad \text{for all vectors } \xi$$

Thus, a_0 has three principle eigenvectors and only positive eigenvalues.

Lemma (Voigt-Reiss Inequality): a_0 lies between the harmonic and arithmetic averages. More precisely, if

$$\hat{a} = \left(\frac{1}{|Y|} \int_Y (a(x,y))^{-1} \, dy \right)^{-1} \quad \text{and} \quad \bar{a} = \frac{1}{|Y|} \int_Y a(x,y) \, dy$$

then

$$\xi^T \hat{a} \xi \le \xi^T a_0 \xi \le \xi^T \bar{a} \xi$$

Theorem: If the first order corrector is defined as

$$p_\epsilon^1 = p_0 + \epsilon \sum_j \omega_j(x, x/\epsilon) \, \partial_j p_0(x) = p_0(x) + \epsilon \, p_1(x, x/\epsilon)$$

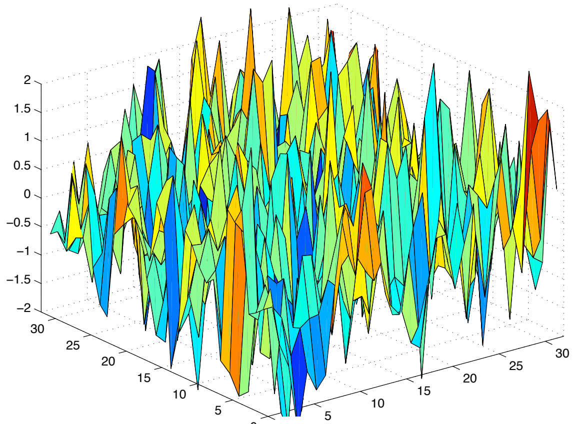
then

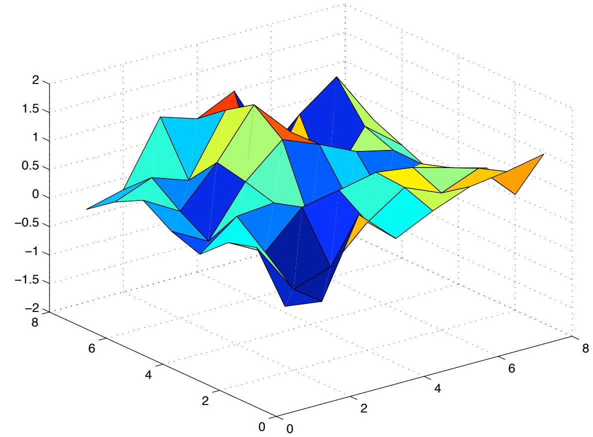
$$\|p_\epsilon - p_0\|_0 \le C\epsilon$$
$$\|\nabla(p_\epsilon - p_\epsilon^1)\|_0 \le C\sqrt{\epsilon}$$

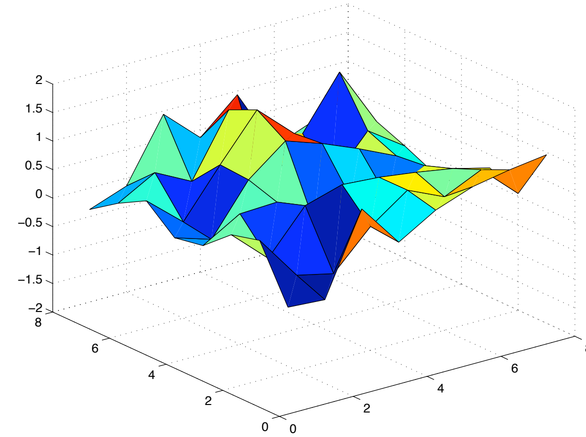
A Numerical Result using Homogenization

In our small 2-D problem, we obtain the following.

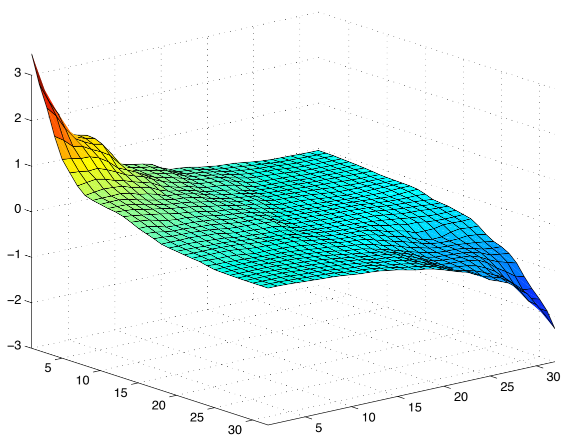
Log-permeability and xx and yy local averages ($xy = yx$ set to 0):

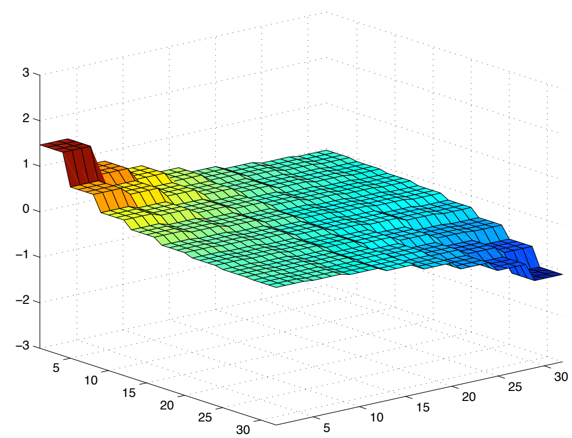


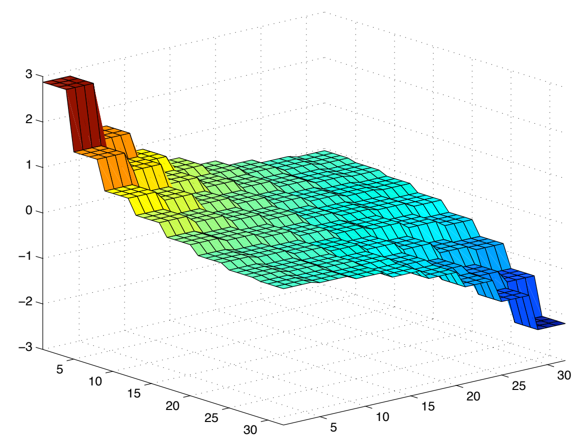




Computed pressure:







32×32 | 8×8 homogenized avg | 8×8 harmonic average

Relative errors: Homogenized 0.36, Harmonic 0.40

1. p_0 is approximated coarsely, and so has no microstructure, and

$$\mathbf{u}_0 = -a_0 \nabla p_0 \not\approx \mathbf{u}_\epsilon$$

2. $p_\epsilon^1 \approx p_\epsilon$ has microstructure, and

$$\mathbf{u}_\epsilon^1 = -a_\epsilon \nabla p_\epsilon^1 \approx \mathbf{u}_\epsilon$$

   but then

$$\nabla \cdot \mathbf{u}_\epsilon^1 \not\approx \nabla \cdot \mathbf{u}_\epsilon$$

   This means that the local conservation principle is not satisfied.

3. In the two-scale separation case, given $a_\epsilon(x)$, what is $a(x, y)$?
4. What about non-two-scale separation cases?

However, we use homogenization theory as a guide for the general case!

Multiscale Numerics

Objective. We want to solve the problem in a way that:

- does not fully incorporate the problem dynamics (i.e., solves some global coarse scale problem to resolution $h > \epsilon$),
- yet captures significant features of the solution, by taking into account the micro-structure (to resolution $h_f < \epsilon$).

Multiscale Methods

(Sorry, this is a very incomplete list!)

- **Multiscale finite elements**
  1. Babuška, Caloz & Osborn 1994
  2. Hou & Wu 1997
  3. Hou, Wu & Cai 1999
  4. Efendiev, Hou & Wu 2000
  5. Strouboulis, Babuška & Copps 2001
  6. Chen & Hou 2003
  7. Aarnes 2004
  8. Aarnes, Krogstad & Lie 2006
- **Multiscale finite volumes**
  1. Jenny, Lee & Tchelepi 2003
  2. He & Ren 2004
  3. Ginting 2004
  4. Hesse, Mallison & Tchelepi 2008
- **Heterogeneous multiscale methods**
  1. E & Engquist 2003
- **Variational multiscale analysis**
  1. Hughes 1995
  2. Hughes, Feijóo, Mazzei & Quincy 1998
  3. Arbogast, Minkoff & Keenan 1998
  4. Brezzi 1999
  5. Arbogast 2004
  6. Arbogast & Boyd 2006
- **Multiscale multilevel methods**
  1. Moulton, Dendy & Hyman 1998
  2. Xu, Zikatanov 2004
  3. Graham & Scheichl 2007
  4. Van lent, Scheichl & Graham 2009
- **Multiscale mortar methods**
  1. Arbogast, Pencheva, Wheeler & Yotov 2007
- **Multiscale basis optimization**
  1. Rath 2007 (Ph.D. dissertation)

Remark. These are all similar as a general method!

1. **Localization.** The full PDE problem is decomposed into many small, local, coarse element subproblems (of scale $h > \epsilon$).
2. **Fine-scale effects.** The local subproblems are given appropriate boundary conditions and solved on the fine scale $h_f < \epsilon$ (to resolve variations in a_ϵ) to define a coarse scale multiscale finite element or finite volume basis.
3. **Global coarse-grid problem.** This h-scale coarse basis is used to approximate the solution globally.
4. **Fine-grid reconstruction.** The finite element basis encapsulates an h_f-scale fine representation of the solution.

Remarks.

- The problem is fully resolved on the fine scale.
- The problem is *not* fully coupled. The global problem is a reduced degree-of-freedom system.
- Computational efficiency comes from divide-and-conquer:
  - (a) Small, localized subproblems are easily solved;
  - (b) The coupled global problem has only a few degrees of freedom per coarse element, and so is relatively easily solved.

The Nonmixed System: Multiscale Finite Elements

(Define appropriate finite elements)

The differential problem:

$$\begin{cases} -\nabla \cdot a_\epsilon \nabla p = f & \text{in } \Omega \\ -a_\epsilon \nabla p \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

A variational problem: Let

$$X = H^1/\mathbb{R} \qquad \text{(The function space)}$$
$$A_\epsilon(p, w) = (a_\epsilon \nabla p, \nabla w) \qquad \text{(A bilinear form)}$$
$$F(w) = (f, w) \qquad \text{(A linear form)}$$

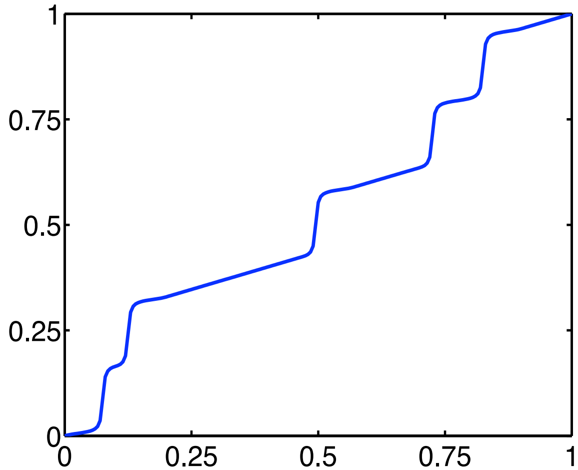
Find $p \in X = H^1/\mathbb{R}$ such that

$$A_\epsilon(p, w) = F(w) \quad \forall\ w \in X$$

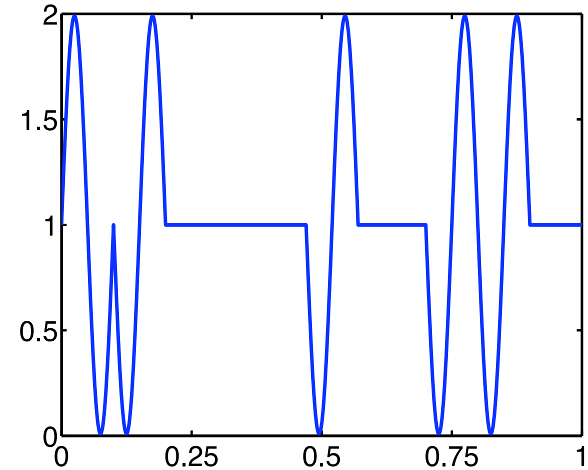
Galerkin's method: Let $X_h \subset X$ be a finite dimensional subspace.
Find $p_h \in X_h$ such that

$$A_\epsilon(p_h, w) = F(w) \quad \forall\ w \in X_h$$

(Babuška and Osborn, 1983; Hou and Wu, 1997)



True solution p



Coefficient $a > 0$

Differential problem.

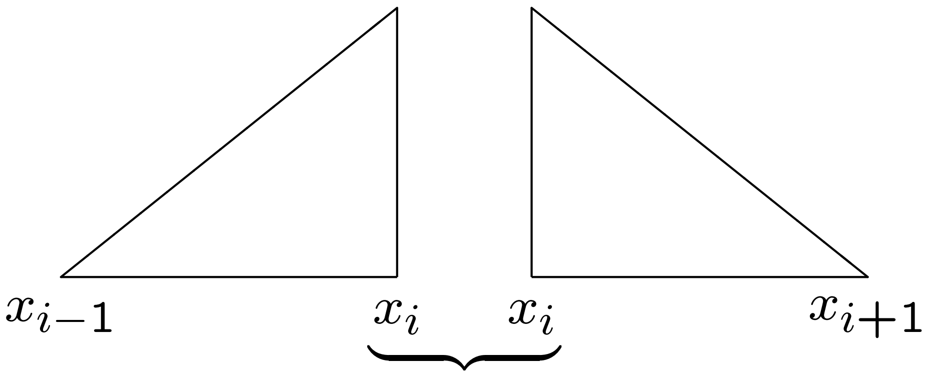
$$\begin{cases} -(ap')' = 0, & 0 < x < 1 \\ p(0) = 0 \text{ and } p(1) = 1 \end{cases}$$

Variational Form. Let $X = H_0^1(0,1) = \left\{ w \in H^1 : w(0) = w(1) = 0 \right\}$
Find $p \in X + x$ such that

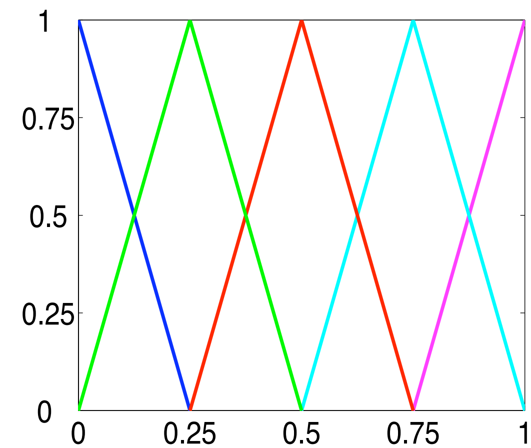
$$(ap', w') = 0 \quad \forall\, w \in X$$

Choose a uniform grid of five points: $x_i = i/4$, $i = 0, 1, 2, 3, 4$.

Standard finite elements $\bar{X}_h$. At x_i, define



$q_i(x_{i-1}) = 0$ $\quad$ $q_i(x_i) = 1$ $\quad$ $q_i(x_{i+1}) = 0$

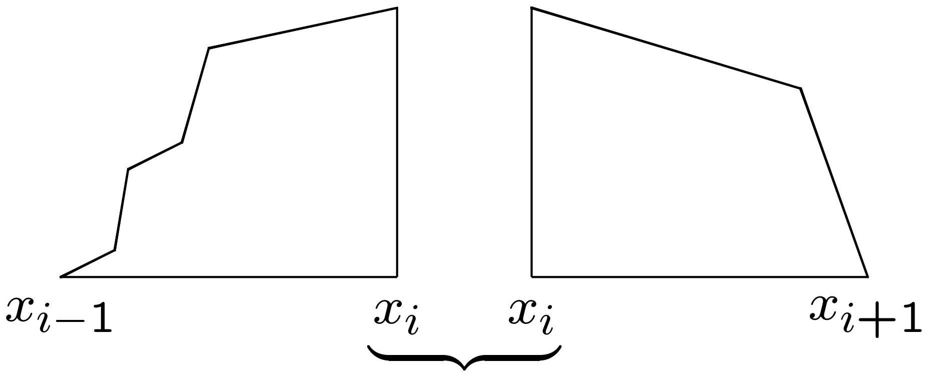


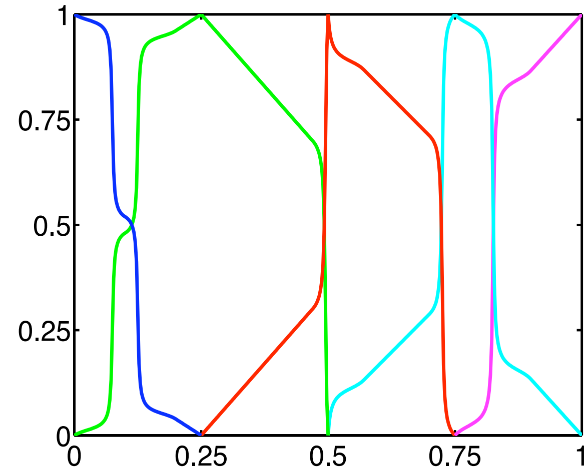
- Set q_i on the element boundary
- Linearly interpolate
- Join the pieces together continuously

A Simple Example—3

Localize X to the element $E = (x_{i-1}, x_i)$ as $X(E) = H_0^1(E)$

Multiscale finite elements X_h. At x_i, define





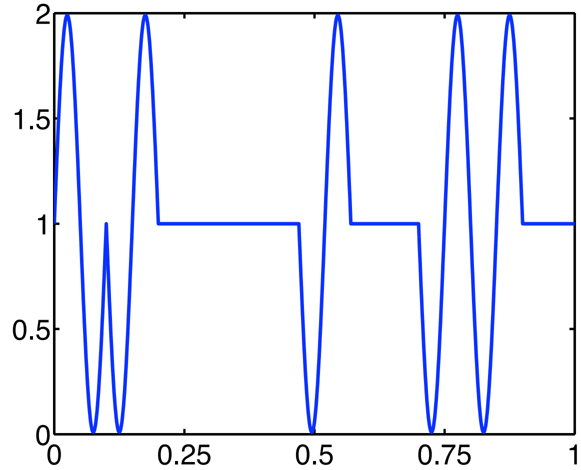
$q_i(x_{i-1}) = 0 \qquad q_i(x_i) = 1 \qquad q_i(x_{i+1}) = 0$

- Set q_i on the element boundary
- Solve the homogeneous problem on each element E:
  Find $q_i \in X(E) + \ell_i(x)$ such that

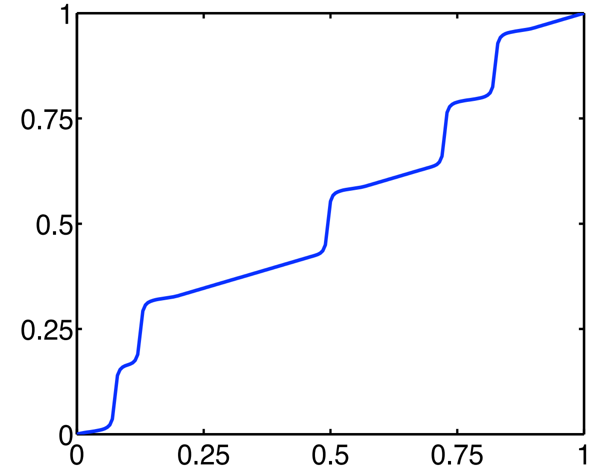
$$(a q_i', w')_E = 0 \quad \forall\ w \in X(E)$$

  where E is (x_{i-1}, x_i) or (x_i, x_{i+1}), using the appropriate linear function $\ell_i(x)$ for the BC's.
- Join the pieces together continuously

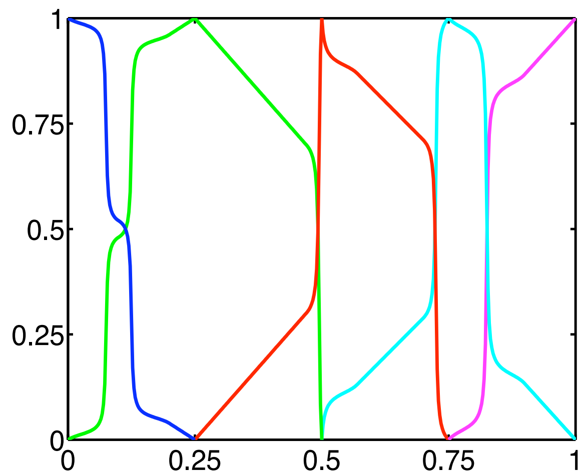
A Simple Example—4



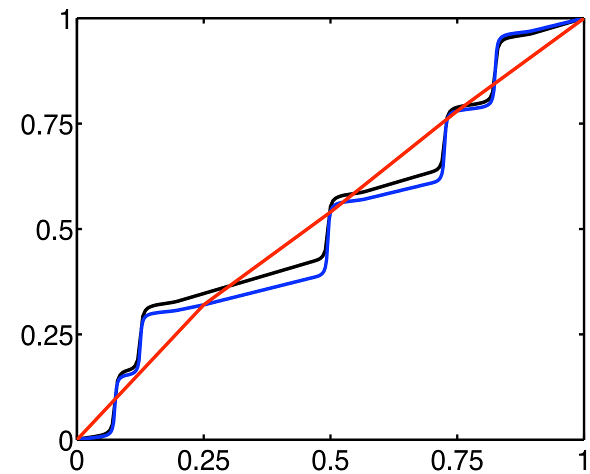
Coefficient a



True solution p



Multiscale basis functions



Multiscale vs. Standard solution

Remark: Actually, the multiscale solution is exact in 1-D.

Standard finite elements.

- Set $\bar{q}_i = \ell_i(x)$ on the element boundary, where ℓ_i is an appropriate simple polynomial on ∂E
- Use some polynomial interpolation
- Join the pieces together continuously to form $\bar{X}_h = \text{span}\{\bar{q}_i\}$

Multiscale finite elements.

- Set $q_i = \ell_i(x)$ on the element boundary, where ℓ_i is an appropriate simple function on ∂E (such as a polynomial)
- Solve the homogeneous problem on each element E:
  Find $q_i \in X(E) + \ell_i(x)$ such that

$$A_\epsilon(q_i, w)_E = 0 \quad \forall\, w \in X(E)$$

  That is, solve the Dirichlet problems (on a fine grid)

$$\begin{cases} -\nabla \cdot a_\epsilon \nabla q_i = 0 & \text{in } E \\ q_i = \ell_i & \text{on } \partial E \end{cases}$$

- Join the pieces together continuously to form $X_h = \text{span}\{q_i\}$

We took the standard variational form and modified the finite elements to incorporate Multiscale effects:

Multiscale space: $X_h = \text{span}\{q_i\}$ from solving local problems
Find $q_i \in X(E) + \ell_i(x)$ such that

$$A_\epsilon(q_i, w)_E = 0 \quad \forall\ w \in X(E)$$

Multiscale method: Using the standard variational form
Find $p_h \in X_h$ such that

$$A_\epsilon(p_h, w) = F(w) \quad \forall\ w \in X_h$$

Remark: The approach has a lot of flexibility, and there exist many variants of the above procedure.

$$q_i = \bar{q}_i + (q_i - \bar{q}_i) \equiv \bar{q}_i + q_i'$$

Find $q_i \in X(E) + \bar{q}_i$ such that

$$A_\epsilon(q_i, w)_E = 0 \quad \forall\ w \in X(E)$$

$\Longrightarrow$

Find $q_i' \in X(E)$ such that

$$A_\epsilon(q_i', w')_E = A_\epsilon(\bar{q}_i, w')_E \quad \forall\ w' \in X(E)$$

- The q_i' are "bubble functions", defined locally in $X(E) = H_0^1(E)$.
- The q_i' are fine-scale and contain the microstructure information.
- The $\bar{q}_i$ are coarse-scale.

Theorem: Let $X_h' = \text{span}\{q_i'\}$. Then

$$X_h = \text{span}\{\bar{q}_i + q_i'\} \subsetneq \bar{X}_h \oplus X_h'$$

is a Hilbert space direct sum decomposition into coarse and fine scales.

The Nonmixed System:

Variational Multiscale Method

(Modify the variational form)

The differential problem:

$$\begin{cases} -\nabla \cdot a_\epsilon \nabla p = f & \text{in } \Omega \\ -a_\epsilon \nabla p \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

A two-scale variational problem: Let

$$X = \bar{X} \oplus X' = H^1/\mathbb{R} \quad \text{(The two-scale function space)}$$
$$A_\epsilon(p, w) = (a_\epsilon \nabla p, \nabla w) \quad \text{(A bilinear form)}$$
$$F(w) = (f, w) \quad \text{(A linear form)}$$

Find $p = \bar{p} + p' \in \bar{X} \oplus X'$ such that

$$A_\epsilon(\bar{p} + p', \bar{w}) = F(\bar{w}) \quad \forall\ w \in \bar{X} \quad \text{(Coarse scales)}$$
$$A_\epsilon(\bar{p} + p', w') = F(w') \quad \forall\ w \in X' \quad \text{(Fine scales)}$$

Remark: This is the same problem. It is merely viewed in two scales.

Rewrite the fine scale equation as

$$A_\epsilon(p', w') = F(w') - A_\epsilon(\bar{p}, w') \quad \forall\ w \in X'$$

This is a well defined problem for p'. It implicitly defines an *affine* upscaling operator taking $\bar{X}$ to $\rightarrow X'$.

Linear part: $\widehat{p}' : \bar{X} \rightarrow X'$ satisfies

$$A_\epsilon(\widehat{p}'(\bar{q}), w') = -A_\epsilon(\bar{q}, w') \quad \forall\ w \in X'$$

Constant part: $\tilde{p}' \in X'$ satisfies

$$A_\epsilon(\tilde{q}', w') = F(w') \quad \forall\ w \in X'$$

Upscaling operator: $\widehat{p}'(\cdot) + \tilde{p}' : \bar{X} \rightarrow X'$

$$p' = \widehat{p}'(\bar{p}) + \tilde{p}'$$

Given coarse scales, we can fine fine scales.

Now the coarse scale equation is simply

$$A_\epsilon\Big(\bar{p}+\widehat{p}'(\bar{p}),\bar{w}\Big) = F(\bar{w}) - A_\epsilon(\tilde{p}',\bar{w}) \quad \forall\ w \in \bar{X}$$

The effect of the fine scales is now manifest.

The upscaling operator says

$$A_\epsilon\Big(\widehat{p}'(\bar{p}),\widehat{p}'(\bar{w})\Big) = -A_\epsilon\Big(\bar{p},\widehat{p}'(\bar{w})\Big)$$

so, symmetrizing, we have

$$A_\epsilon\Big(\bar{p}+\widehat{p}'(\bar{p}),\bar{w}+\widehat{p}'(\bar{w})\Big) = F(\bar{w}) - A_\epsilon(\tilde{p}',\bar{w}) \quad \forall\ w \in \bar{X}$$

Variational Multiscale Method: (for the differential problem)

$$\mathcal{A}_\epsilon(\bar{p},\bar{w}\Big) = \mathcal{F}(\bar{w}) \quad \forall\ w \in \bar{X}$$

where

$$\mathcal{A}_\epsilon(\bar{p},\bar{w}\Big) = A_\epsilon\Big(\bar{p}+\widehat{p}'(\bar{p}),\bar{w}+\widehat{p}'(\bar{w})\Big)$$
$$\mathcal{F}(\bar{w}) = F(\bar{w}) - A_\epsilon(\tilde{p}',\bar{w})$$

Remark: The bilinear and linear forms are both modified.

To be useful, we need to localize the fine scales. Take

$$X' = \bigoplus_E X(E) = \bigoplus_E H_0^1(E)$$

Then

$$\bar{X} = X/X' \simeq \{q|_e : e \text{ is a coarse edge}\}$$

Thus $\bar{X}$ is determined by values on $\partial E\ \forall E$.

We use the standard space $\bar{X}_h = \{\bar{q}_h\}$ and the multiscale fine space

$$X'_h = \text{span}\{q'_h\} \subset X'$$

That is, X' is localized and

$$\bar{X}_h \oplus X'_h \subsetneq \bar{X} \oplus X' = H^1/\mathbb{R}$$

Version 1: Find $p_h = \bar{p}_h + p'_h \in \bar{X}_h \oplus X'_h$ such that

$$A_\epsilon(p_h, w) = F(w) \quad \forall\ w \in \bar{X}_h \oplus X'_h$$

But $\bar{X}_h \oplus X'_h$ is a large space. In fact, $\bar{p}_h$ and p'_h are related, and the solution is in a much smaller space.

Theorem: Since Galerkin methods minimize energy, the multiscale solution minimizes energy in the large space $\bar{X}_h \oplus X'_h$. For these methods, if one specifies the value of the finite elements on ∂E, then the best approximation comes from using the finite element that minimizes energy within E.

Version 2: By solving for the upscaling operator, we obtain
Find $\bar{p}_h \in \bar{X}_h$ such that

$$\mathcal{A}_\epsilon(\bar{p}_h, \bar{w}) = \mathcal{F}(\bar{w}) \quad \forall\ \bar{w} \in \bar{X}_h$$

Now $\bar{X}_h$ is very small, but we must find the upscaling operator to relate $\bar{q}_h$ and $p'_h(\bar{q}_h)$. Given a basis

$$\bar{X}_h = \text{span}\{\bar{q}_i\}$$

we solve a local Dirichlet problems for $\bar{q}_i$ on element E

$$A_\epsilon(\bar{q}_i + \widehat{p}'(\bar{q}_i), w')_E = 0 \quad \forall\ w \in X(E)$$

These are the same problems as in the multiscale finite element case, so

$$\boxed{X_h = \text{span}\{\bar{q}_i + \widehat{p}'(\bar{q}_i)\}}$$

Version 3: Find $p_h \in X_h$ such that

$$A_\epsilon(p_h, w) = F(w) - A_\epsilon(\tilde{p}', w) \quad \forall\ \bar{w} \in X_h$$

Theorem: Up to treatment of f (i.e., $\tilde{p}'$), the two approaches are the same in this basic setting.

Remark: Unlike multiscale finite elements, the variational multiscale method naturally handles nonzero f. Henceforth we will use this correction in the multiscale finite element method as well.

We take standard finite elements and use the modified variational form that incorporates the multiscale effects:

Standard space and upscaling operator: $\bar{X}_h = \text{span}\{\bar{q}_i\}$
Solve a local Dirichlet problems for $\bar{q}_i$ on element E

$$A_\epsilon(\bar{q}_i + \widehat{p}'(\bar{q}_i), w')_E = 0 \quad \forall\ w \in X(E)$$

and for

$$A_\epsilon(\tilde{p}', w')_E = F(w')_E \quad \forall\ w \in X(E)$$

Variational multiscale method 1: Find $\bar{p}_h \in \bar{X}_h$ such that

$$\mathcal{A}_\epsilon(\bar{p}_h, \bar{w}) = \mathcal{F}(\bar{w}) \quad \forall\ \bar{w} \in \bar{X}_h$$

Finally

$$p_h = \bar{p}_h + \widehat{p}'(\bar{p}_h) + \tilde{p}'$$

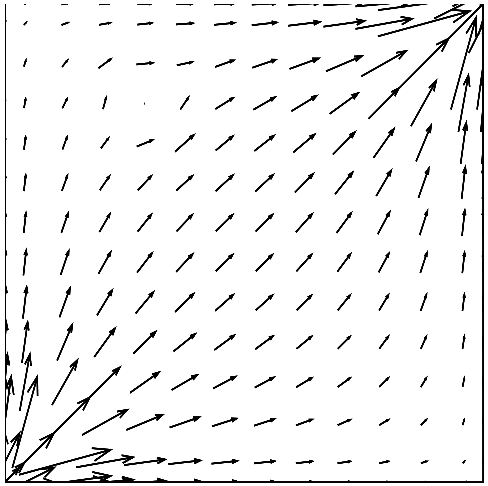
is your fine scale reconstruction.

Variational multiscale method 2: Find $p_h \in X_h = \text{span}\{\bar{q}_i + \widehat{p}'(\bar{q}_i)\}$ so that

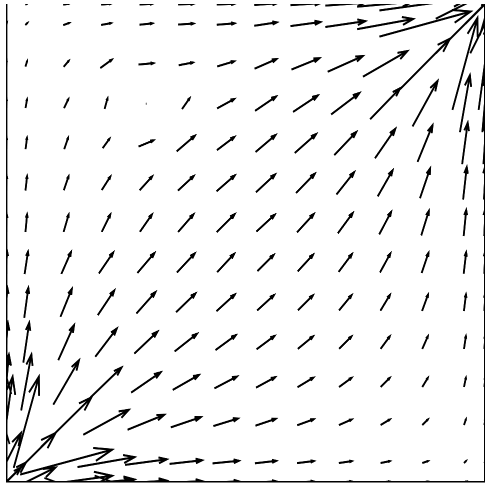
$$A_\epsilon(p_h, w) = F(w) - A_\epsilon(\tilde{p}', w) \quad \forall\ \bar{w} \in X_h$$

Some Numerical Examples of mixed multiscale numerics

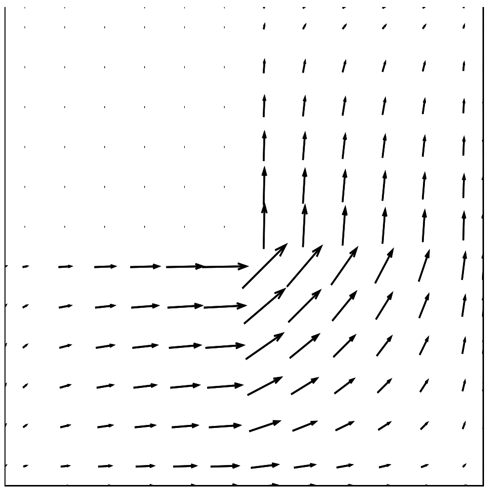
Low Premeability Spot (10^{-16})



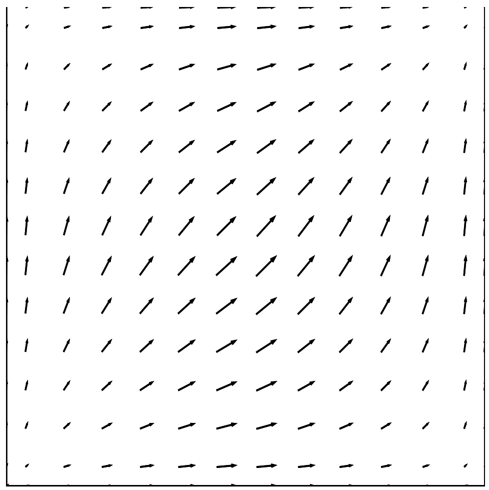
12×12
fine
scale



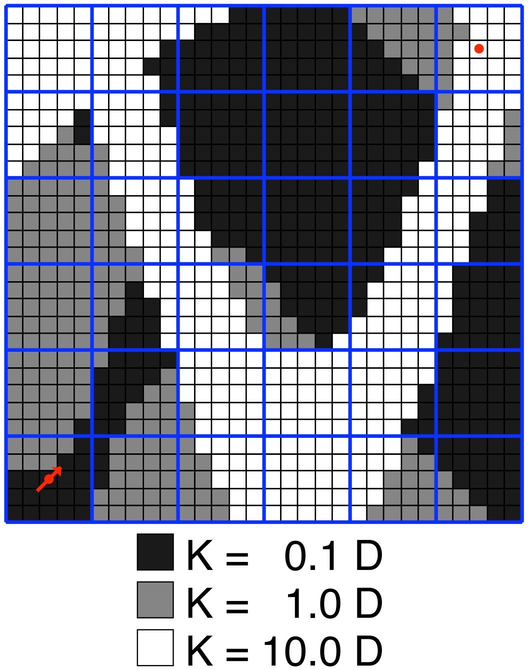
2×2
upscaled



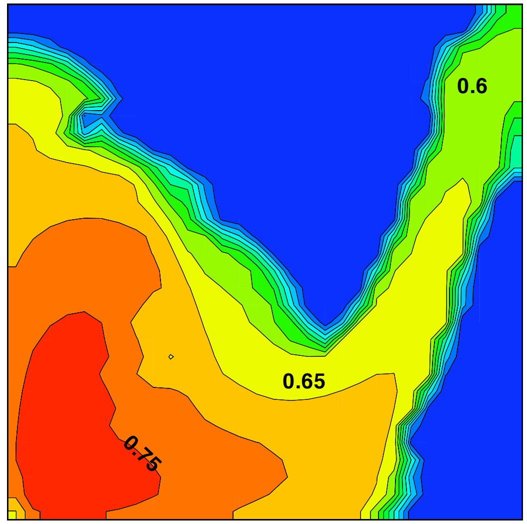
2×2
coarse
scale



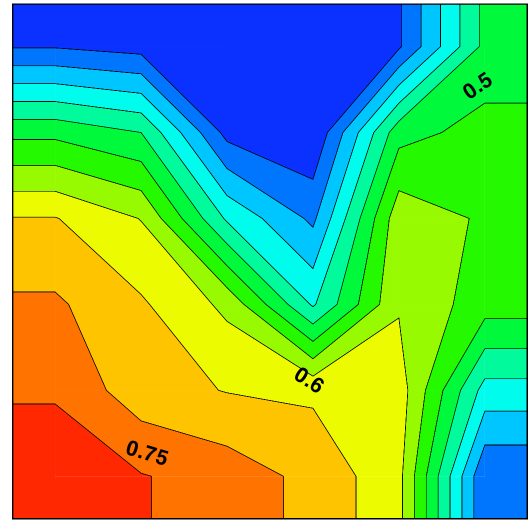
2×2
coarse
scale
using
coarse
average
a



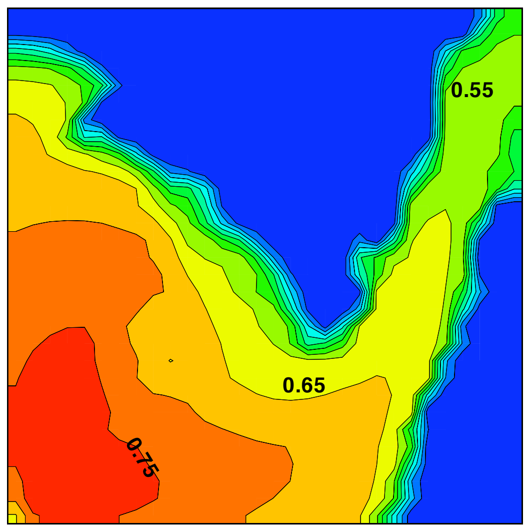
Permeability field K
(White & Horne, 1987)



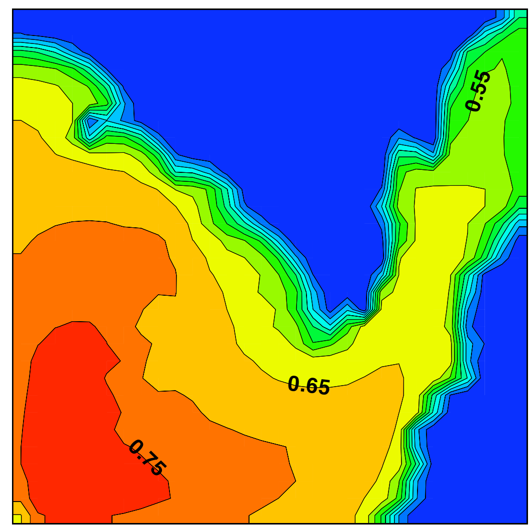
Fine 30×30



Average K 6×6

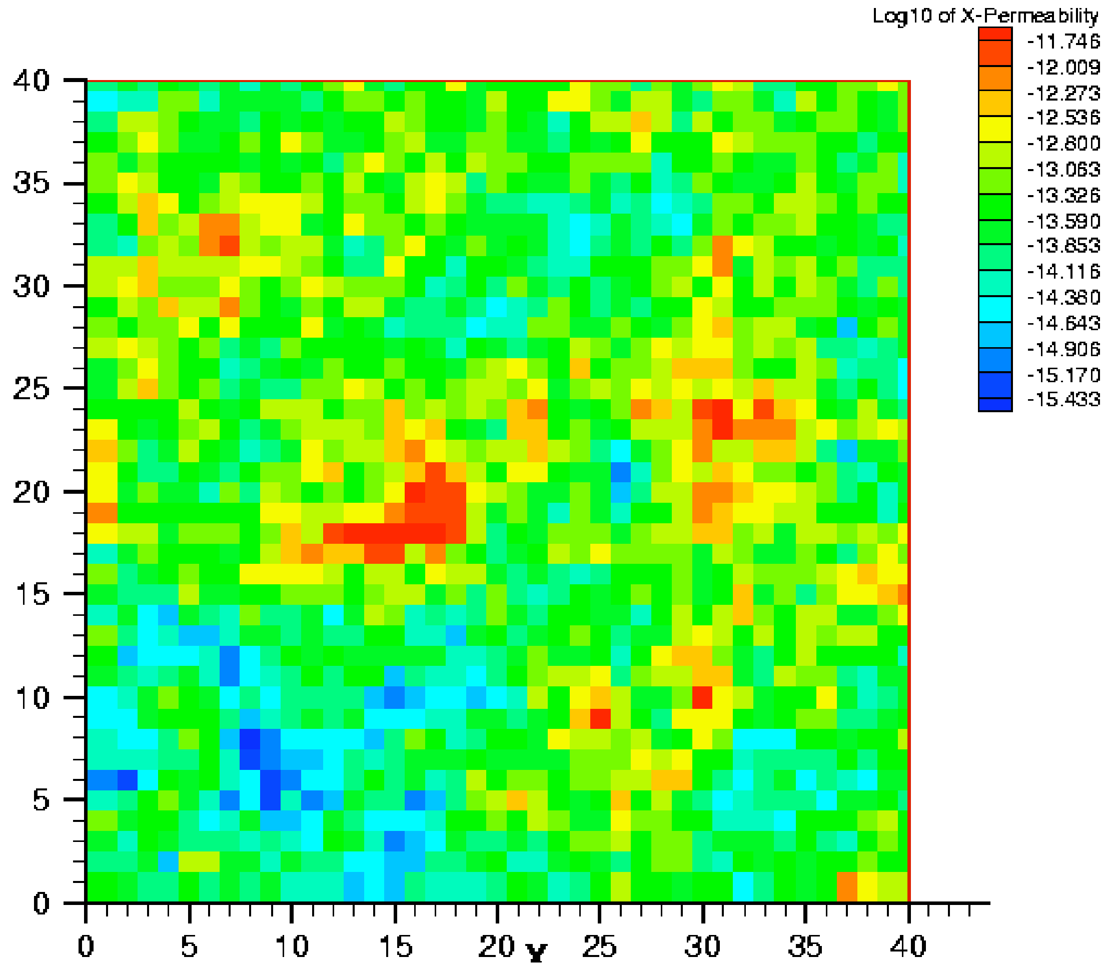


Upscaled to 6×6



Upscaled to 3×3

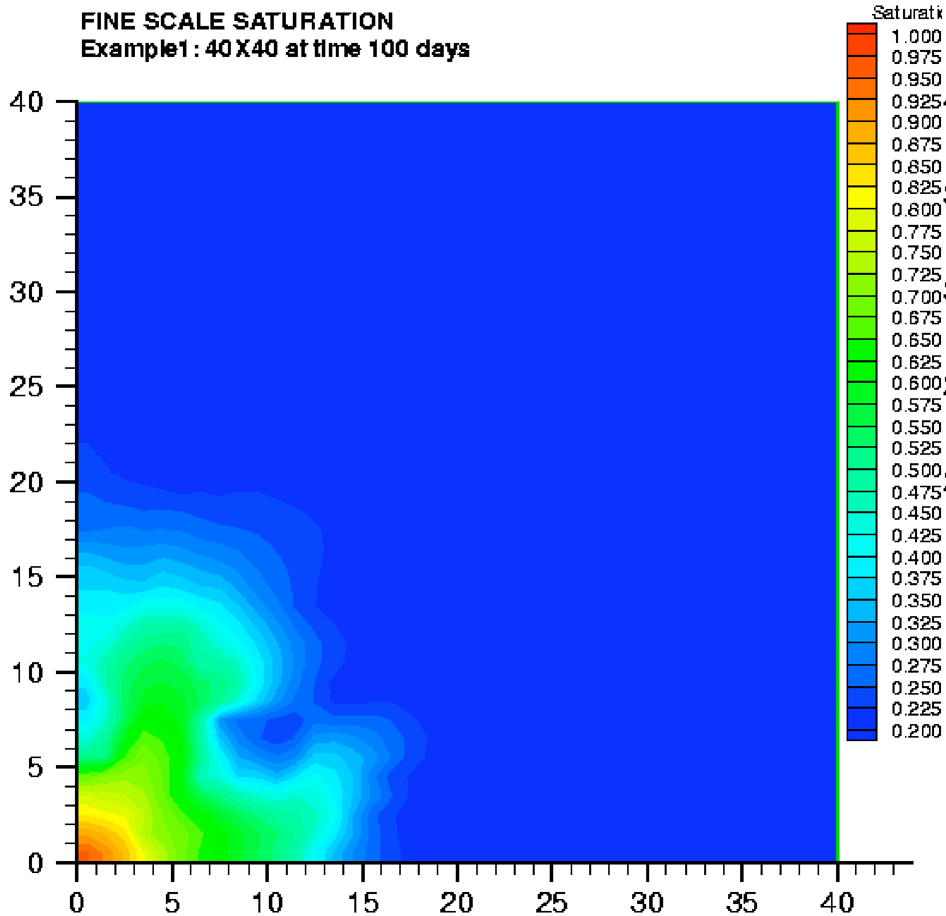
Logarithm of the permeability



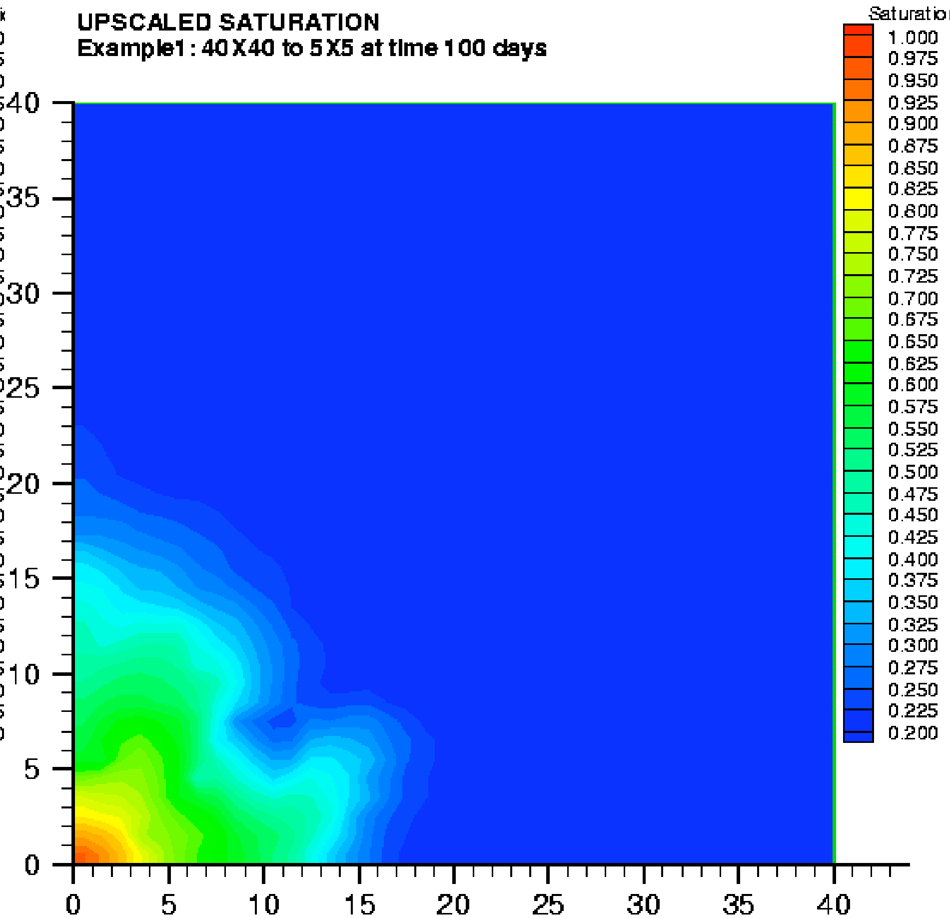
Fine 40×40

A Quarter Five-spot Oil Reservoir Waterflood—2

Water saturation contours at 100 days

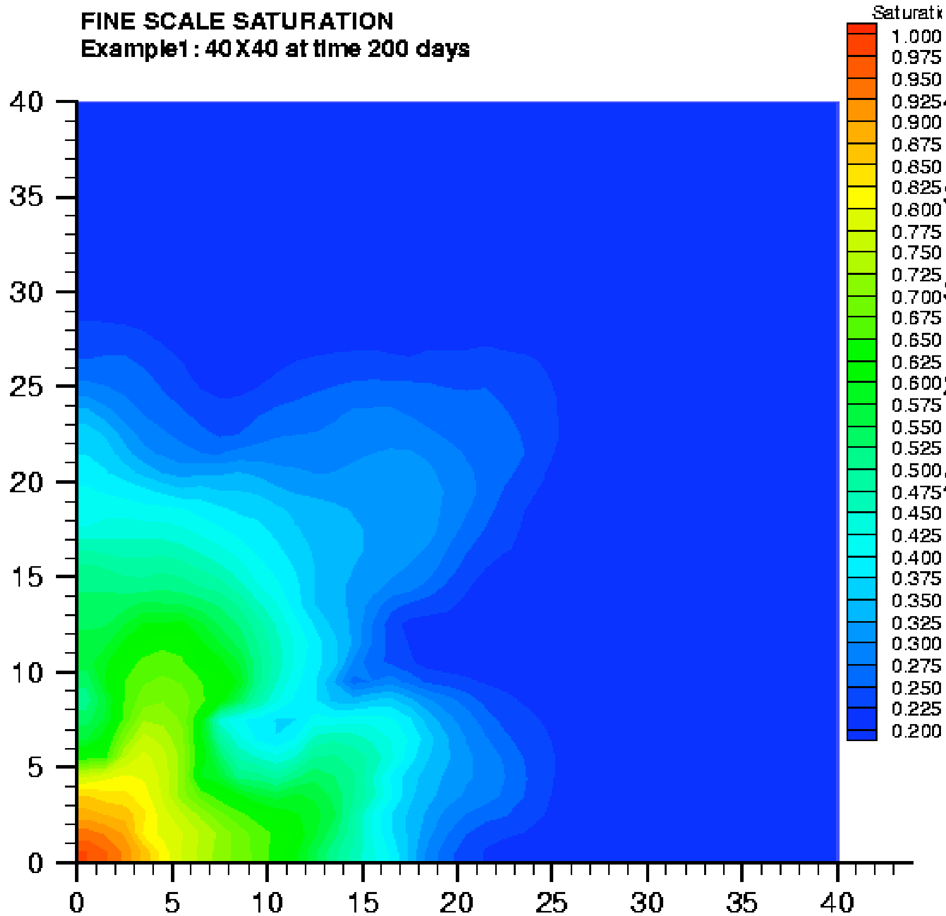


Fine 40 $\times$ 40

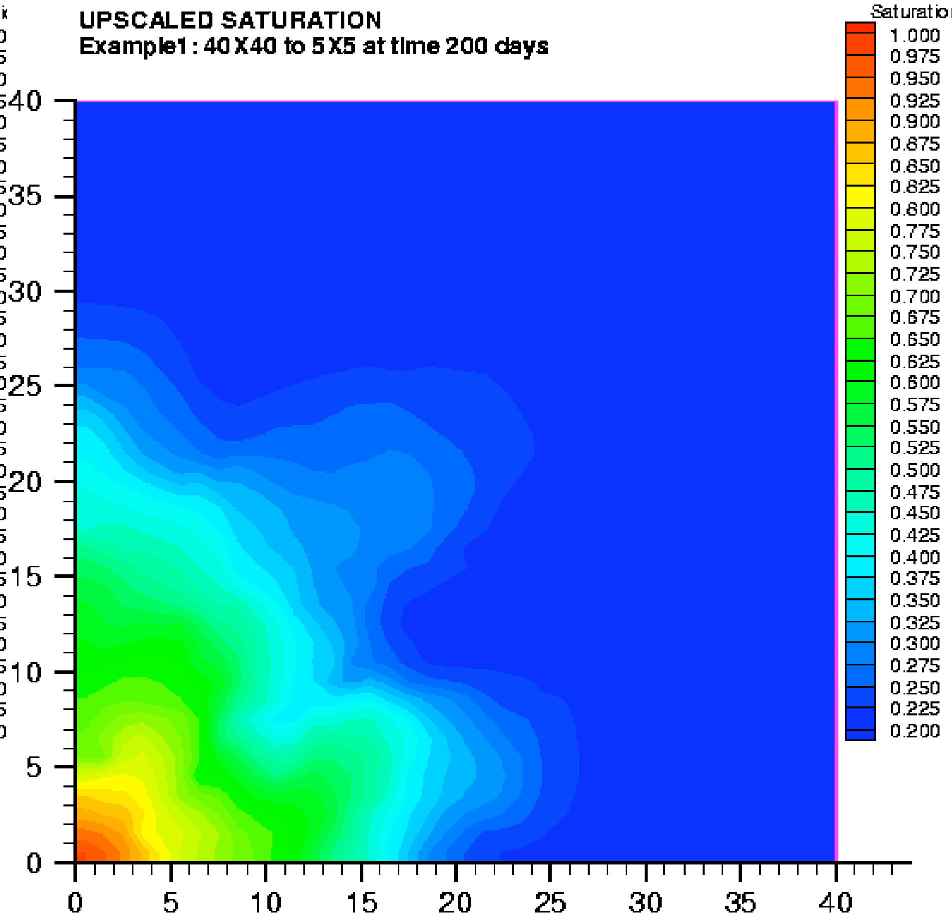


Fully upscaled to 5 $\times$ 5

Water saturation contours at 200 days

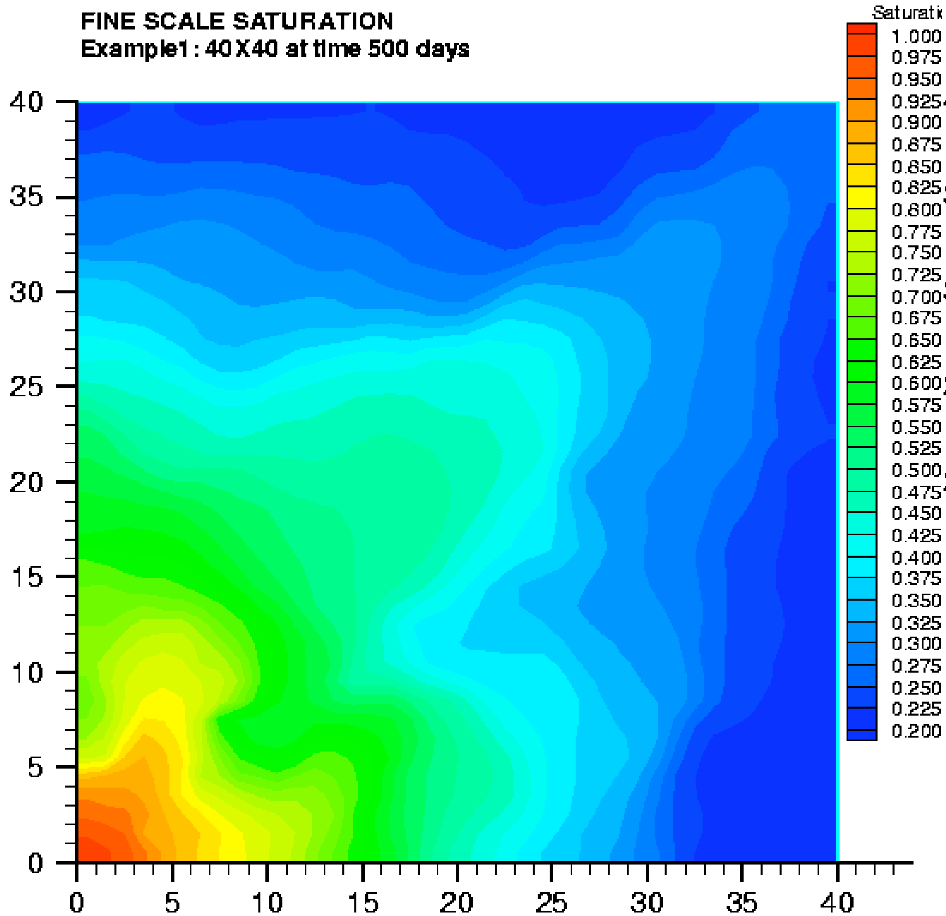


Fine 40×40

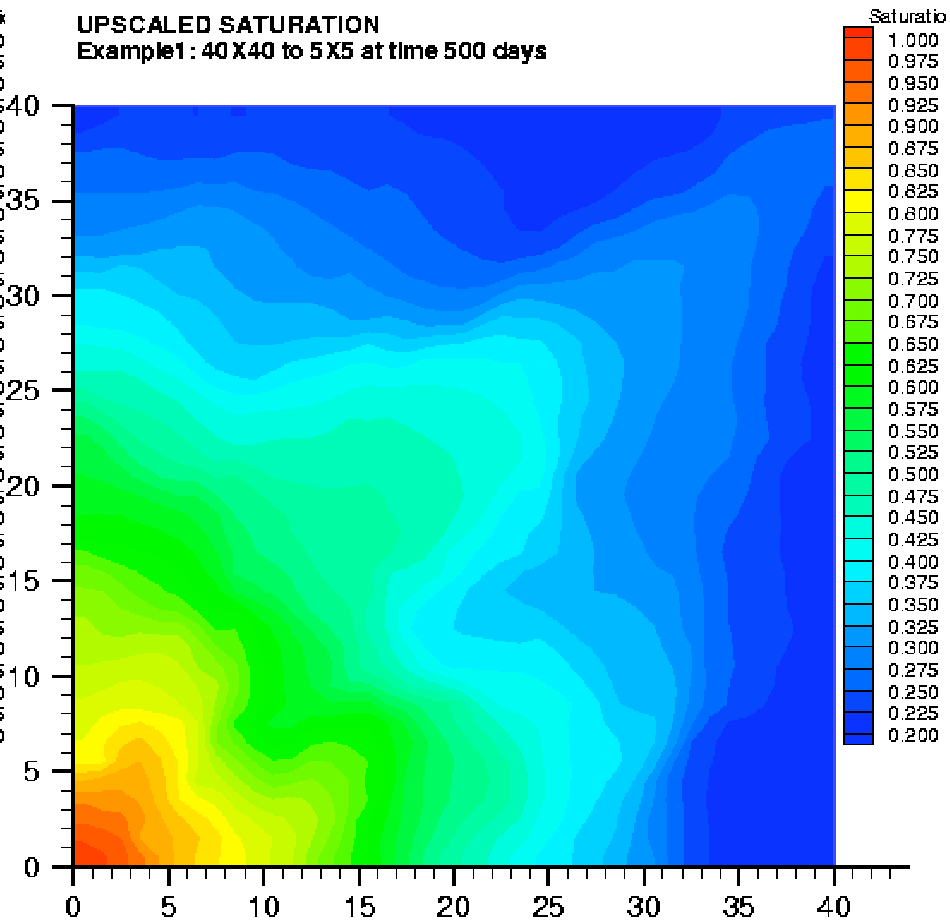


Fully upscaled to 5×5

Water saturation contours at 500 days



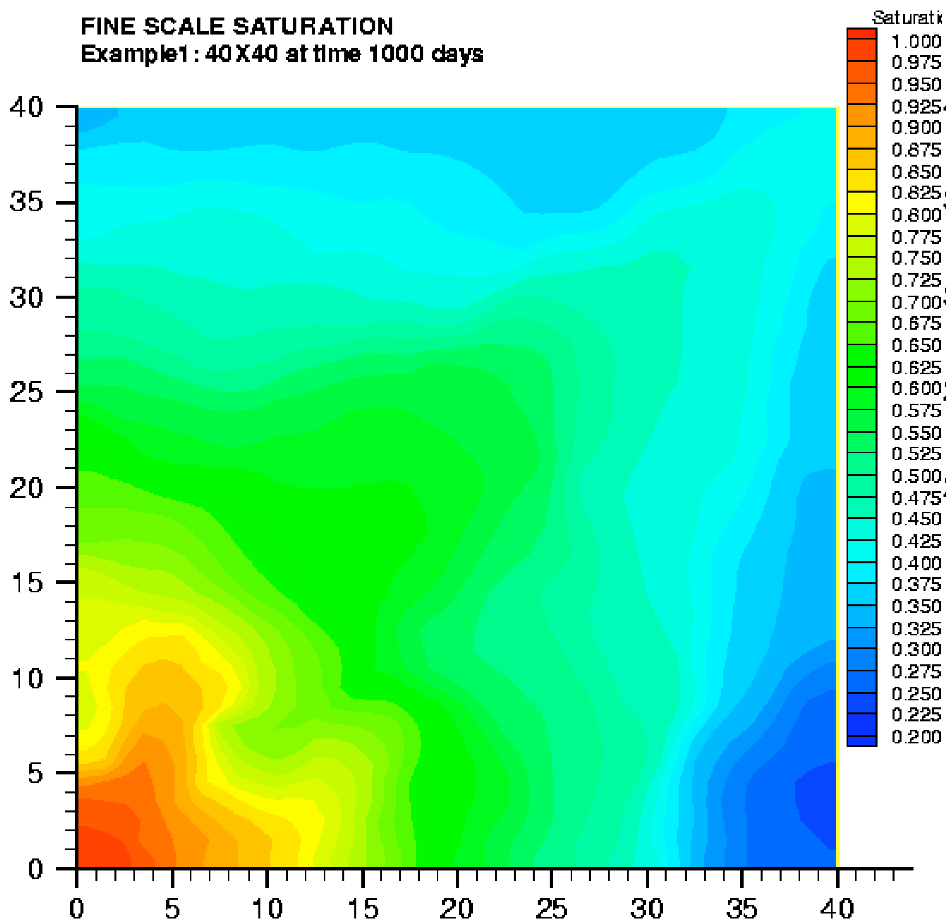
Fine 40 × 40



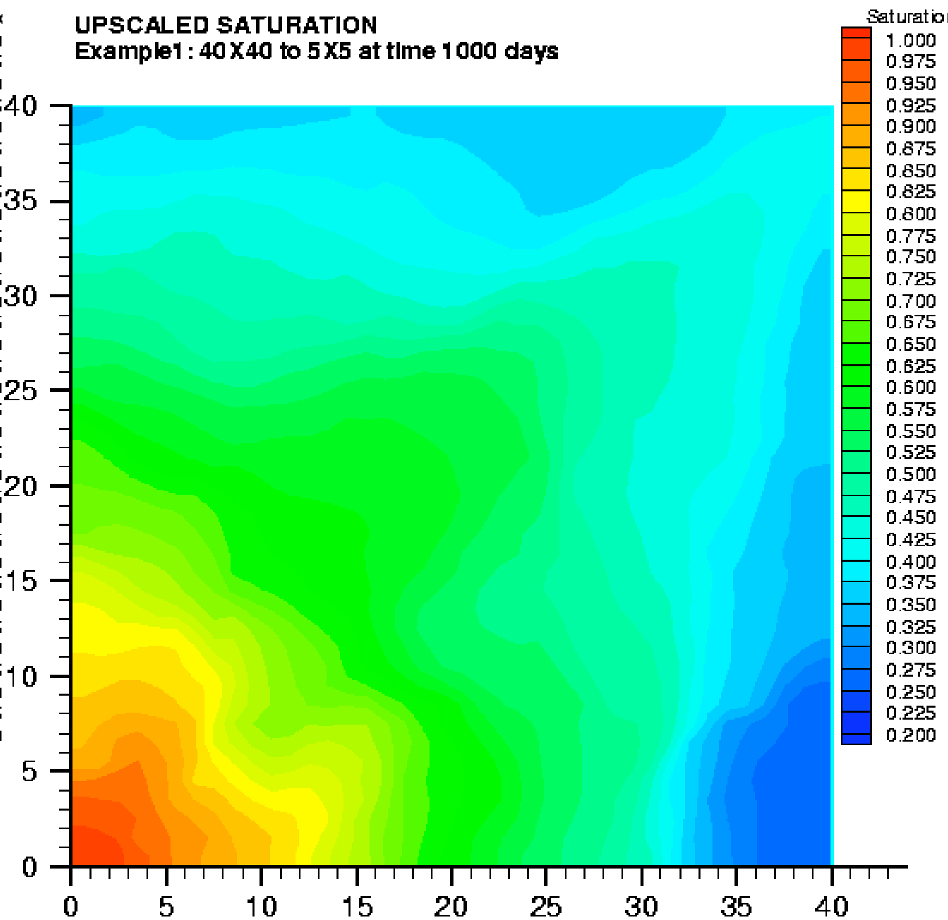
Fully upscaled to 5 × 5

A Quarter Five-spot Oil Reservoir Waterflood—5

Water saturation contours at 1000 days



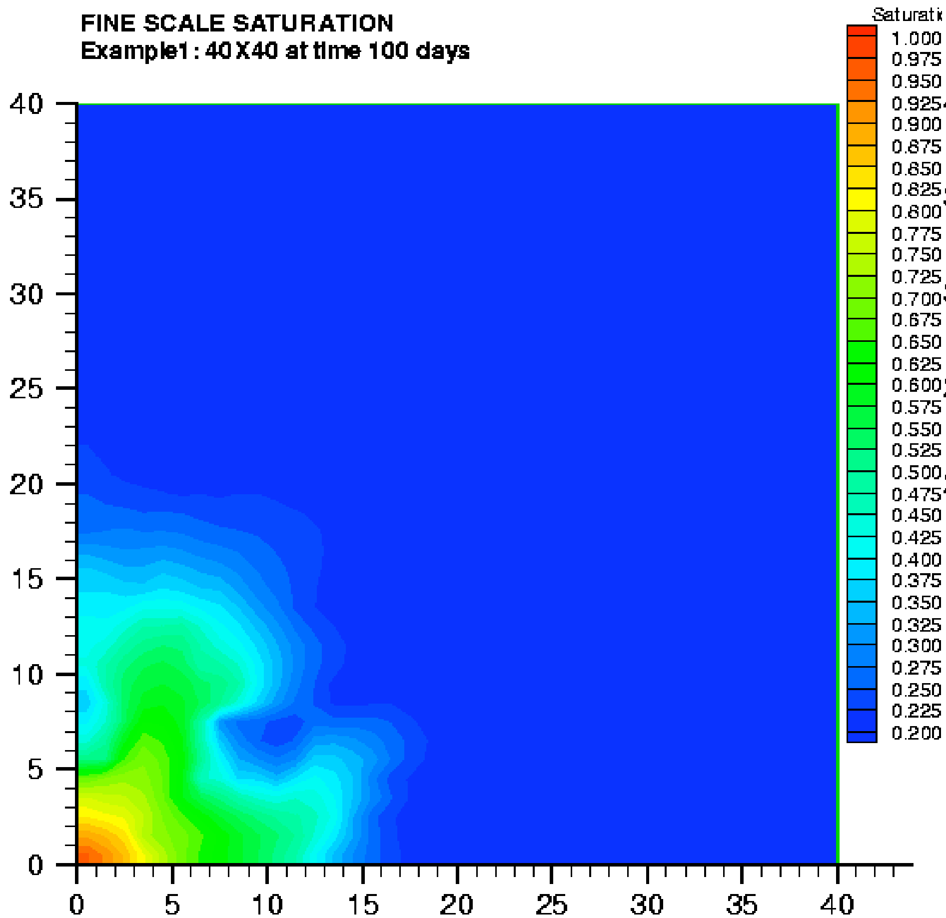
Fine 40×40



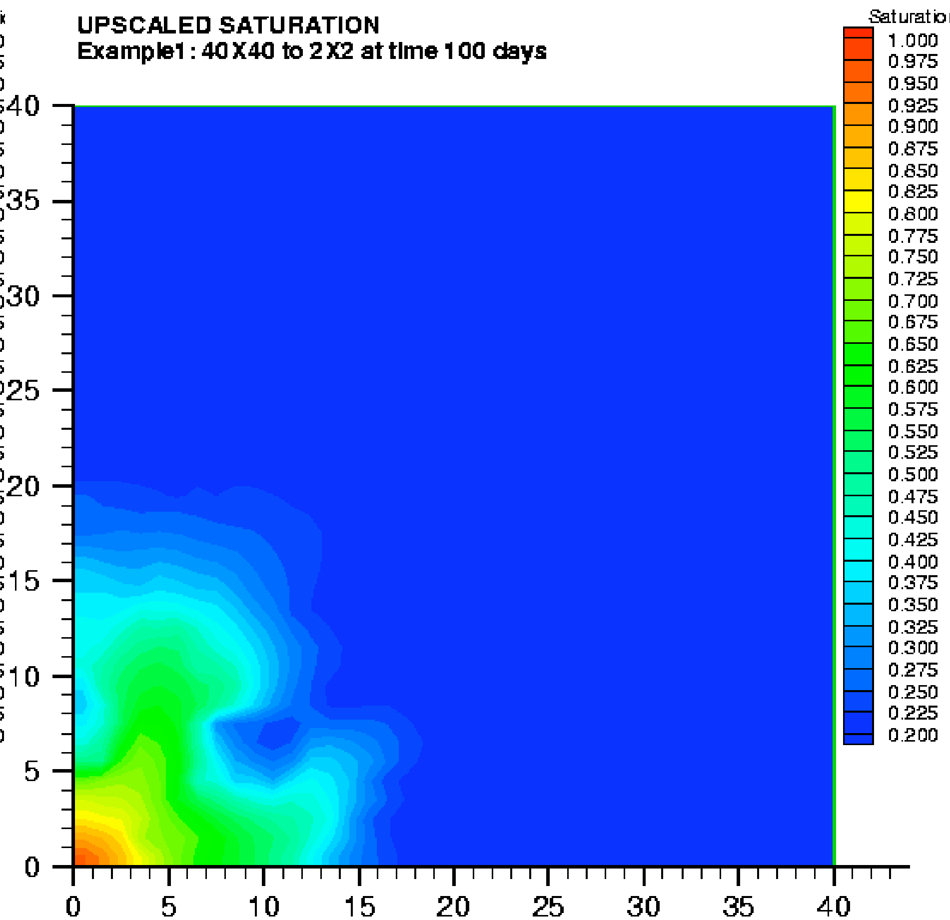
Fully upscaled to 5×5

A Quarter Five-spot Oil Reservoir Waterflood—6

Water saturation contours at 100 days



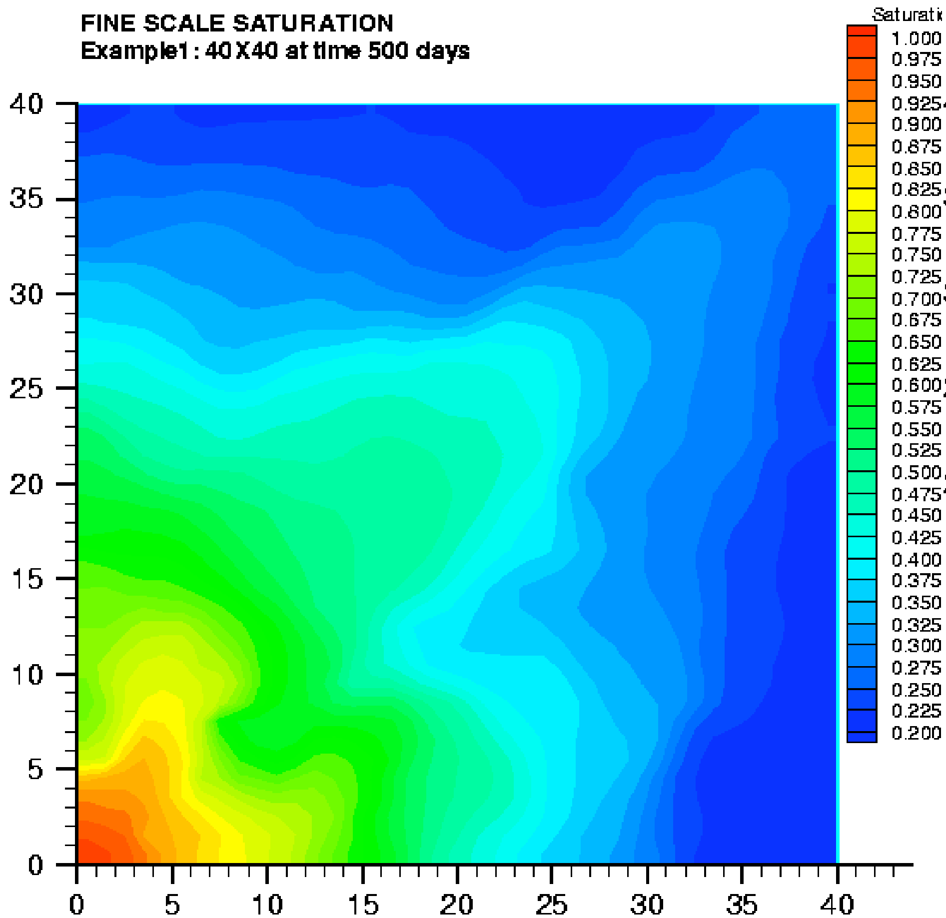
Fine 40 $\times$ 40



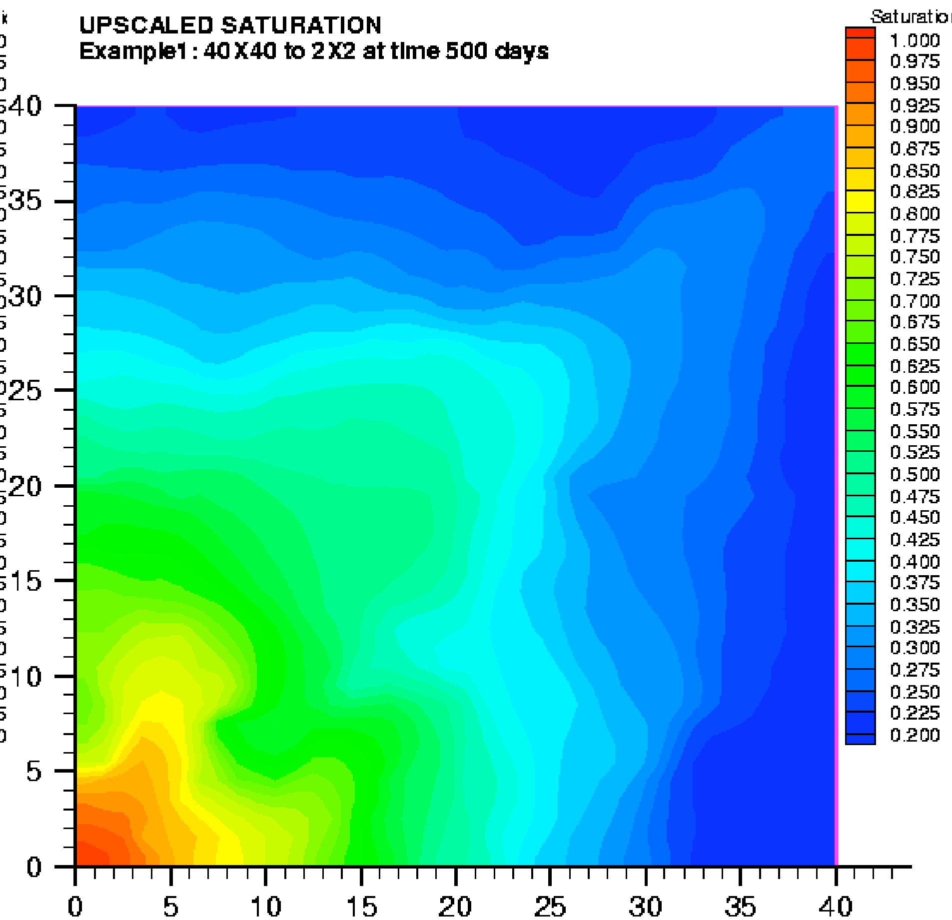
Fully upscaled to 2 $\times$ 2

A Quarter Five-spot Oil Reservoir Waterflood—7

Water saturation contours at 500 days

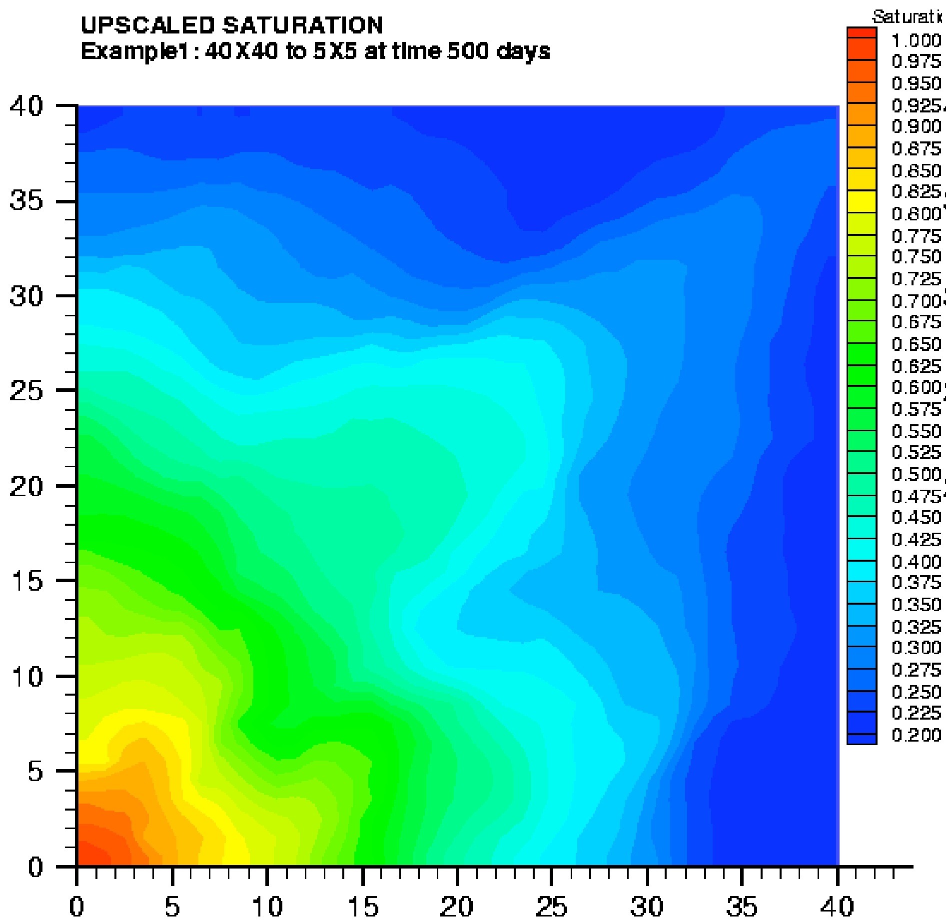


Fine 40 × 40

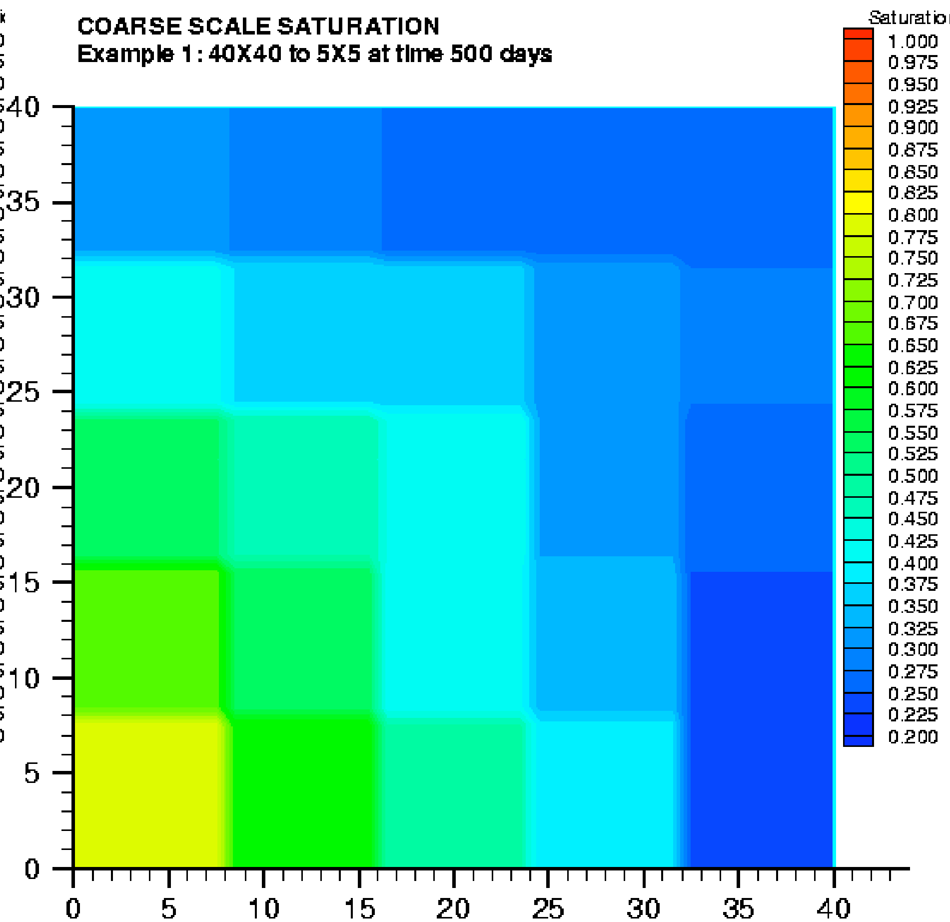


Fully upscaled to 2 × 2

Water saturation contours at 500 days



Fully upscaled to 5×5



Coarse 5×5

Summary and Conclusions

Summary and Conclusions

1. Heterogeneity can be difficult to resolve
   - Derivatives scale as ϵ^{-1}
   - Direct simulation system is computationally too expensive
2. Effective macroscopic parameter upscaling has difficulty with
   - Anisotropy
   - Nonlinearities
3. Homogenization is mathematically rigorous, but
   - Fails to give accurate locally conservative velocities
   - Requires local periodicity (two-scale separation)
4. Multiscale numerics for nonmixed system to handle heterogeneity:
   - Multiscale finite elements—define nonpolynomial finite elements
   - Variational multiscale method—modify the variational form
5. Examples show mixed multiscale numerics can be very effective