Mixed Multiscale Methods for Heterogeneous Elliptic Problems

Part 3: Mixed Multiscale Mortar Methods

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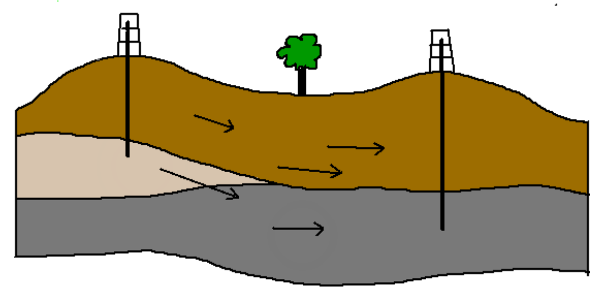
The differential problem:

$$\begin{cases} \mathbf{u} = -a_\epsilon \nabla p & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = f & \text{in } \Omega \\ \mathbf{u} \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$



Flow in porous media

The mixed variational problem:

Find $p \in W = L^2/\mathbb{R}$ and $\mathbf{u} \in \mathbf{V} = H_0(\text{div})$ such that

$$(a_\epsilon^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V} \quad \text{(Darcy's law)}$$

$$(\nabla \cdot \mathbf{u}, w) = (f, w) \quad \forall\, w \in W \quad \text{(conservation)}$$

Remark: The mixed form preserves the conservation equation, and so allows locally conservative approximations. This is a critical property in many applications.

Difficulty: Fine-scale variation in a_ϵ (the *permeability*) leads to fine-scale variation in the solution $(\mathbf{u}, p)$.

Solution: Divide and Conquer!
Use ideas from domain decomposition to control the scales.

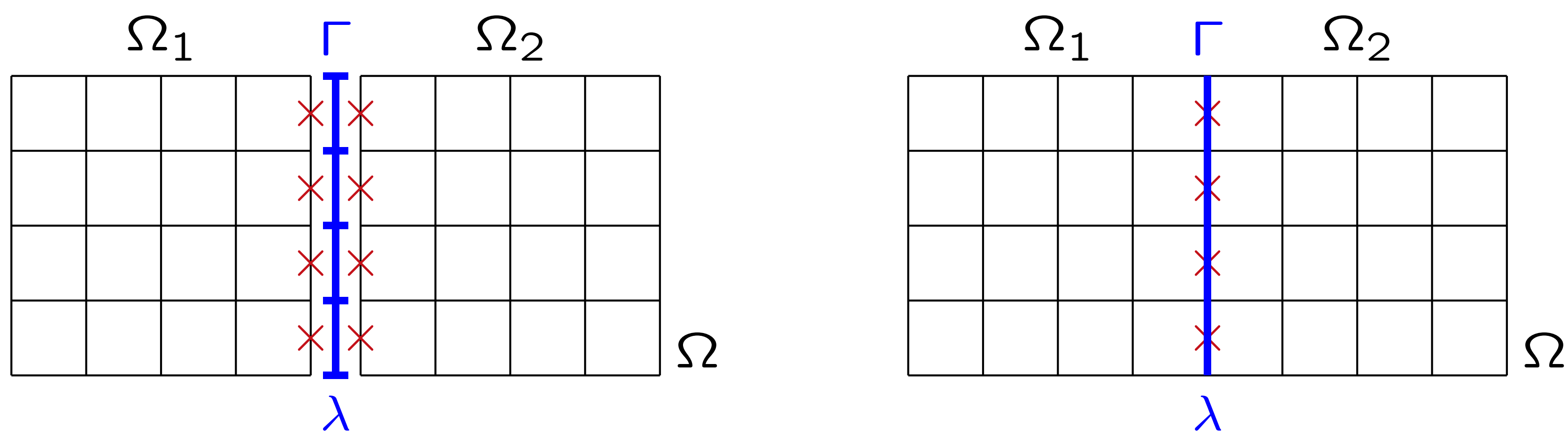
Domain Decomposition and Mortar Methods

Domain Decomposition for Mixed Methods

Glowinski and Wheeler (1988) defined domain decomposition for mixed methods by iterating on the Dirichlet-to-Neumann map.

Algorithm: Given the pressure λ on the subdomain interfaces Γ, one computes the flow locally. Based on the flux mismatch on Γ (i.e., the jump in $\mathbf{u} \cdot \nu$), one updates λ using conjugate gradients.

Once converged, the full fine-scale problem is solved.



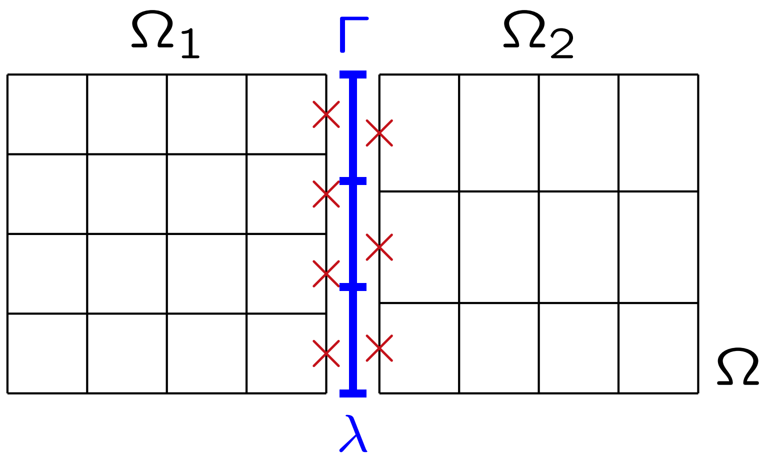
Advantages:

- Allows great flexibility in handling interdomain multiphysics (different physical models in different subdomains).
- Well suited to parallel computation.
- Will show it allows us to handle interdomain multiscale aspects.

Mortar Domain Decomposition Mixed Methods

Bernardi, Maday, and Patera (1994) defined mortar methods to glue the subdomains together weakly when the subdomain grids do not match.

Mixed Mortar Methods: Arbogast, Cowsar, Wheeler, & Yotov (2000) extended the mortar idea to mixed methods, using a continuous or discontinuous linear mortar λ.



The idea was to use grid spacings of $\mathcal{O}(h)$ for all grids.

Theorem. Approximate $\mathbf{u}$ and p in $\mathbb{P}_{k-1}(\mathcal{T}_h^{\Omega})$ and λ in $\mathbb{P}_k(\mathcal{T}_h^{\Gamma})$. Then

$$\begin{aligned}
\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0 &\le C\|f\|_k h^k &= \mathcal{O}(h^k)\\
\|\mathbf{u}-\mathbf{u}_h\|_0 &\le C\|p\|_{k+1} h^k &= \mathcal{O}(h^k)\\
\|p-p_h\|_0 &\le C\Big(\|p\|_{k+1}+\|f\|_k\Big)\, h^k &= \mathcal{O}(h^k)
\end{aligned}$$

Since λ lives on Γ, we lose 1/2-derivative going to Ω, or $h^{-1/2}$. Thus we need polynomials of degree 1 (actually 1/2) more.

The mixed mortar method is similar to standard multiscale techniques.

- Multiscale finite elements.
- Variational multiscale Method.

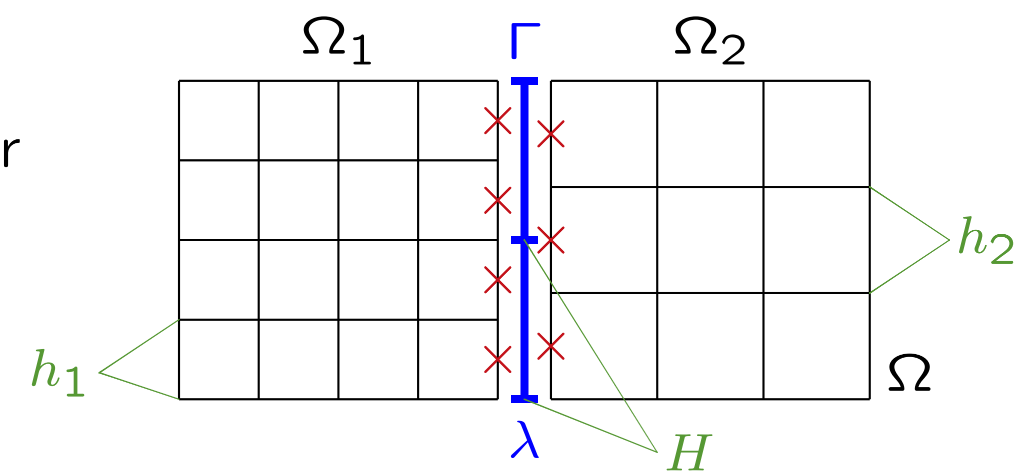
1. **Localization.** Divide Ω into many small subdomains (or coarse elements of scale H), over which the original PDE is imposed.
2. **Fine-scale effects (resolution).** The subdomains are given Dirichlet boundary conditions $p = \lambda$ on Γ and solved on the fine scale h to define the local solution.
3. **Global coarse-grid interface problem (coupling).** The weak flux mismatch (jump in $\mathbf{u} \cdot \nu$) on Γ is used to define a better λ on scale h, and we iterate the previous step until convergence is attained.
4. **Fine-grid representation of the solution.** With mortar grid scale $h < \epsilon$, we obtain a fully resolved and fully coupled approximation.

Idea: Let us relax the coupling dynamics as in multiscale methods!

The Multiscale Mixed Mortar Method

1. **Localization.** Divide Ω into many small subdomains (or coarse elements of scale H), over which the original PDE is imposed.
2. **Fine-scale effects.** The subdomains are given Dirichlet boundary conditions $p = \lambda$ on Γ and solved on the fine scale h to define the local solution.
3. **Global coarse-grid (interface) problem.** The weakly defined flux mismatch (jump in $\mathbf{u} \cdot \nu$) on Γ is used to define a better λ on scale $H > h$, and we iterate the previous step until convergence is attained.
4. **Fine-grid representation of the solution.** We obtain a fully resolved and well coupled approximate solution if λ is approximated in a higher order space.

By using a higher order mortar approximation, we compensate for the coarseness of the grid and maintain good (fine scale) overall accuracy.



Multiscale Resolution and Subdomain Coupling

1. The subdomain problems are fully resolved on the fine scale.
2. The system dynamics are *not* fully coupled between subdomains.
   - The global problem is a higher order but reduced degree-of-freedom system.
   - Computational efficiency comes from divide-and-conquer:
     (a) Small, localized subproblems are easily solved;
     (b) The coupled global problem involves far fewer degrees of freedom than the full fine-grid system (a few per coarse element), and so is relatively easily solved.

Detailed Description of the Method

Differential Equations:

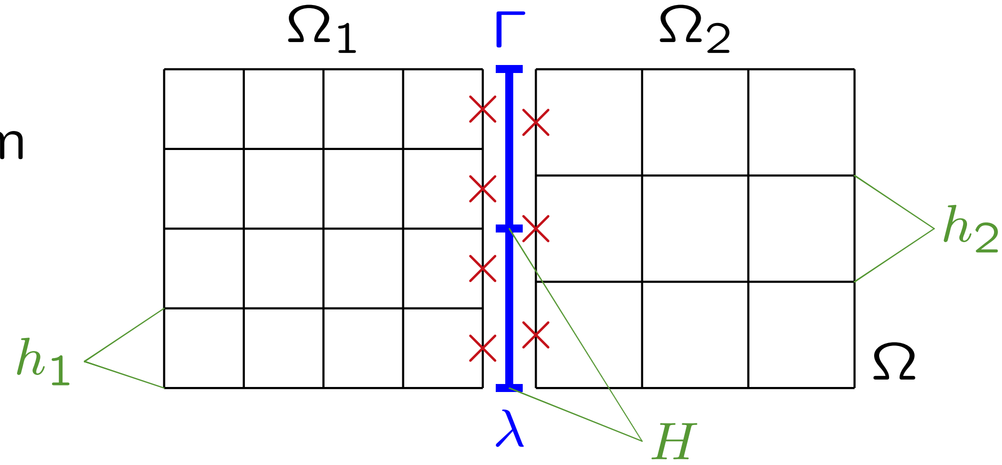
$$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j \qquad \Gamma = \bigcup_{i<j} \Gamma_{ij} \qquad \Gamma_i = \partial\Omega_i \cap \Gamma$$

$$\begin{cases} a^{-1}\mathbf{u} = -\nabla p & \text{in } \Omega_i & \text{(subdomain Darcy's law)} \\ \nabla\cdot\mathbf{u} = f & \text{in } \Omega_i & \text{(subdomain conservation)} \\ \mathbf{u}_i\cdot\nu_i + \mathbf{u}_j\cdot\nu_j = 0 & \text{on } \Gamma_{ij} & \text{(conservation on interface } \Gamma) \\ p|_{\Omega_i} = p|_{\Omega_j} & \text{on } \Gamma_{ij} & \text{(continuity of } p \text{ on } \Gamma) \\ p = 0 & \text{on } \partial\Omega & \text{(BC for simplicity)} \end{cases}$$

Variational form: Find $\mathbf{u} \in H(\text{div};\Omega_i)$, $p \in L^2(\Omega_i)$, $\lambda = p \in H^{1/2}(\Gamma_{ij})$:

$$\begin{cases} (a^{-1}\mathbf{u},\mathbf{v})_{\Omega_i} = (p,\nabla\cdot\mathbf{v})_{\Omega_i} - \langle\lambda,\mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} & \forall\,\mathbf{v}\in H(\text{div};\Omega_i) \\ (\nabla\cdot\mathbf{u},w)_{\Omega_i} = (f,w)_{\Omega_i} & \forall\,w\in L^2(\Omega_i) \\ \sum_i \langle\mathbf{u}\cdot\nu_i,\mu\rangle_{\Gamma_i} = 0 & \forall\mu\in H^{1/2}(\Gamma_{ij}) \end{cases}$$

Remark. The last equation enforces continuity of flux on Γ.



Key idea. On the interface

- Use only a few degrees of freedom (manage the linear algebra).
- Use higher order approximation (maintain accuracy).

Finite element spaces.

- **Subdomain.** $\mathbf{V}_{h,i} \times W_{h,i}$ is usual mixed space on mesh $h > 0$ on Ω_i.
- **Mortar.** $M_{H,ij}$ is continuous or discontinuous on mesh $H > h$ on Γ_{ij}.

Mortar method: Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, $\lambda_H \in M_H$ such that

$$\begin{cases} (a^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda_H, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} & \forall\, \mathbf{v} \in \mathbf{V}_{h,i} \\ (\nabla\cdot\mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i} & \forall\, w \in W_{h,i} \\ \sum_i \langle \mathbf{u}_h\cdot\nu_i, \mu\rangle_{\Gamma_i} = 0 & \forall \mu \in M_H \end{cases}$$

Remark. The last equation enforces weak continuity of flux on Γ.

An Interface Problem

Define the bilinear and linear forms on M_H by

$$d_H(\lambda, \mu) = \sum_i d_{H,i}(\lambda, \mu) = -\sum_i \langle \hat{\mathbf{u}}_h(\lambda) \cdot \nu_i, \mu \rangle_{\Gamma_i}$$

$$g_H(\mu) = \sum_i g_{H,i}(\mu) = \sum_i \langle \tilde{\mathbf{u}}_h \cdot \nu_i, \mu \rangle_{\Gamma_i}$$

where $(\hat{\mathbf{u}}_h(\lambda), \hat{p}_h(\lambda)) \in \mathbf{V}_h \times W_h$ solves (λ given, $f = 0$)

$$\begin{cases} (a^{-1}\hat{\mathbf{u}}_h(\lambda), \mathbf{v})_{\Omega_i} = (\hat{p}_h(\lambda), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} & \forall\, \mathbf{v} \in \mathbf{V}_{h,i} \\ (\nabla \cdot \hat{\mathbf{u}}_h(\lambda), w)_{\Omega_i} = 0 & \forall\, w \in W_{h,i} \end{cases}$$

and $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{V}_h \times W_h$ solves ($\lambda = 0$, f given)

$$\begin{cases} (a^{-1}\tilde{\mathbf{u}}_h, \mathbf{v})_{\Omega_i} = (\tilde{p}_h, \nabla \cdot \mathbf{v})_{\Omega_i} & \forall\, \mathbf{v} \in \mathbf{V}_{h,i} \\ (\nabla \cdot \tilde{\mathbf{u}}_h, w)_{\Omega_i} = (f, w)_{\Omega_i} & \forall\, w \in W_{h,i} \end{cases}$$

Theorem.

$$d_H(\lambda_H, \mu) = g_H(\mu) \quad \forall \mu \in M_H$$

if, and only if,

$$\boxed{\mathbf{u}_h = \hat{\mathbf{u}}_h(\lambda_H) + \tilde{\mathbf{u}}_h} \quad \text{and} \quad \boxed{p_h = \hat{p}_h(\lambda_H) + \tilde{p}_h}$$

(Glowinski & Wheeler, 1988; A., Cowsar, Wheeler & Yotov, 2000)

Interface problem. Find $\lambda_H \in M_H$ such that

$$d_H(\lambda_H, \mu) = g_H(\mu) \quad \forall \mu \in M_H$$

Theorem. The interface bilinear form $d_H(\cdot, \cdot)$ is symmetric and positive definite on M_H.

Thus, our problem reduces to a symmetric and positive definite linear system, and it can be solved by conjugate gradient iteration (for example). The computations involve:

- Once solving for $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ to get $g_H(\mu)$.
- Many times solving for $\left(\hat{\mathbf{u}}_h(\lambda_H^i), \hat{p}_h(\lambda_H^i)\right)$ to get $d_H(\lambda_H^i, \mu)$.

This seems like a natural way to obtain the solution.

Relation to Multiscale Finite Elements

Recall the bilinear form

$$d_H(\lambda,\mu) = \sum_i d_{H,i}(\lambda,\mu) = -\sum_i \langle \hat{\mathbf{u}}_h(\lambda)\cdot\nu_i, \mu\rangle_{\Gamma_i}$$

where $(\hat{\mathbf{u}}_h(\lambda), \hat{p}_h(\lambda)) \in \mathbf{V}_h \times W_h$ solves

$$\begin{cases} (a^{-1}\hat{\mathbf{u}}_h(\lambda), \mathbf{v})_{\Omega_i} = (\hat{p}_h(\lambda), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} & \forall\, \mathbf{v}\in\mathbf{V}_{h,i} \\ (\nabla\cdot\hat{\mathbf{u}}_h(\lambda), w)_{\Omega_i} = 0 & \forall\, w \in W_{h,i} \end{cases}$$

Proposition.

$$d_H(\lambda,\mu) = \Big(a^{-1}\hat{\mathbf{u}}_h(\lambda), \hat{\mathbf{u}}_h(\mu)\Big)$$

Proof: Take $\mathbf{v} = \hat{\mathbf{u}}_h(\mu)$ and use symmetry. □

Remark. We note that

$$\begin{aligned} d_{H,i}(\mu,\mu) &= (a^{-1}\hat{\mathbf{u}}_h(\mu), \hat{\mathbf{u}}_h(\mu))_{\Omega_i} \\ &\sim (a\nabla\hat{p}_h(\mu), \nabla\hat{p}_h(\mu))_{\Omega_i} = \|a^{1/2}\hat{p}_h(\mu)\|_1^2 \end{aligned}$$

So $\|\cdot\|_{d_H} = d_H(\cdot,\cdot)^{1/2}$ is a norm, essentially a discrete $H^{1/2}$-norm on Γ.

Mortar Degrees of Freedom

Let $\{\mu_\ell\}$ be a basis for $M_H = \text{span}\{\mu_\ell\}$. Define

$$\mathbf{v}_\ell = \hat{\mathbf{u}}_h(\mu_\ell) \qquad \text{and} \qquad w_\ell = \hat{p}_h(\mu_\ell)$$

Then

$$\lambda_H = \sum_\ell \lambda_\ell \mu_\ell \quad \text{and} \quad \mathbf{u}_h = \sum_\ell \lambda_\ell \mathbf{v}_\ell + \tilde{\mathbf{u}}_h \quad \text{and} \quad p_h = \sum_\ell \lambda_\ell w_\ell + \tilde{p}_h$$

Find $\{\lambda_\ell\}$ such that

$$\sum_\ell \lambda_\ell \, d_H(\mu_\ell, \mu_k) = g_H(\mu_k) \quad \forall k$$

is equivalent to

$$\sum_\ell \lambda_\ell (a^{-1}\mathbf{v}_\ell, \mathbf{v}_k) = (f, \hat{p}_h(\mu_k)) - (a^{-1}\tilde{\mathbf{u}}_h, \hat{\mathbf{u}}_h(\mu_k))$$

That is, find $(\mathbf{u}_h, p_h) \in \text{span}\{(\mathbf{v}_\ell, w_\ell)\} + (\tilde{\mathbf{u}}_h, \tilde{p}_h)$ such that

$$(a^{-1}\mathbf{u}_h, \mathbf{v}_k) = (f, w_k)$$

This is another natural way to solve the problem!

Let

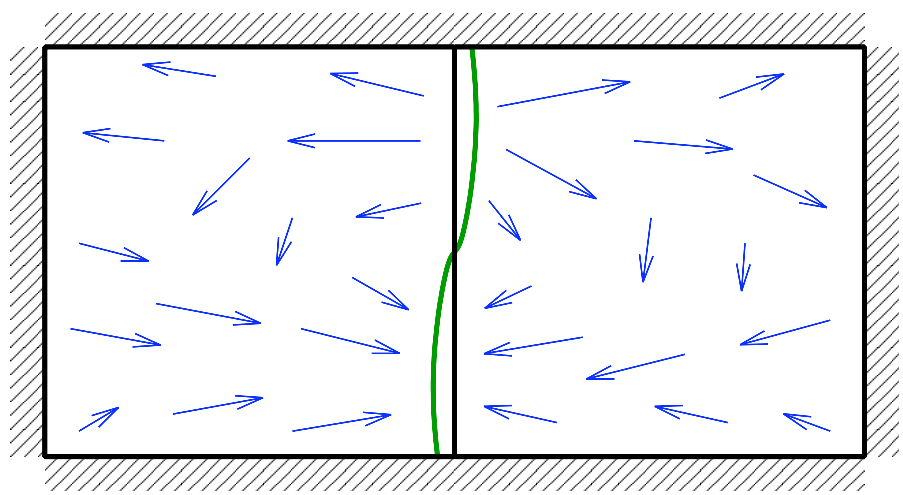
$$N_{h,H} = \text{span}\left\{ \begin{pmatrix} \mathbf{v}_\ell \\ w_\ell \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} \hat{\mathbf{u}}_h(\mu_\ell) \\ \hat{p}_h(\mu_\ell) \end{pmatrix} \right\} \subset \begin{pmatrix} \mathbf{V}_h \\ W_h \end{pmatrix}$$

Multiscale finite element formulation. Find $\begin{pmatrix} \mathbf{u}_h \\ p_h \end{pmatrix} \in N_{h,H} + \begin{pmatrix} \tilde{\mathbf{u}}_h \\ \tilde{p}_h \end{pmatrix}$ so that

$$(a^{-1}\mathbf{u}_h, \mathbf{v}) = (f, w) \quad \forall \begin{pmatrix} \mathbf{v} \\ w \end{pmatrix} \in N_{h,H}$$

Remarks. This is an unusual multiscale finite element method!

- We couple pressures and velocities.
- We allow flow on all edges.
- We add a constant term to the solution, as is typical of variational multiscale methods.
- Our multiscale finite elements are locally defined over the subdomains (i.e., the course elements).



Relation to the Variational Multiscale Method

Define the Weakly continuous velocities:

$$\mathbf{V}_w = \{\mathbf{v} \in \mathbf{V}_h : \langle \mathbf{v} \cdot \nu, \mu \rangle = 0 \ \forall \mu \in M_H\}$$

The method reduces to: Find $\mathbf{u}_h \in \mathbf{V}_w$, $p_h \in W_h$, such that

$$\begin{cases} (a^{-1}\mathbf{u}_h, \mathbf{v}) = \sum_i (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} & \forall\, \mathbf{v} \in \mathbf{V}_w \\ \sum_i (\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w) & \forall\, w \in W_h \end{cases}$$

Hilbert Space Decomposition

Define the Weakly zero velocities:

$$\mathbf{V}'_w = \{\mathbf{v} \in \mathbf{V}_w : \langle \mathbf{v} \cdot \nu, \mu \rangle_{\Gamma_i} = 0 \ \forall \mu \in M_H \text{ and } \forall i\}$$

Note that (if constants are in M_H), on each Ω_i,

$$\nabla \cdot \mathbf{V}'_w = \{w \in W : w \text{ is orthogonal to constants}\} = W'_w$$

Decomposition: With $\bar{\mathbf{V}}_w \simeq \mathbf{V}_w / \mathbf{V}'_w$,

$$\boxed{\mathbf{V}_w = \bar{\mathbf{V}}_w \oplus \mathbf{V}'_w}$$

Moreover, on each Ω_i,

$$\nabla \cdot \bar{\mathbf{V}}_w = \{w \in W : w \text{ is constant}\} = \bar{W}_w$$

The method is: Find $\mathbf{u}_h = \bar{\mathbf{u}}_w + \mathbf{u}'_w \in \mathbf{V}_w$, $p_h = \bar{p}_w + p'_w \in W_h$, such that

$$\begin{cases} (a^{-1}(\bar{\mathbf{u}}_w + \mathbf{u}'_w), \bar{\mathbf{v}}_w) = \sum_i (\bar{p}_w, \nabla \cdot \bar{\mathbf{v}}_w)_{\Omega_i} & \forall\, \bar{\mathbf{v}}_w \in \bar{\mathbf{V}}_w \\ \sum_i (\nabla \cdot \bar{\mathbf{u}}_w, \bar{w}_w)_{\Omega_i} = (f, \bar{w}_w) & \forall\, \bar{w}_w \in \bar{W}_w \end{cases}$$

and

$$\begin{cases} (a^{-1}(\bar{\mathbf{u}}_w + \mathbf{u}'_w), \mathbf{v}'_w) = \sum_i (p'_w, \nabla \cdot \mathbf{v}'_w)_{\Omega_i} & \forall\, \mathbf{v}'_w \in \mathbf{V}'_w \\ \sum_i (\nabla \cdot \mathbf{u}'_w, w'_w)_{\Omega_i} = (f, w'_w) & \forall\, w'_w \in W'_w \end{cases}$$

Since the fine-scale equation is well posed, it defines an affine closure operator and we obtain

$$\Big(a^{-1}\underbrace{(\bar{\mathbf{u}}_w + \hat{\mathbf{u}}'_w(\bar{\mathbf{u}}_w))}_{\hat{\mathbf{u}}'_w \cdot \nu \neq 0}, \bar{\mathbf{v}}_w\Big) = \sum_i (\bar{p}_w, \nabla \cdot \bar{\mathbf{v}}_w)_{\Omega_i} - (a^{-1}\tilde{\mathbf{u}}'_w, \bar{\mathbf{v}}_w) \quad \forall\, \bar{\mathbf{v}}_w \in \bar{\mathbf{V}}_w$$

This is formally the same variational multiscale method as before, but now we use nonconforming elements with greater flexibility near $\partial\Omega_i$.

A Nonstandard Multiscale Error Analysis

A Technical Condition

Let $\mathcal{Q}_{h,i}$ be $L^2(\Gamma_i)$-projection onto $\mathbf{V}_{h,i} \cdot \nu_i|_{\Gamma_i}$.

Assumption. There exists C, independent of h and H, such that

$$\|\mu\|_{0,\Gamma_{ij}} \leq C(\|\mathcal{Q}_{h,i}\mu\|_{0,\Gamma_{ij}} + \|\mathcal{Q}_{h,j}\mu\|_{0,\Gamma_{ij}}) \qquad \forall \mu \in M_{H,ij}$$

This condition says that the mortar space cannot be too rich compared to the normal traces of the subdomain velocity spaces. This is not a problem in our case $h < H$.

Theorem. There exists C, independent of h and H, such that

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le C\Big\{\|\mathbf{u}\|_k h^k + \|p\|_{m+1/2} H^{m-1/2} + \|\mathbf{u}\|_{k+1/2} h^k H^{1/2}\Big\} = \mathcal{O}(h^k + H^{m-1/2})$$

$$\|p-p_h\|_0 \le C\Big\{\|p\|_k h^k + \|p\|_{m+1/2} H^{m+1/2} + (\|f\|_k + \|\mathbf{u}\|_k) h^k H + \|\mathbf{u}\|_{k+1/2} h^k H^{3/2}\Big\} = \mathcal{O}(h^k + H^{m+1/2})$$

$$|||p-p_h||| \le C\Big\{\|p\|_{m+1/2} H^{m+1/2} + (\|f\|_k + \|\mathbf{u}\|_k) h^k H + \|\mathbf{u}\|_{k+1/2} h^k H^{3/2}\Big\} = \mathcal{O}(h^k H + H^{m+1/2})$$

Assume a is diagonal and Raviart-Thomas spaces on rectangles, then

$$|||\mathbf{u}-\mathbf{u}_h||| \le C\Big\{\|p\|_{m+1/2} H^{m-1/2} + \|\mathbf{u}\|_{k+1/2} h^k H^{1/2}\Big\} = \mathcal{O}(h^k H^{1/2} + H^{m-1/2})$$

Remarks. Optimal velocity superconvergence for RT0 ($k=1$)

- $H = \mathcal{O}(h^{1/(m-1)})$ gives error $\mathcal{O}(h^{1+1/2(m-1)})$.
- quadratic mortars ($m=3$) use $H = \mathcal{O}(h^{1/2})$ for error $\mathcal{O}(h^{5/4})$.
- cubic mortars ($m=4$) use $H = \mathcal{O}(h^{1/3})$ for error $\mathcal{O}(h^{7/6})$.

The a-priori error estimates suffer from the typical polynomial approximation scale problem. Assume $h < \epsilon < H$.

- $h < \epsilon$ fully resolves the problem.
- $H > \epsilon$ does *not* resolve the problem.

We have

$$|\nabla p| = \mathcal{O}(\epsilon^{-1}) \qquad \text{and} \qquad |D^k p| = \mathcal{O}(\epsilon^{-k})$$

so

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C\Big[(h/\epsilon)^{k+1} + (H/\epsilon)^{m+1/2} + (h/\epsilon)^{k+1}(H/\epsilon)^{1/2}\Big]$$

$$\|p - p_h\|_0 \leq C\Big[(h/\epsilon)^{k+1} + (H/\epsilon)^{m+3/2} + (h/\epsilon)^{k+1}(H/\epsilon)^{3/2}\Big]$$

Adaptivity. We turn to an a-posteriori error analysis and an iterative grid refinement process to resolve the coupling dynamics.

For all $E \in \mathcal{T}_h^\Omega$ and $\tau \in \mathcal{T}_H^\Gamma$,

$$\omega_E^2 = \|a^{-1}\mathbf{u}_h + \nabla p_h\|_E^2 h_E^2 + \|f - \nabla\cdot\mathbf{u}_h\|_E^2 h_E^2 \quad \text{(Residuals)}$$
$$+ \|\lambda_H - p_h\|_{\partial E\cap\Gamma}^2 h_E \quad \text{(Pressure mismatch)}$$
$$\omega_\tau^2 = \sum_{E\in E_\tau} \|[\mathbf{u}_h\cdot\nu]\|_{\partial E\cap\tau}^2 H_\tau^3 \quad \text{(Flux mismatch)}$$

Saturation Assumptions. We need sufficient resolution that the coarse approximation contains some "reasonable" information about the solution, so that we can detect inadequate resolution. These saturation assumptions are justified by the a-priori error theorem.

Theorem. There exists a constant C, independent of h and H, such that

$$\|p - p_h\|_0 \le C\Big\{\sum_{E\in\mathcal{T}_h^\Omega}\omega_E^2 + \sum_{\tau\in\mathcal{T}_H^\Gamma}\omega_\tau^2\Big\}^{1/2}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C\Big\{\sum_{E\in\mathcal{T}_h^\Omega}h_E^{-2}\omega_E^2 + \sum_{\tau\in\mathcal{T}_H^\Gamma}H_\tau^{-2}\omega_\tau^2\Big\}^{1/2}$$

Explicit Residual-Based Estimators: Lower Bounds

Theorem. There exists a constant C, independent of h and H, such that

$$\sum_{E\in\mathcal{T}_h^\Omega} \omega_E^2 + \sum_{\tau\in\mathcal{T}_H^\Gamma} \omega_\tau^2 \le C\Big\{\|p-p_h\|_0^2 + \sum_{E\in\mathcal{T}_h^\Omega} h_E^2\|\mathbf{u}-\mathbf{u}_h\|_{H(\text{div};E)}^2$$

$$+ \sum_{E\in\mathcal{T}_h^\Omega} h_E\|\lambda-\lambda_H\|_{\partial E\cap\Gamma}^2 + \sum_{\tau\in\mathcal{T}_H^\Gamma}\sum_{E\in E_\tau} h_E^{-1}H_\tau^3\|\mathbf{u}-\mathbf{u}_h\|_{H(\text{div};E)}^2\Big\}$$

Remark. Generally, the terms after $\|p-p_h\|_0^2$ are of higher order. For RT0 and quadratic mortars, the optimal choice $H=\mathcal{O}(h^{2/5})$ gives

$$C_1\Big\{\sum_{E\in\mathcal{T}_h^\Omega} \omega_E^2 + \sum_{\tau\in\mathcal{T}_H^\Gamma} \omega_\tau^2 + \mathcal{O}(h^{2k+2.2})\Big\}^{1/2}$$

$$\le \|p-p_h\|_0 \le C_2\Big\{\sum_{E\in\mathcal{T}_h^\Omega} \omega_E^2 + \sum_{\tau\in\mathcal{T}_H^\Gamma} \omega_\tau^2 + \mathcal{O}(h^{2k+3})\Big\}^{1/2}$$

Thus, the error $\|p-p_h\|_0$ is dominated above and below by our local residual estimators, for small enough h, so this quantity is an efficient and reliable indicator of the pressure error.

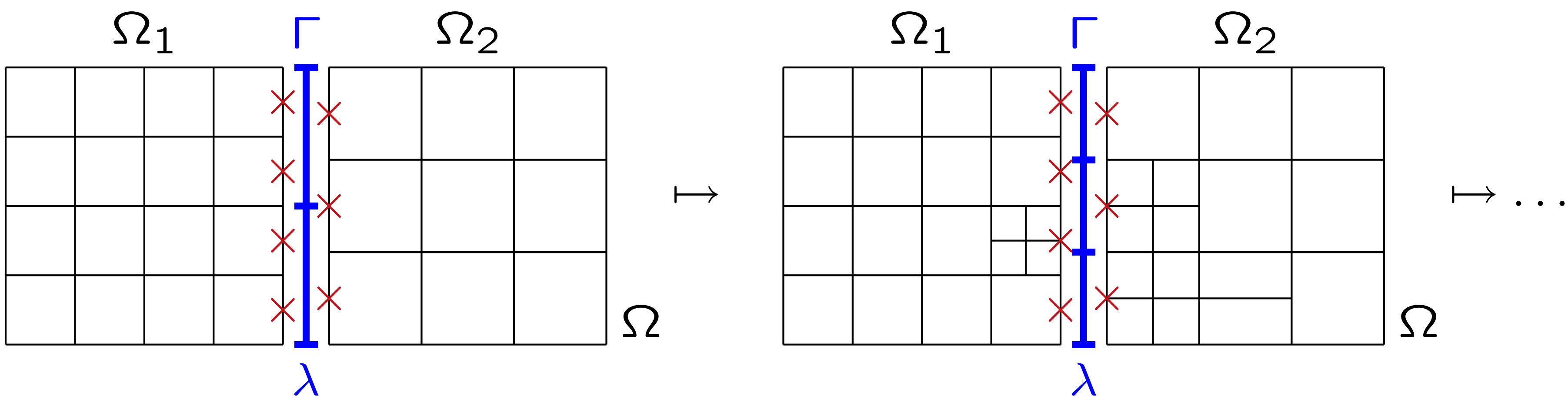
Some Remarks

1. We also developed more complex implicit error estimators which require solving local (element) boundary value problems. These give optimal upper and lower bounds for the velocity error.
2. Goal-oriented adaptivity would also be natural in this context, and would generally result in much less mortar refinement.
3. Pencheva, Vohralik, Wheeler, and Wildey (2010) recently improved the estimators to avoid saturation assumptions.

A Nonstandard Multiscale Analysis

Our analysis is *not* a standard multiscale analysis.

- A-posteriori error indicators are computed from the input data and the computed solution.
- The error indicators are used to drive adaptive mesh refinement (AMR).
- Through AMR iteration, the numerical solution is obtained on appropriate subdomain and mortar grids.
- We detect the multiscale nature of the solution through this a-posteriori analysis of intermediate approximation results.



Grid refinement algorithm

1. Solve the problem on a coarse subdomain and mortar grid.
2. For each subdomain Ω_i:
   (a) Compute $\omega_i = \left(\sum_{E \in \mathcal{T}_h^{\Omega_i}} \omega_E^2 + \sum_{\tau \in \mathcal{T}_H^{\Gamma_i}} \omega_\tau^2 \right)^{1/2}$.
   (b) If $\omega_i > \frac{1}{2} \max_j \omega_j$, refine $\mathcal{T}_h^{\Omega_i}$.
3. For each interface $\Gamma_{i,j}$, if either Ω_i or Ω_j has been refined $m-1$ times, refine $\mathcal{T}_H^{\Gamma_{ij}}$.
4. Solve the problem on the refined grid. If either the desired error tolerance or the maximum refinement level has been reached, exit; otherwise, go to Step 2.

Remarks.

- We employ pressure error estimators, since it is efficient and reliable.
- The mortar grids are refined if either adjacent subdomain grid is refined sufficiently many times.

Improvements via Homogenization and a Standard Multiscale Analysis

Weakly continuous velocities: Let

$$\mathbf{V}_{h,H,0} = \left\{ \mathbf{v} \in \mathbf{V}_h : \sum_i \langle \mathbf{v}_i \cdot \nu_i, \mu \rangle_{\Gamma_i} = 0 \quad \forall\, \mu \in M_H \right\}.$$

If we work over the weakly continuous velocities, we have

$$\begin{cases} (a_\epsilon^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{v}) = \sum_i \left\{ (p - p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle p, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} \right\} & \forall\, \mathbf{v} \in \mathbf{V}_{h,H,0} \\ \sum_i (\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w)_{\Omega_i} = 0 & \forall\, w \in W_h \end{cases}$$

Troublesome term: A "nonconforming error" term arises because p is *not* continuous. However, $\mathbf{v}$ is weakly continuous, so

$$\langle p, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} = \langle p - w, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} \quad \text{for any } w \in M_H$$

leads to coarse H-level approximation.

- This is a pure approximation problem.
- Polynomial approximation gives $\mathcal{O}(H^{m-1/2})$.

Idea: Can we use multiscale ideas to improve this?

Suppose that a_ϵ is locally periodic of period ϵ. Then

$$a_\epsilon(x) = a(x, x/\epsilon)$$

where $a(x,y)$ is periodic in y of period 1 on the unit cube Y.

Let a_0 be the homogenized permeability matrix, defined by

$$a_{0,ij}(x) = \int_Y a(x,y)\left(\delta_{ij} - \frac{\partial \omega^j(x,y)}{\partial y_i}\right) dy$$

where, for fixed x, $\omega^j(x,y)$ is the Y-periodic solution of

$$-\nabla_y \cdot (a \nabla_y \omega^j) = \nabla \cdot (a \mathbf{e}_j)$$

Homogenized solution: Let $(\mathbf{u}_0, p_0)$ solve

$$\begin{cases} \mathbf{u}_0 = -a_0 \nabla p_0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_0 = f & \text{in } \Omega \\ p_0 = 0 & \text{on } \partial\Omega \end{cases}$$

Then $(\mathbf{u}_0, p_0)$ is a smooth "approximation" of $(\mathbf{u}, p)$.

Define the first order corrector

$$p_1^\epsilon(x) = p_0(x) - \epsilon \sum_j \omega^j(x, x/\epsilon) \frac{\partial p_0}{\partial x_j}$$

Theorem. $\|\nabla[p - p_1^\epsilon]\|_0 \le C\Big\{\epsilon\|\nabla p_0\|_1 + \sqrt{\epsilon|\partial\Omega|}\,\|\nabla p_0\|_{0,\infty}\Big\}$

Structure: Although p is not smooth, it is a fixed operator (based on the microstructure) of a smooth function p_0. Thus we should approximate

$$p(x) \approx p_1^\epsilon(x) = \left(1 - \epsilon \sum_j \omega^j(x, x/\epsilon) \frac{\partial}{\partial x_j}\right) p_0(x) \approx \left(1 - \epsilon \sum_j \omega^j(x, x/\epsilon) \frac{\partial}{\partial x_j}\right) q(x)$$

where q is some polynomial.

A Multiscale Mortar Space: Let

$$M_H = \left\{ \left(1 - \epsilon \sum_j \omega^j(x, x/\epsilon) \frac{\partial}{\partial x_j}\right) q\Big|_\Gamma \;:\; q \in \mathbb{P}_{m-1}(\mathcal{T}_H^{\Gamma^*}) \right\}$$

wherein Γ is extended into the domain to form Γ^*.

Theorem. There exists a constant C, independent of h and H, such that

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C\Big\{\|\mathbf{u}\|_k h^k + \|p_0\|_{m+1/2} H^{m-1/2} + \sqrt{\epsilon}\Big\}$$
$$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_0 \le C\|\nabla \cdot \mathbf{u}\|_k h^k$$

Remarks.

- $h < \epsilon$, so the first term is small (i.e., fully resolved).
- p_0 is smooth, so the second term is small.
- $\sqrt{\epsilon}$ is the standard term from the boundary corrector from homogenization theory, and is assumed small.
- There is no numerical resonance (i.e., no ϵ/H term)!

Some Numerical Results

- Domain Ω is the unit square or cube.
- Use lowest order Raviart-Thomas spaces (RT0).
- Use Dirichlet boundary conditions on the left and right edges, and Neumann on the rest of the boundary.
- The domain is equally divided into four (or eight in 3-D) subdomains.
- Solve using congugate gradients and a balancing preconditioner.
- Continuous and discontinuous quadratic and linear mortars ($m = 3, 2$).
- We use the scaling $H = h^{1/2}$ for $m = 3$ and $H = 2h$ for $m = 2$, which is optimal for superconvergent velocities.

$\mathbb{P}(\mathcal{T}_H^\Gamma)$	H	$\|\|p - p_h\|\|$	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - l_H\|\|\|$	
						Full a	Diag a
2	$h^{1/2}$	1	1	1.5	1.25	1.25	1.5
1	$2h$	1	1	2	1.5	1.5	2

A 2-D smooth problem with known analytic solution

$$p(x,y) = x^3y^4 + x^2 + \sin(xy)\cos(y)$$

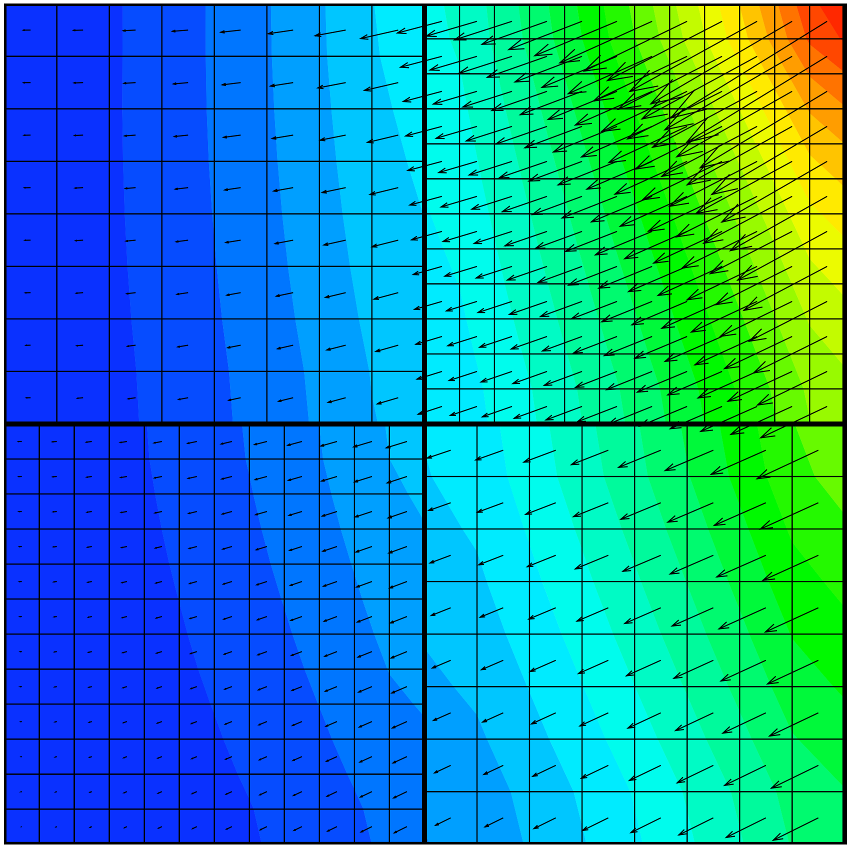
and full tensor coefficient

$$a = \begin{pmatrix} (x+1)^2 + y^2 & \sin(xy) \\ \sin(xy) & (x+1)^2 \end{pmatrix}.$$

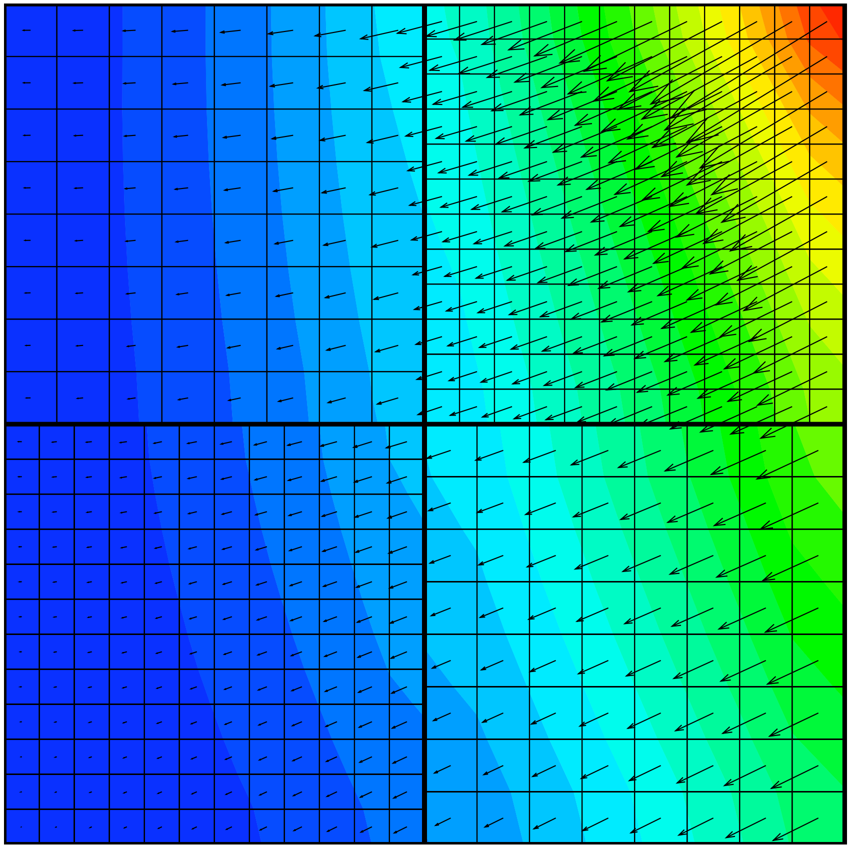
- $\Omega = (0,1)^2$
- Use lowest order Raviart-Thomas spaces (RT0).
- Use Dirichlet boundary conditions on the left and right edges, and Neumann on the rest of the boundary.
- We use the scaling $H = h^{1/2}$ for $m = 3$ and $H = 2h$ for $m = 2$, which is optimal for superconvergent velocities.

$\mathbb{P}(\mathcal{T}_H^\Gamma)$	H	$\|\|p - p_h\|\|$	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - l_H\|\|\|$	
						Full a	Diag a
2	$h^{1/2}$	1	1	1.5	1.25	1.25	1.5
1	$2h$	1	1	2	1.5	1.5	2

Computed pressure and velocity on nonmatching grids.



Discontinuous quadratic mortars



Discontinuous linear mortars

Discontinuous quadratic mortars and nonmatching grids.

$1/h$	Iter	Cond	$\|\|p - p_h\|\|$	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - l_H\|\|\|$
4	8	18.8	2.64E-1	2.03E-1	4.62E-2	2.13E-2	4.45E-2
16	7	2.5	6.37E-2	4.86E-2	2.83E-3	1.82E-3	2.72E-3
64	7	2.3	1.59E-2	1.21E-2	1.75E-4	1.59E-4	1.69E-4
256	8	3.0	3.98E-3	3.03E-3	1.09E-5	1.68E-5	1.06E-5
Rate			1.01	1.01	2.01	1.72	2.01
Theor			1.00	1.00	1.50	1.25	1.25

Discontinuous linear mortars and nonmatching grids.

$1/h$	Iter	Cond	$\|\|p - p_h\|\|$	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - l_H\|\|\|$
4	4	1.31	2.63E-1	2.04E-1	4.54E-2	2.35E-2	4.55E-2
16	7	2.12	6.37E-2	4.86E-2	2.82E-3	2.30E-3	2.86E-3
64	8	3.27	1.59E-2	1.21E-2	1.75E-4	2.38E-4	1.78E-4
256	8	5.02	3.98E-3	3.03E-3	1.09E-5	2.74E-5	1.11E-5
Rate			1.01	1.01	2.01	1.63	2.00
Theor			1.00	1.00	2.00	1.50	1.50

Conclusions.

- Convergence rates hold, and may be better than advertised.
- The solution procedure is efficient (# iterations $\sim$ constant).

Ex. 2—A Discontinuous Permeability—1

A 2-D discontinuous problem with

$$a = \begin{cases} I & \text{for } 0 \le x < 1/2 \\ 10I & \text{for } 1/2 < x \le 1 \end{cases}$$

and known continuous solution with continuous normal flux at $x = 1/2$

$$p(x,y) = \begin{cases} x^2y^3 + \cos(xy), & 0 \le x \le 1/2, \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos\left(\frac{2x+9}{20}y\right), & 1/2 \le x \le 1, \end{cases}$$

Continuous quadratic mortars and nonmatching grids.

$1/h$	Iter	Cond	$\|\|p - p_h\|\|$	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - l_H\|\|\|$
4	9	111.0	1.84E-2	6.20E-2	1.13E-3	4.58E-2	3.27E-3
16	14	25.5	4.37E-3	1.50E-2	8.07E-5	3.67E-3	2.40E-4
64	15	24.1	1.09E-3	3.73E-3	5.37E-6	6.45E-4	2.45E-5
256	16	30.3	2.72E-4	9.26E-4	3.70E-7	1.27E-4	2.97E-6
Rate			1.01	1.01	1.93	1.40	1.68
Theor			1.00	1.00	1.50	1.25	1.50

Continuous linear mortars and nonmatching grids.

$1/h$	Iter	Cond	$\|\|p - p_h\|\|$	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - l_H\|\|\|$
4	5	16.8	1.84E-2	9.57E-2	1.28E-3	7.04E-2	5.23E-3
16	14	25.5	4.37E-3	1.75E-2	8.20E-5	7.76E-3	3.53E-4
64	22	43.2	1.09E-3	3.85E-3	5.10E-6	9.06E-4	2.18E-5
256	23	62.6	2.72E-4	9.28E-4	3.19E-7	1.11E-4	1.36E-6
Rate			1.01	1.11	2.00	1.55	1.99
Theor			1.00	1.00	2.00	1.50	2.00

Conclusion. We see similar results to the continuous case.

Domain decomposition studies.

- Fine grid 256×256.
- Coarse grid 2×2, 4×4, or 8×8 subdomains.
- Mortar grids consistent with the optimal velocity superconvergence ($H = h^{1/2}$ for quadratic mortars and $H = 2h$ for linear mortars).

Continuous quadratic mortars.

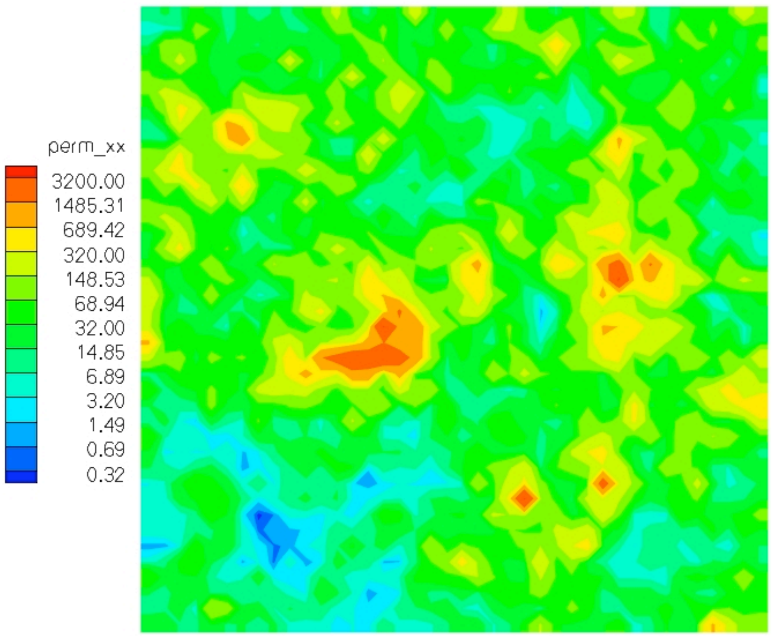
Dom	Iter	$\|\|p - p_h\|\|$	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - l_H\|\|\|$
2×2	16	4.97E-3	4.31E-3	1.61E-5	2.43E-5	1.37E-5
4×4	23	4.97E-3	4.31E-3	1.62E-5	5.20E-5	2.48E-5
8×8	23	4.97E-3	4.31E-3	1.63E-5	9.28E-5	3.83E-5

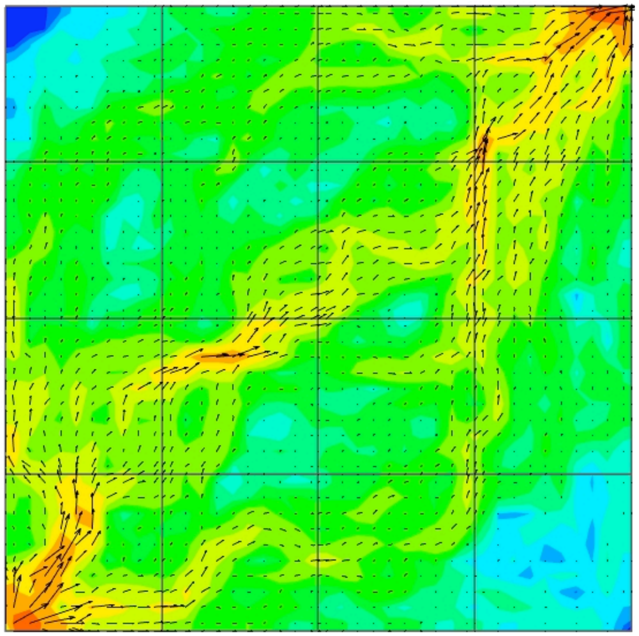
Continuous linear mortars.

Dom	Iter	$\|\|p - p_h\|\|$	$\|\|\mathbf{u} - \mathbf{u}_h\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - l_H\|\|\|$
2×2	23	4.97E-3	4.31E-3	1.61E-5	2.27E-5	1.35E-5
4×4	36	4.97E-3	4.31E-3	1.61E-5	2.78E-5	2.27E-5
8×8	39	4.97E-3	4.31E-3	1.61E-5	3.74E-5	3.41E-5

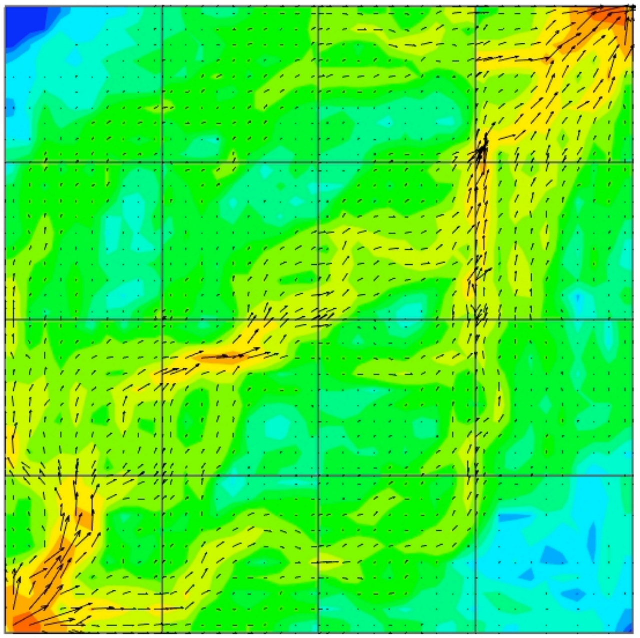
Conclusion. Larger subdomains give better results, and require less work.

- $\Omega = (0, 40)^2$
- Single mortar per coarse interface, $H = 10$, $h = 1$:
  - Subdomain is 10×10.
  - Coarse problem is 4×4.
- Use lowest order Raviart-Thomas spaces (RT0).
- Inject fluid in lower left, extract in upper right, no-flow boundary conditions.
- Compare to fine-scale RT0 solution.
- Permeability varies by 10^4.

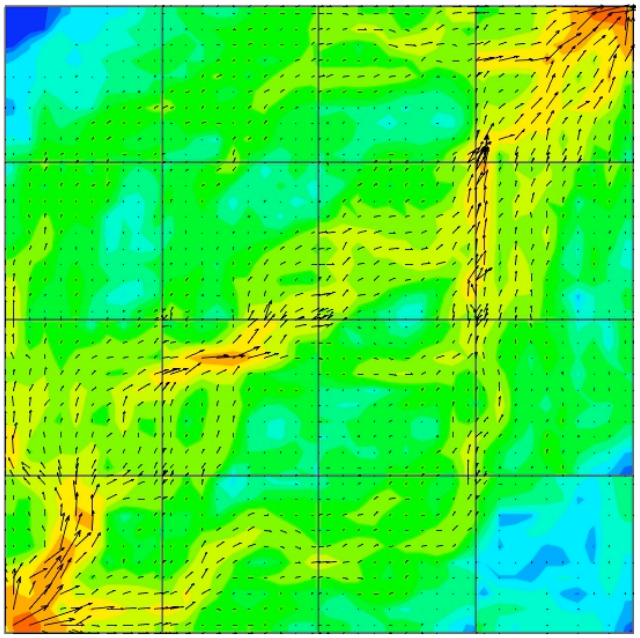




RT0 (Fine)

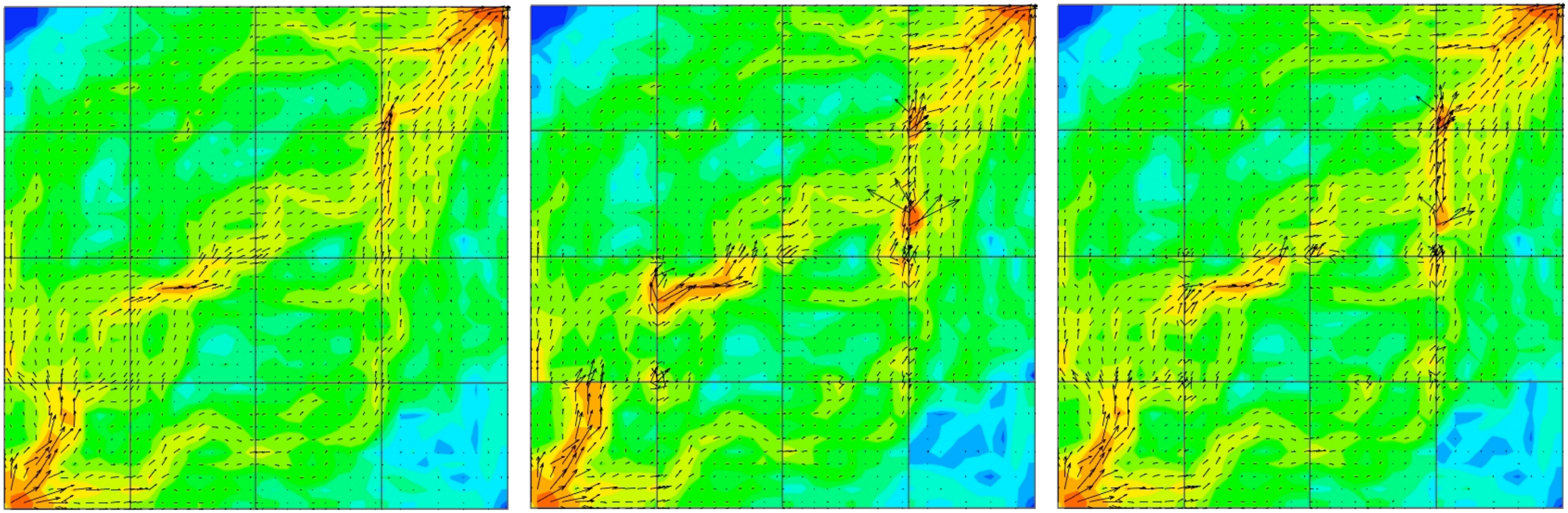


HMS-OS



HMS

Speed (log scale) and Velocity



RT0 (Fine) P1 P2

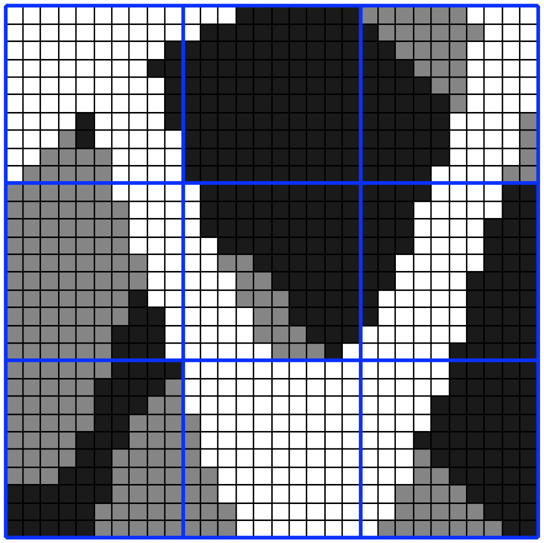
Speed (log scale) and Velocity

Ex. 3—A Heterogeneous Quarter-Five Spot—4

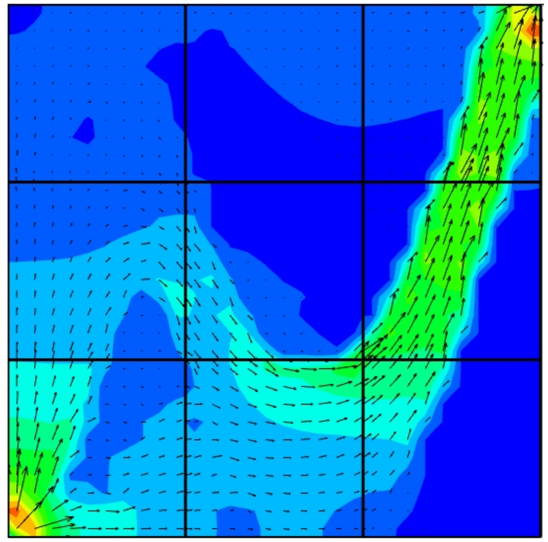
Relative Errors with Respect to
the Fine-scale RT0 Solution

Method	Pressure		Velocity	
	L2 err	max err	L2 err	max err
P1	0.199	0.145	0.416	0.804
P2	0.043	0.035	0.256	0.527
HMS	0.011	0.014	0.107	0.143
HMS-OS	0.009	0.013	0.082	0.136

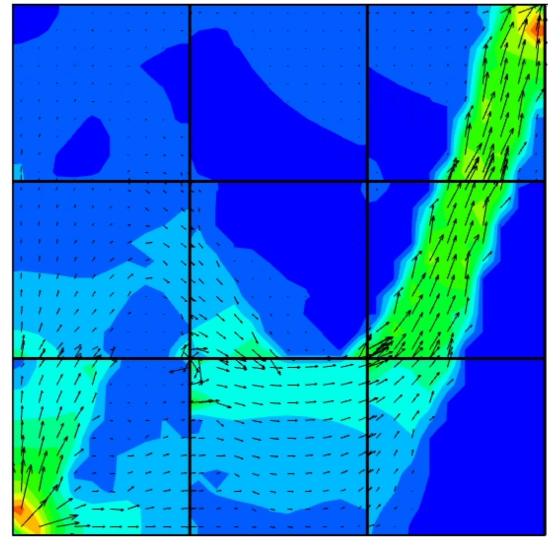
Ex. 4—Simple Channel—1



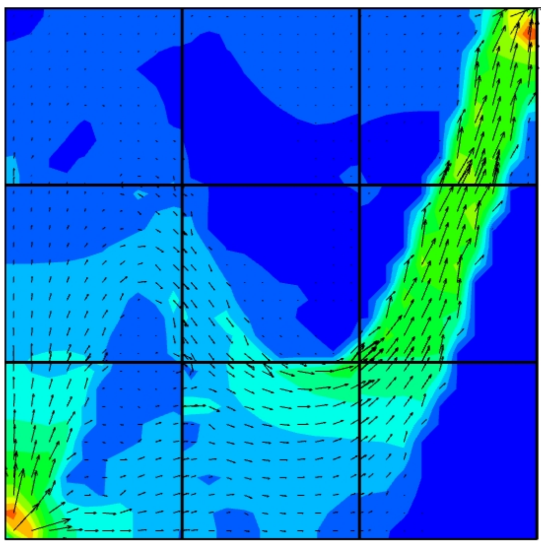
Permeability



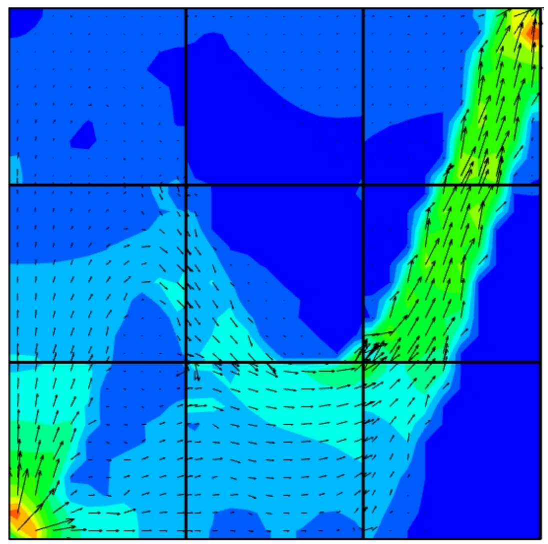
RT0



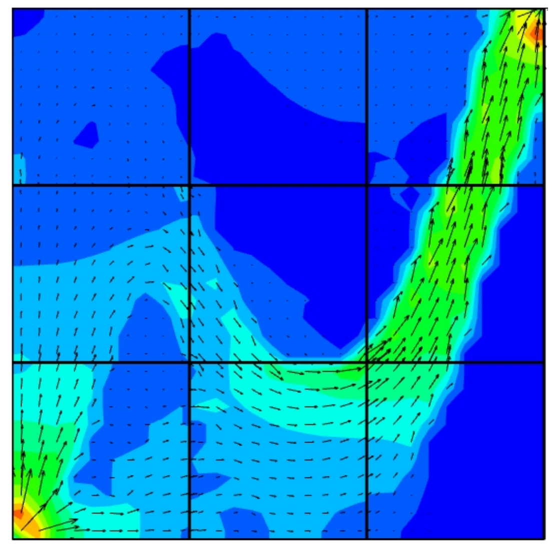
P1



P2



HMS



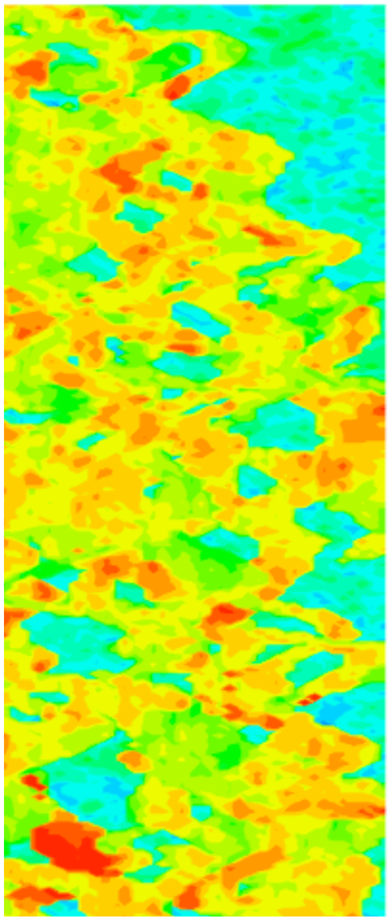
HMS-OS

- Permeability: 10 (white), 1 (gray), and 0.1 on 30×30 grid
- Speed and velocity on 3×3 coarse grid with 10×10 subgrid

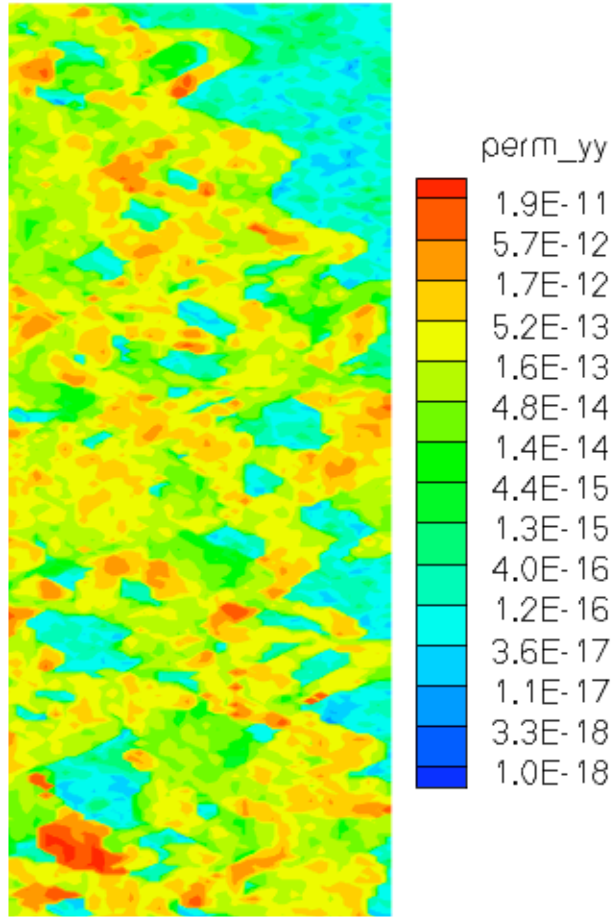
Ex. 4—Simple Channel—2

Method	Pressure Error ℓ^2	Pressure Error ℓ^∞	Velocity Error ℓ^2	Velocity Error ℓ^∞
P1	0.0432	0.0168	0.168	0.360
P2	0.0104	0.0033	0.062	0.176
HMS	0.0074	0.0021	0.105	0.289
HMS-OS	0.0087	0.0034	0.067	0.120

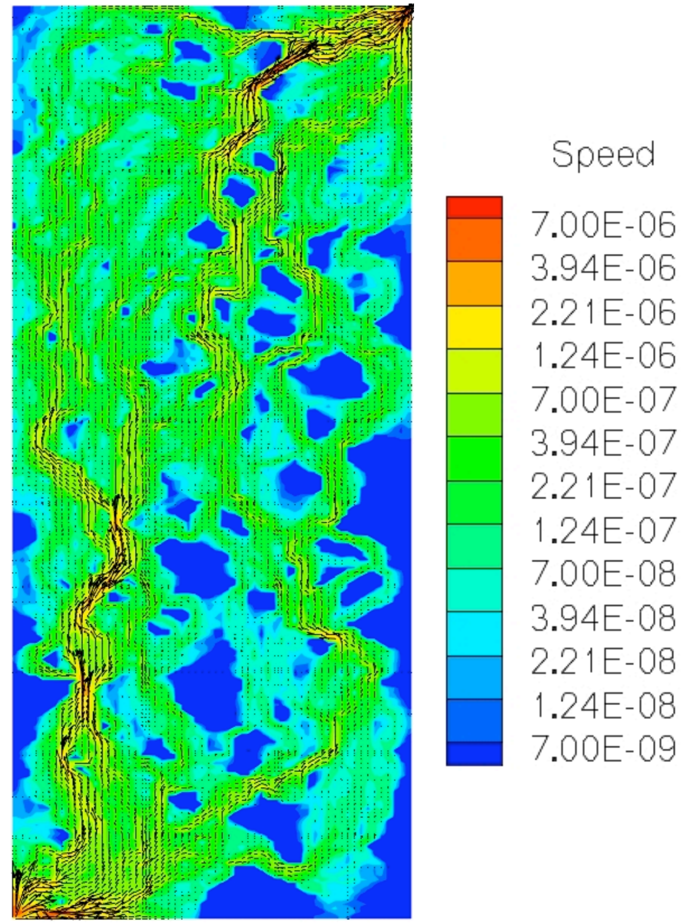
Ex. 5—SPE10–Layer 85 Test Example—1



x-permeability



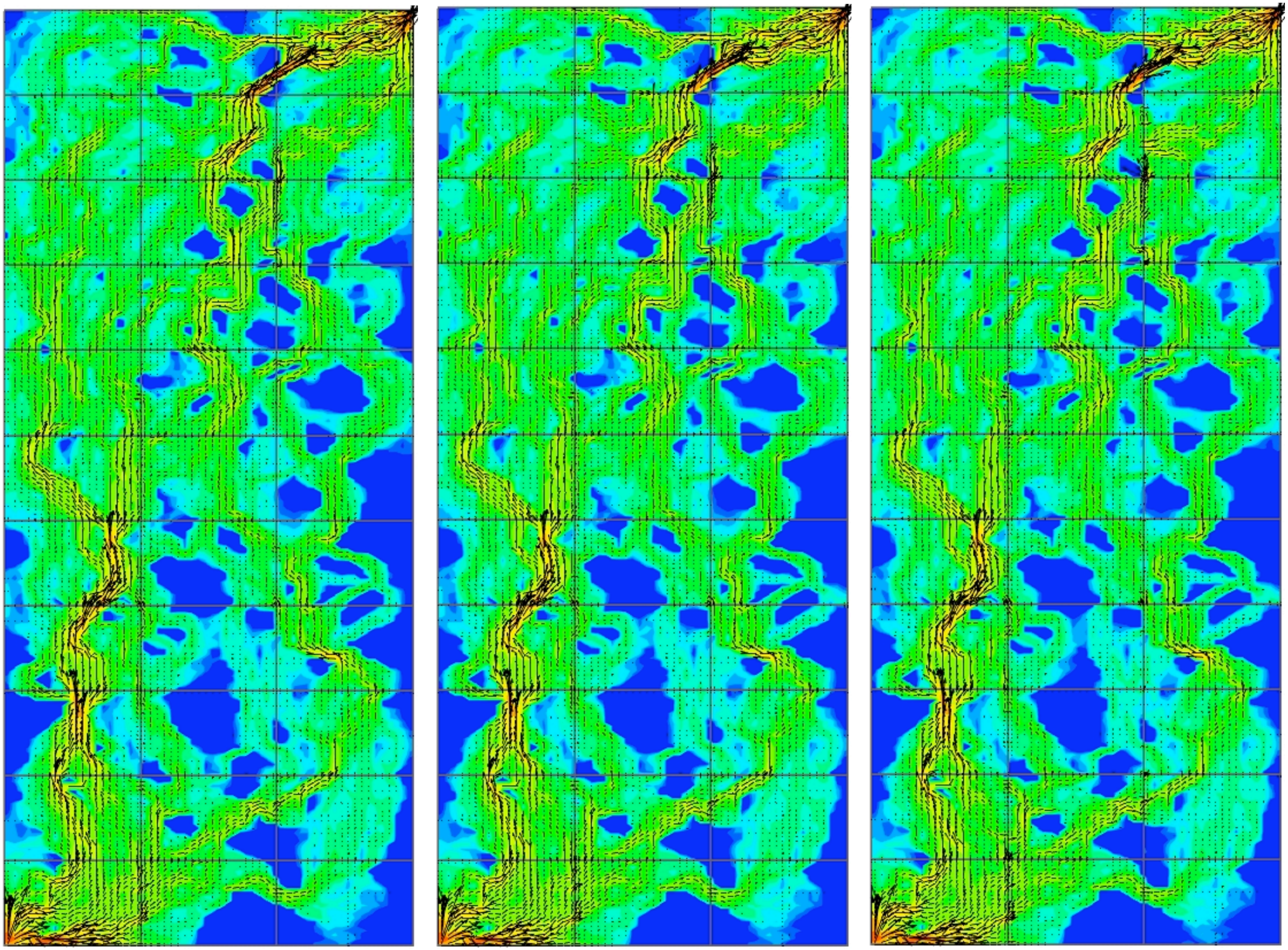
y-permeability



RT0 (Fine)

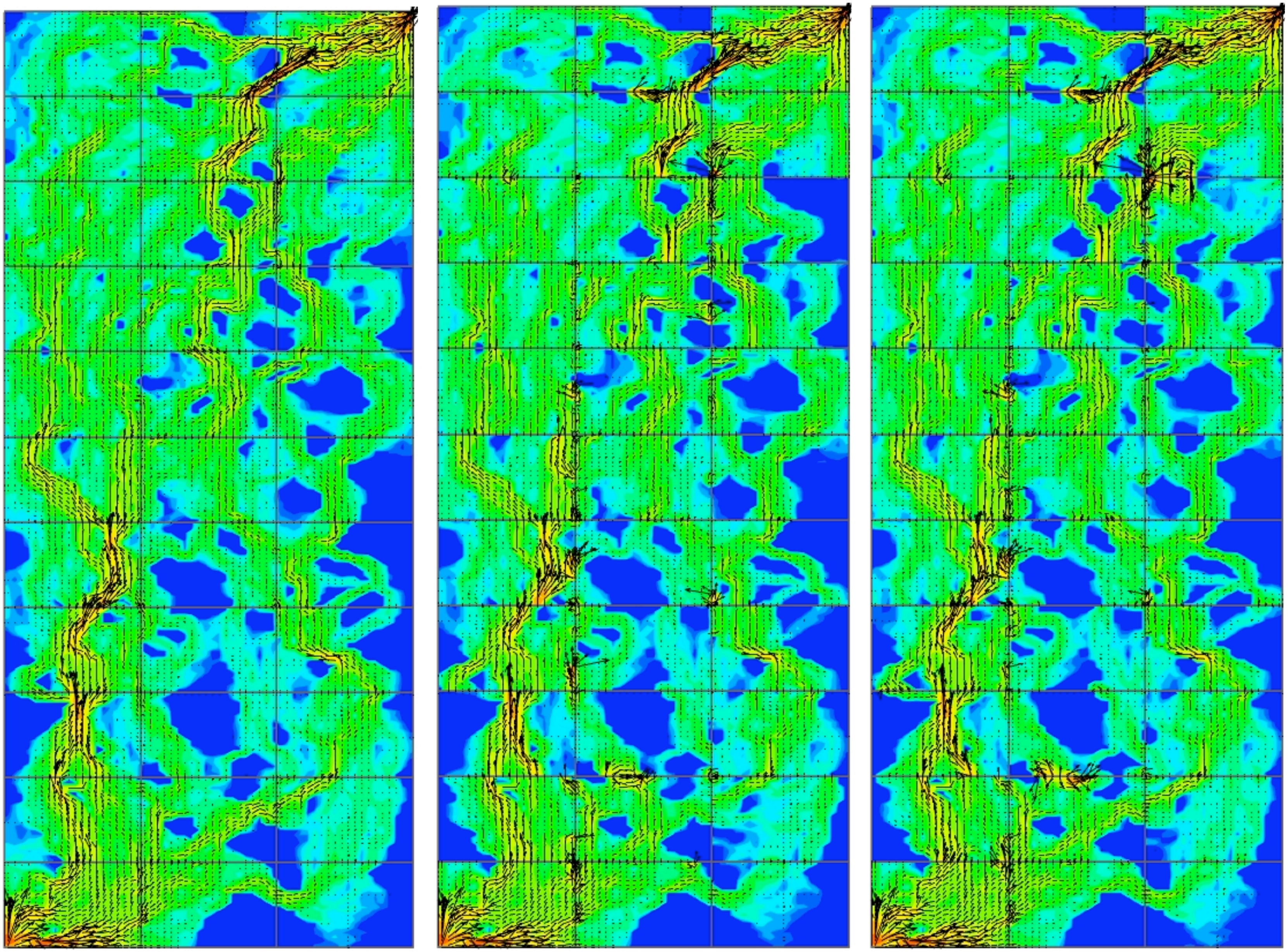
- Single mortar per coarse interface
  - Coarse problem is 3×11.
  - Subdomain is 20×20.
- Use lowest order Raviart-Thomas spaces (RT0).
- Inject fluid in lower left, extract in upper right, no-flow BC's.

Ex. 5—SPE10–Layer 85 Test Example—2



RT0 (Fine) HMS-OS HMS

Speed and Velocity



RT0 (Fine) P1 P2

Speed and Velocity

Ex. 5—SPE10–Layer 85 Test Example—4

Relative Errors with Respect to
the Fine-scale RT0 Solution

Method	Pressure		Velocity	
	L2 err	max err	L2 err	max err
P1	0.064	0.027	0.399	0.743
P2	0.036	0.020	0.406	0.904
HMS	0.010	0.007	0.160	0.380
HMS-OS	0.006	0.003	0.106	0.318

Summary and Conclusions

Summary and Conclusions

1. A multiscale mortar mixed method was defined, involving:
   - **Localization** into small subdomains (or coarse elements);
   - **Fine-scale effects** through full resolution of local subproblems;
   - **Global coarse-grid problem** for the mortar unknowns coupling the subdomains through weak velocity continuity;
   - **Fine-grid construction** of an approximate solution.
2. The multiscale mortar mixed method
   - uses novel multiscale finite elements;
   - is a nonconforming approximation in the variational multiscale method.
3. An nonstandard multiscale analysis of errors shows:
   - good a-priori estimates for resolved systems;
   - computable a-posteriori error indicators;
   - that adaptive mesh refinement can be used to capture microscales in the solution as needed.
4. Homogenization theory suggests a multiscale mortar space.
   - A standard multiscale analysis shows good multiscale convergence with no numerical resonance.
5. Numerical results confirm the theory and demonstrate the effectiveness of the method.