Imaging in random acoustic waveguides

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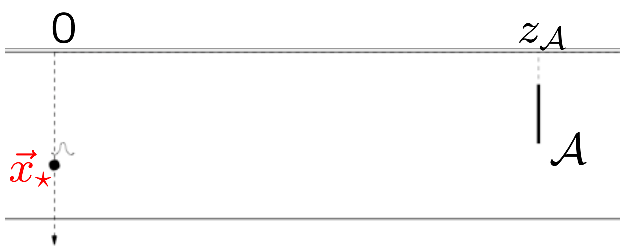
Source localization with passive array of receivers



Array data: $p(t, \vec{x}_r), \quad \vec{x}_r = (r, z_{\mathcal{A}}), \ r \in \mathcal{A}$

Problem: Determine source location $\vec{x}_\star = (x_\star, 0)$ (source cross-range $x_\star$ and the range $z_{\mathcal{A}}$ to the array).

- This is the simplest imaging problem. Can be extended to imaging reflectors with active arrays of sources and receivers.

Difficulty: We consider waveguides with fluctuating sound speed. The fluctuations are typically small $(1\% - 3\%)$ but their cumulative long range effect is strong $\rightsquigarrow$ $p(t, \vec{x}_r)$ loses coherence.

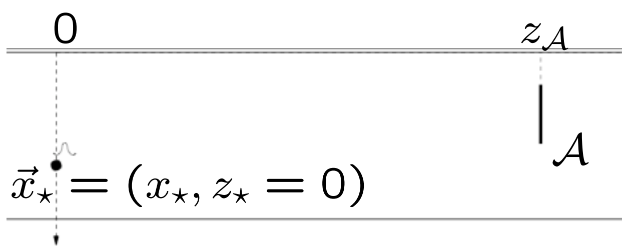
Goal of talk

• Using mathematical analysis based on modeling the wave speed fluctuations with random processes:

1. Understand how the pressure field received at the array loses coherence $\rightsquigarrow$ how and why widely used imaging methods fail.

2. Show how imaging can still be done with incoherent data.

Mathematical setup. Planar waveguide.



$\vec{x} = (x, z), \quad x \in (0, X), \quad z \in \mathbb{R}.$

$$
\begin{aligned}
-\frac{1}{c^2(\vec{x})}\frac{\partial^2 p(t,\vec{x})}{\partial t^2} + \Delta p(t,\vec{x}) &= f(t)\frac{\partial}{\partial z}\delta(\vec{x}-\vec{x}_\star), \\
p(t,\vec{x}) &\equiv 0, \qquad t \le 0 \\
p(t,\vec{x}) &= 0, \qquad x \in \{0, X\}.
\end{aligned}
$$

- The sound speed model is*

$$\frac{c_o^2}{c^2(\vec{x})} = 1 + \varepsilon\nu(\vec{x}), \quad \varepsilon \ll 1$$

$\nu(\vec{x})$ is a bounded, mean zero random process, stationary and decorrelating fast enough in z.

*For simplicity c_o = constant but $c_o(x)$ could be considered. Typical $\varepsilon = 1$–3%

Step 1: Write the mathematical model of the data recorded at the array: $p(t, \vec{x}_r)$ for $\vec{x}_r = (r, z_{\mathcal{A}})$ and $r \in \mathcal{A}$.

Unperturbed waveguides ($\varepsilon = 0$)

- We have $p(t, \vec{x}) = \int \frac{d\omega}{2\pi} \widehat{p}(\omega, \vec{x}) e^{-i\omega t}$ where

$$\left(\frac{\omega^2}{c_o^2} + \frac{\partial^2}{\partial x^2} \right) \widehat{p}(\omega, \vec{x}) + \frac{\partial^2 \widehat{p}(\omega, \vec{x})}{\partial z^2} = \widehat{f}(\omega) \delta(x - x_\star) \delta'(z)$$

$$\widehat{p}(\omega, \vec{x}) = 0 \quad \text{for} \quad x \in \{0, X\}, \qquad \vec{x} = (x, z),$$
$$\widehat{p}(\omega, \vec{x}) = \text{ bounded \& outgoing at } z \to \pm\infty.$$

- Separation of variables $\rightsquigarrow$ solution in terms of eigenfunctions*

$$\phi_j(x) = \sqrt{\frac{2}{X}} \sin\left(\frac{\pi j x}{X}\right) \quad \text{for} \quad j = 1, 2, \ldots \quad \text{and eigenvalues}$$

$$\mu_j = \left(\frac{\omega}{c_o}\right)^2 - \left(\frac{\pi j}{X}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2 \left[1 - \left(\frac{j\lambda}{2X}\right)^2\right], \qquad \frac{\omega}{c_o} = \frac{2\pi}{\lambda}.$$

*If $c_o = c_o(x) \rightsquigarrow$ slight complication that ϕ_j are frequency dependent.

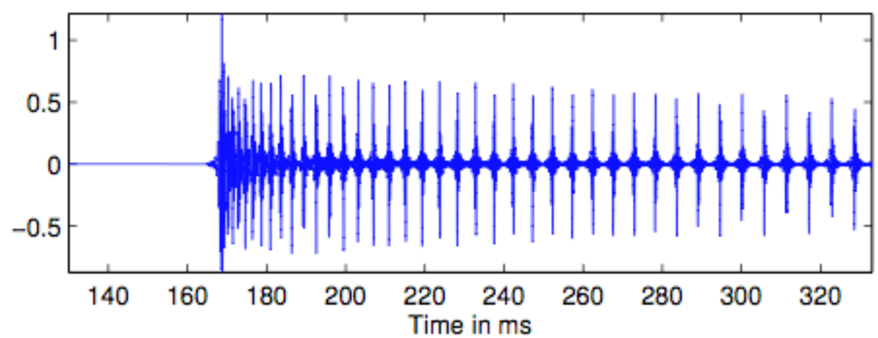
Data model at receiver $\vec{x}_r = (r, z_{\mathcal{A}})$ in unpert. waveguide

$$\widehat{p}(\omega,\vec{x}_r) = \frac{\widehat{f}(\omega)}{2}\Big[\sum_{j=1}^{N(\omega)} \phi_j(x_\star)\phi_j(r)e^{i\beta_j(\omega)z_{\mathcal{A}}} + \underbrace{\sum_{j>N(\omega)} \phi_j(x_\star)\phi_j(r)e^{-\beta_j(\omega)z_{\mathcal{A}}}}_{\text{evanescent}}\Big]$$

with modal wavenumbers

$$\beta_j(\omega) = \begin{cases} \frac{2\pi}{\lambda}\sqrt{1-\left(\frac{j\lambda}{2X}\right)^2}, & j = 1,\dots N(\omega) = \left\lfloor \frac{2X}{\lambda} \right\rfloor \\ \frac{2\pi}{\lambda}\sqrt{\left(\frac{j\lambda}{2X}\right)^2 - 1}, & j > N(\omega). \end{cases}$$

Numerics* $p(t,\vec{x}_r)$



*Setup: $c_o = 1.5$km, pulse bandwidth $1.5 - 4.5$kHz ($\lambda_c = 0.5$m). The waveguide is $20\lambda_c$ deep and $z_{\mathcal{A}} = 494\lambda_c$.

Model in random waveguides

$$\left[\frac{\omega^2}{c_o^2} + \frac{\partial^2}{\partial x^2} + \varepsilon\nu(\vec{x})\frac{\omega^2}{c_o^2}\right]\widehat{p}(\omega,\vec{x}) + \frac{\partial^2 \widehat{p}(\omega,\vec{x})}{\partial z^2} = \widehat{f}(\omega)\delta(x-x_\star)\delta'(z)$$

- For each z we can expand $\widehat{p}(\omega,\vec{x})$ in orthonormal basis $\left\{\phi_j(x)\right\}_{j\geq 1}$

$$\widehat{p}(\omega,\vec{x}) = \sum_{j=1}^{N(\omega)} \phi_j(x)\Big[a_j(\omega,z)e^{i\beta_j(\omega)z} + b_j(\omega,z)e^{-i\beta_j(\omega)z}\Big] + \sum_{j>N(\omega)} \phi_j(x)\widehat{P}_j^e(\omega,z)$$

- Here a_j, b_j and $\widehat{P}_j^e$ satisfy a coupled system* of stochastic ODE's driven by stationary random processes

$$C_{j,l}(z) = \int_0^X dx\, \nu(\vec{x})\phi_j(x)\phi_l(x), \qquad j,l = 1,2,\ldots \qquad \vec{x} = (x,z).$$

*Kohler, Papanicolaou-1977; Garnier, Papanicolaou - 2007

Model in random waveguide

$$\left(\partial_z^2 + \beta_j^2\right)\left(a_j e^{i\beta_j z} + b_j e^{-i\beta_j z}\right) + \varepsilon\left(\frac{\omega}{c_o}\right)^2 \sum_{l=1}^{N} C_{jl}\left(a_l e^{i\beta_l z} + b_l e^{-i\beta_l z}\right)$$
$$+\varepsilon\left(\frac{\omega}{c_o}\right)^2 \sum_{l>N} C_{jl}\widehat{P}_l^{\varepsilon} = 0, \qquad j = 1, \ldots N,$$

$$\left(\partial_z^2 - \beta_j^2\right)\widehat{P}_j^{\varepsilon} + \varepsilon\left(\frac{\omega}{c_o}\right)^2\left[\sum_{l=1}^{N} C_{jl}\left(a_l e^{i\beta_l z} + b_l e^{-i\beta_l z}\right) + \sum_{l>N} C_{jl}\widehat{P}_l^{\varepsilon}\right] = 0.$$

- Boundary cond: $a_j(\omega, z = 0)$ and $\widehat{P}_j^{e}(\omega, z = 0)$ given by source excitation. As $z \to \infty$, the field is outgoing and $\widehat{P}_j^{\varepsilon}(\omega, z) \to 0$.

- To get well posed problem ask*: $(\partial_z a_j)e^{i\beta_j z} + (\partial_z b_j)e^{-i\beta_j z} = 0$.

- Eliminating the evanescent $\widehat{P}_j^{e}(\omega, z) \rightsquigarrow$ closed first order system for $\{a_j(\omega, z), b_j(\omega, z)\}_{j=1,\ldots N(\omega)}$ driven by random $\{C_{jl}(z)\}$.

*Kohler, Papanicolaou-1977; Garnier, Papanicolaou - 2007

Model in random waveguides

• The stochastic ODE system is studied with the asymptotic ($\varepsilon \to 0$) limit tools of Khasminskii, Blakenship, Papanicolaou, Stroock, Varadhan.

∘ For ranges $\ll O(\varepsilon^{-2})$ the fluctuations are negigible.

∘ The fluctuations play a role at ranges $\sim O(\varepsilon^{-2})$

∘ As $\varepsilon \to 0$, negligible coupling between a_j and b_j for smooth z-autocorrelation of fluctuations $\rightsquigarrow$ forward scattering approx*.

$\rightsquigarrow$ Closed first order system of stochastic ODE's for $\{a_j\}_{j=1,\dots N(\omega)}$

*Kohler, Papanicolaou-1977

The random transfer matrix (Green's function) $T^{\varepsilon}_{jl}(\omega, z)$

$$a_j(\omega, z/\varepsilon^2) \approx \sum_{l=1}^{N(\omega)} T^{\varepsilon}_{jl}(\omega, z) a_l(\omega, 0), \qquad a_l(\omega, 0) = \frac{\hat{f}(\omega)}{2}\phi_l(x_\star),$$

where

$$\begin{aligned} \frac{\partial}{\partial z} T^{\varepsilon}(\omega, z) &= \left[\frac{1}{\varepsilon}\mathbb{P}\left(\omega, \frac{z}{\varepsilon^2}\right) + \mathbb{E}\left(\omega, \frac{z}{\varepsilon^2}\right) + \dots\right] T^{\varepsilon}(\omega, z), \qquad z > 0, \\ T^{\varepsilon}(\omega, 0) &= I. \end{aligned}$$

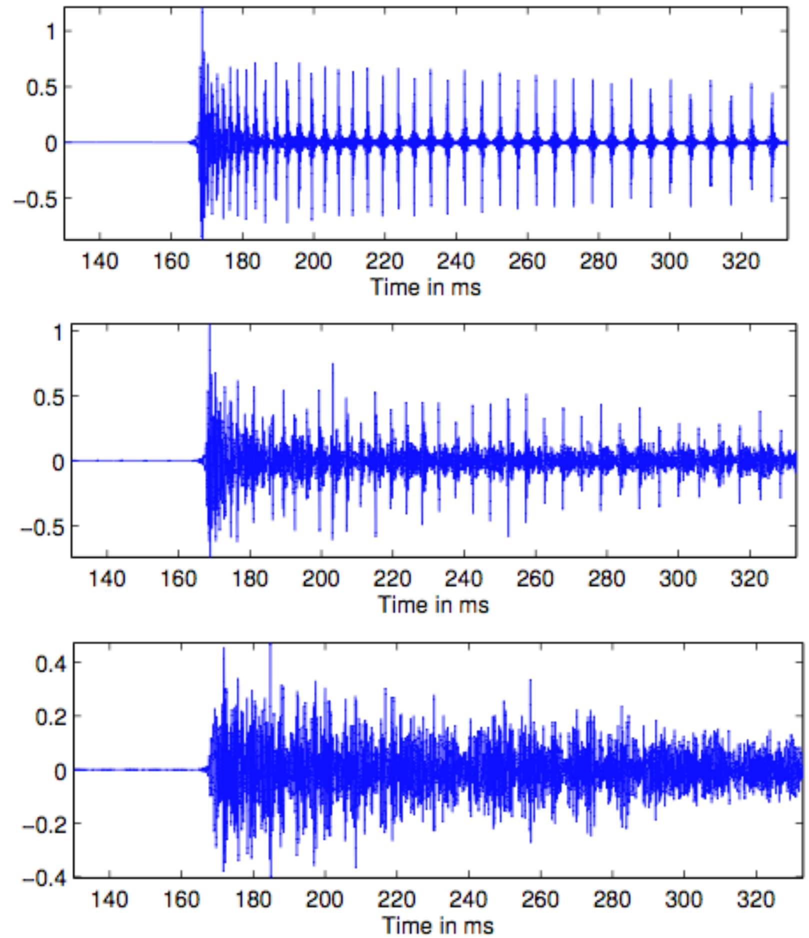
- Leading coupling: $\mathbb{P}_{jl}(\omega, z) = \frac{i}{2}\left(\frac{\omega}{c_o}\right)^2 \frac{C_{jl}(z)}{\beta_j(\omega)} e^{i[\beta_l(\omega) - \beta_j(\omega)]z}$

- The second order coupling is via the evanescent modes

$$\mathbb{E}_{jl}(\omega, z) = \frac{i}{4}\left(\frac{\omega}{c_o}\right)^4 \sum_{l' > N} \int_{-\infty}^{\infty} ds \frac{C_{jl'}(z) C_{ll'}(z+s)}{\beta_{l'}(\omega)\beta_j(\omega)} e^{i\beta_l(\omega)(z+s) - i\beta_j(\omega) z - \beta_{l'}(\omega)|s|}$$

Data model in random waveguides*

$$p\left(t, r, z_{\mathcal{A}} = Z/\varepsilon^2\right) \approx \int \frac{d\omega}{2\pi}\frac{\widehat{f}(\omega)}{2}\sum_{j,l=1}^{N(\omega)} T^{\varepsilon}_{jl}(\omega, Z)\phi_l(x_\star)\phi_j(r)e^{i\beta_j(\omega)z_{\mathcal{A}}-i\omega t}$$



$\varepsilon = 0\%$

$\varepsilon = 1\%$

$\varepsilon = 3\%$

*Speed fluctuates about $c_o = 1.5$km, with correlation length $= \lambda_c = 0.5$m. Pulse bandwidth $1.5 - 4.5$kHz. The waveguide is $20\lambda_c$ deep and $z_{\mathcal{A}} = 494\lambda_c$.

Statistics of the array data

$$p\left(t, r, z_{\mathcal{A}} = Z/\varepsilon^2\right) \approx \int \frac{d\omega}{2\pi}\frac{\widehat{f}(\omega)}{2}\sum_{j,l=1}^{N(\omega)} T^{\varepsilon}_{jl}(\omega, Z)\phi_l(x_\star)\phi_j(r)e^{i\beta_j(\omega)z_{\mathcal{A}}-i\omega t}$$

- As $\varepsilon \to 0$, $T^{\varepsilon}(\omega, z)$ converges in distribution* to a Markov diffusion process with generator computed explicitly in terms of correlation function of fluctuations.

- All statistical moments of $T^{\varepsilon}(\omega, z)$ can be computed approximately for $\varepsilon \ll 1$.

*Kohler, Papanicolaou - 1977.

Step 2: Analyze coherent part of array data $E\left\{p(t,\vec{x}_r)\right\}$.

This is what imaging methods rely on.

The coherent field

$$E\left\{p\left(t,r,z_{\mathcal{A}}\right)\right\} \approx \int \frac{d\omega}{2\pi}\frac{\widehat{f}(\omega)}{2}\sum_{j,l=1}^{N(\omega)} E\left\{T_{jl}^{\varepsilon}(\omega,Z)\right\}\phi_l(x_\star)\phi_j(r)e^{i\beta_j(\omega)z_{\mathcal{A}}-i\omega t}$$

where $z_{\mathcal{A}} = Z/\varepsilon^2$ and $\lim_{\varepsilon\to 0} E\left\{T_{jl}^{\varepsilon}(\omega,Z)\right\} = \delta_{jl}\, e^{-\mathcal{D}_j(\omega)Z+i\mathcal{O}_j(\omega)Z}$.

• $\mathcal{D}_j(\omega) > 0$ (power spectral densities of fluctuations) is due entirely to direct coupling of propagating modes.

• $\mathcal{O}_j(\omega)$ is also caused by coupling via evanescent modes (they carry negligible energy but cause dispersion).

• The coherent field decays exponentially with range.

The mean intensity and frequency decorrelation

- To compute intensity $E\left\{p^2\left(t, r, z_{\mathcal{A}} = Z/\varepsilon^2\right)\right\}$ we need second moments $E\left\{T^{\varepsilon}_{jl}(\omega, Z)\overline{T^{\varepsilon}_{j'l'}}(\omega', Z)\right\}$.

- We have frequency decorrelation for $|\omega - \omega'| \gg O(\varepsilon^2)$

$$E\left\{T^{\varepsilon}_{jl}(\omega, Z)\overline{T^{\varepsilon}_{j'l'}}(\omega', Z)\right\} \approx E\left\{T^{\varepsilon}_{jl}(\omega, Z)\right\} E\left\{\overline{T^{\varepsilon}_{j'l'}}(\omega', Z)\right\}.$$

- For nearby frequencies $\omega' = \omega - \varepsilon^2 h$,

$$\int \frac{dh}{2\pi} E\left\{T^{\varepsilon}_{jl}(\omega, Z)\overline{T^{\varepsilon}_{j'l'}}(\omega - \varepsilon^2 h, Z)\right\} e^{i\left[\beta_j(\omega) - \beta_{j'}(\omega - \varepsilon^2 h)\right] z_{\mathcal{A}} - iht} \approx$$
$$\delta_{jj'}\delta_{ll'}\mathcal{W}^{(l)}_j(\omega, t, Z) + (1 - \delta_{jj'})\delta_{jl}\delta_{j'l'} \text{ exp. decay in } Z.$$

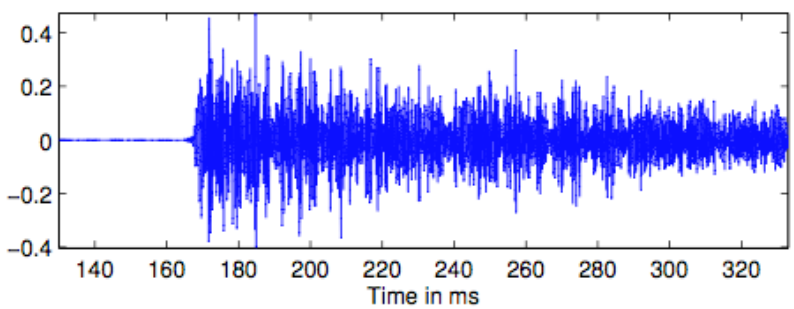
The loss of coherence

• The Wigner transform $\mathcal{W}_j^{(l)}(\omega, t, Z)$ dominates at long ranges and the intensity of the field recorded at $\vec{x}_r = (r, z_{\mathcal{A}} = Z/\varepsilon^2)$ is

$$E\left\{p^2\left(t, r, z_{\mathcal{A}}\right)\right\} \approx \varepsilon^2 \int \frac{d\omega}{2\pi} \frac{|\widehat{f}(\omega)|^2}{4} \sum_{j,l=1}^{N(\omega)} \mathcal{W}_j^{(l)}(\omega, t, Z) \phi_l^2(x_\star) \phi_j^2(r)$$

• In spite of the ε^2 factor, $E\{p^2\} \gg |E\{p\}|^2$ at long ranges, because the latter decays exponentially.

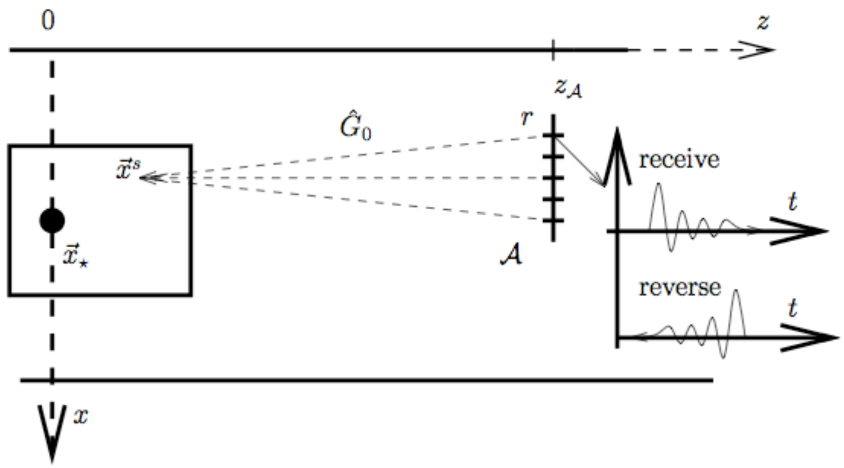
The incoherent field $p - E\{p\}$ becomes dominant at long ranges.



$\varepsilon = 3\%$

Step 3: Analyze how typical imaging methods fail.

Source localization using "time reversal"



$$\mathcal{I}^{\mathrm{TR}}(\vec{x}^s) = \int d\omega \int_{\mathcal{A}} dr\, \overline{\widehat{p}}(\omega, \vec{x}_r) \underbrace{\widehat{G}_o(\omega, \vec{x}_r; \vec{x}^s)}_{\text{Green's function}}$$

• We evaluate the imaging function at points $\vec{x}^s = (x^s, z^s)$ in a search domain and estimate $\vec{x}_\star$ as the peak of $\mathcal{I}^{\mathrm{TR}}$.

Expected to focus at $\vec{x}_\star$ by time reversibility of the wave equation, at least for large enough apertures and if $\widehat{G}_o$ is a good enough approximation of the backpropagation in the real medium.

Matched Field (MF) methods

$$\mathcal{I}^{\mathrm{MF}}(\vec{x}^s) = \int d\omega \left| \int_{\mathcal{A}} dr\, \overline{\widehat{p}}(\omega, \vec{x}_r) \widehat{G}_0(\omega, \vec{x}_r; \vec{x}^s) \right|^2$$

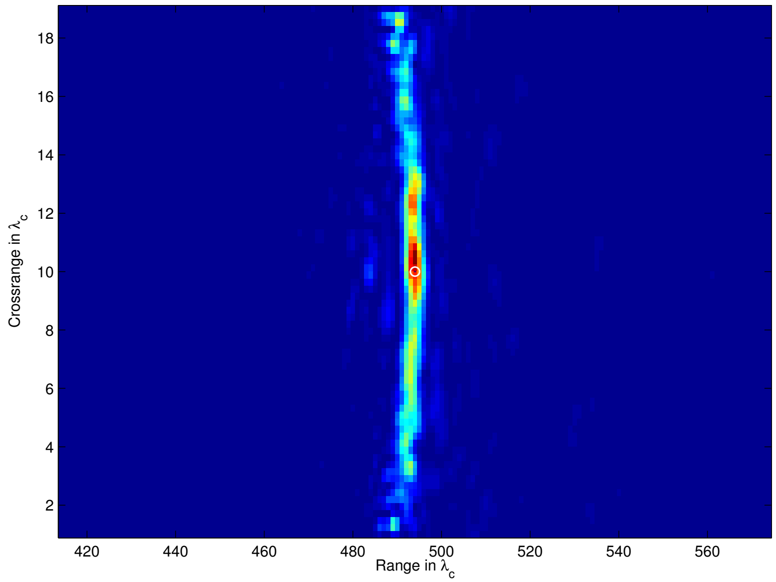
• This is the conventional (Bartlett) MF function. It is known to be more robust than the previous method.

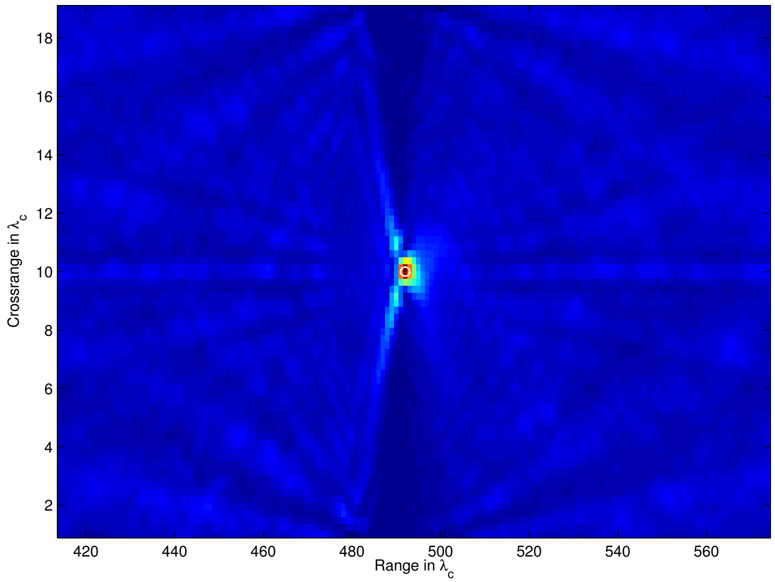
• Variants of MF that use additional data filtering techniques are widely used and have slightly better performance in practice.

- They deal well with additive noise, but rely on coherent data.

$\rightsquigarrow$ sooner or later they will fail similarly at long ranges.

Illustration in media with negligible fluctuations

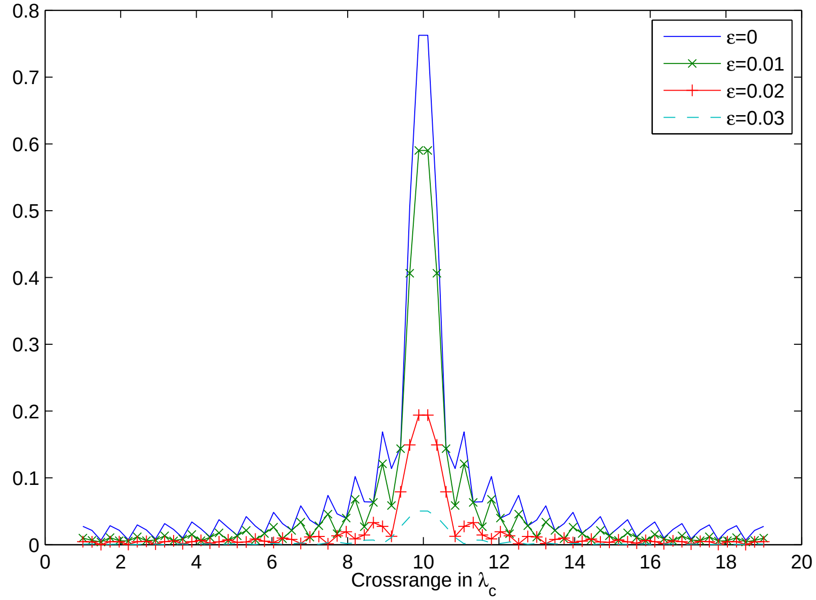




- The imaging functions are computed at 70% aperture and frequency band 2 ± 0.375kHz. The source is in the center.

Next: Let us see what happens when the fluctuations play a role.

Mean of $\mathcal{I}^{\mathsf{TR}}(\vec{x}^s)$ focuses but method statistically unstable

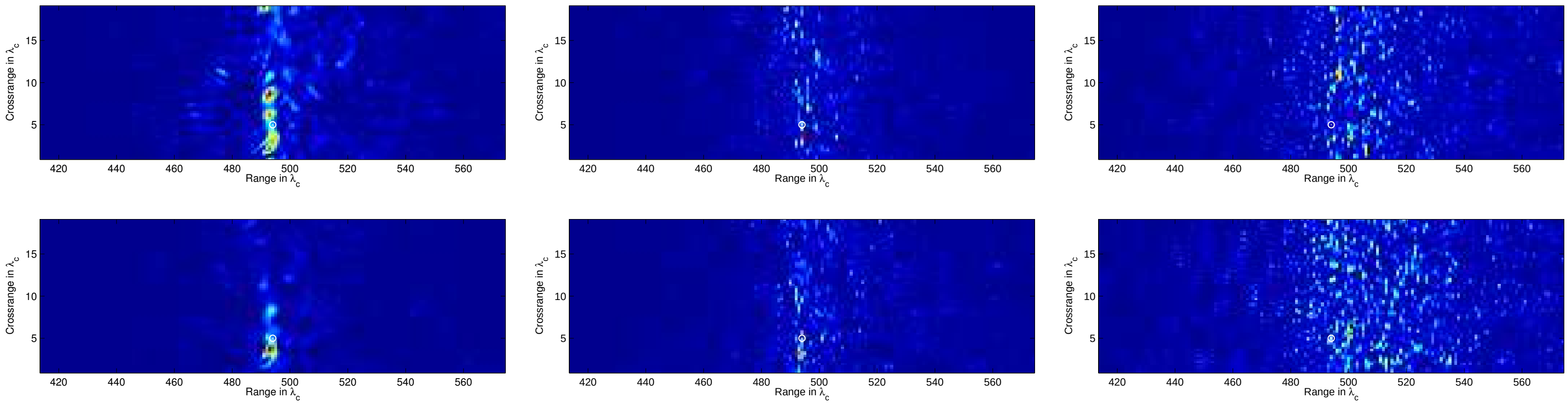


$$\left|E\left\{\mathcal{I}^{\mathsf{TR}}(\vec{x}_\star)\right\}\right| \le C e^{-\mathcal{D}_1(\omega_c)Z}.$$

The relative standard deviation* grows exponentially with range.

$$\frac{\sqrt{E\left\{\left|\mathcal{I}^{\mathsf{TR}}(\vec{x}_\star)\right|^2\right\} - \left|E\left\{\mathcal{I}^{\mathsf{TR}}(\vec{x}_\star)\right\}\right|^2}}{\left|E\left\{\mathcal{I}^{\mathsf{TR}}(\vec{x}_\star)\right\}\right|} \ge \frac{\varepsilon\sqrt{\omega_c/B}}{\sqrt{N(\omega_c)}} e^{\mathcal{D}_1(\omega_c)Z} \underbrace{\mathcal{F}(\omega_c, Z, x_\star)}_{\text{algebraic in} Z}$$

*The frequency band is $|\omega - \omega_c| \le B$.



Full aperture. Left: $\varepsilon = 2\%$, bandwidth: 2 ± 0.375kHz. Middle: $\varepsilon = 2\%$ and full bandwidth. Right: $\varepsilon = 3\%$ and full bandwidth.

• Even though the statistical mean focuses in theory, we cannot observe it due to the statistical instability.

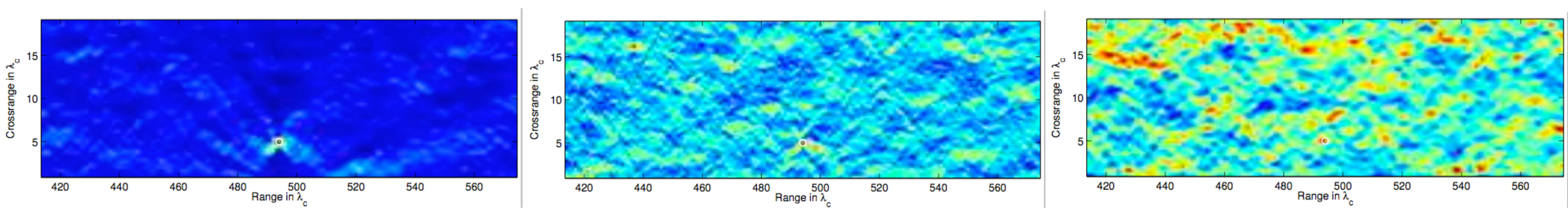
Matched Field

$$E\{\mathcal{I}^{\text{MF}}(\vec{x}^s)\} = \int d\omega\, E\left\{\left|\int_{\mathcal{A}} dr\, \overline{\widehat{p}}(\omega,\vec{x}_r)\widehat{G}_0(\omega,\vec{x}_r;\vec{x}^s)\right|^2\right\}$$

- Using the data model and the second moment formula,

$$E\left\{\overline{\widehat{p}(\omega,\vec{x}_r)}\widehat{p}(\omega,\vec{x}_{r'}\right\} \approx \frac{|\widehat{f}(\omega)|^2}{4}\sum_{j,l=1}^{N(\omega)} \phi_l^2(x_\star)\phi_j(r)\phi_j(r')\int dt\,\mathcal{W}_j^{(l)}(\omega,t,Z).$$

- It is difficult to estimate the source range $z_{\mathcal{A}} = Z/\varepsilon^2$ from $\int dt\,\mathcal{W}_j^{(l)}(\omega,t,Z)$ $\rightsquigarrow$ **MF will not focus.**



The time integral of the mean Wigner transform

$$\int dt\, \mathcal{W}_j^{(l)}(\omega, t, Z) \approx \frac{\beta_l(\omega)}{\beta_j(\omega)} \left\{ e^{\Gamma(\omega) Z} \right\}_{jl}$$

- Here $\Gamma(\omega)$ = negative semidefinite matrix

$$\Gamma_{jj} = -\sum_{j \neq l} \Gamma_{jl}, \quad \Gamma_{jl} = \frac{\omega^4 / c_o^4}{4 \beta_j \beta_l} \int_{-\infty}^{\infty} \cos\left[\left(\beta_j - \beta_l\right) z\right] E\left\{ C_{jl}(0) C_{jl}(z) \right\} dz$$

- As Z grows columns of $e^{\Gamma(\omega) Z} \to \text{span}\{(1, \ldots 1)^T\} = \text{null}[\Gamma(\omega)]$

$$\left| \left\{ e^{\Gamma(\omega) Z} \right\}_{jl} - \frac{1}{N(\omega)} \right| \leq O\left(e^{-Z/L_{eq}} \right), \quad -1/L_{eq} = \text{2-nd eigenval of } \Gamma.$$

Step 4: Imaging at long ranges, where data is incoherent.

Frequency correlation should be exploited for imaging

- Consider

$$\mathcal{F}(\omega,t,r,r') = \int \frac{dh}{2\pi}\,\widehat{p}(\omega,\vec{x}_r)\overline{\widehat{p}(\omega-\varepsilon^2 h,\vec{x}_{r'})}e^{-iht}, \qquad r,r' \in \mathcal{A}.$$

- Due to frequency decorrelation it self-averages over bandwidth

$$\int_{|\omega-\omega_c|\le B} d\omega \mathcal{F}(\omega,t,r,r') \approx \int d\omega \int \frac{dh}{2\pi} E\left\{\widehat{p}(\omega,\vec{x}_r)\overline{\widehat{p}(\omega-\varepsilon^2 h,\vec{x}'_r)}\right\} e^{-iht}$$
$$\sim \|f\|^2 \sum_{j,l=1}^{N(\omega_c)} \phi_l^2(x_\star)\phi_j(r)\phi_j(r')\mathcal{W}_j^{(l)}(\omega_c,t,Z)$$

- Here we assumed a bandwidth $O(\varepsilon^2) \ll B \ll O(1)$.

• We have $\mathcal{W}_j^{(l)}(\omega,t,Z) = \frac{\beta_l(\omega)}{\beta_j(\omega)} W_j^{(l)}(\omega,t,Z)$ where

$$\left[\partial_Z + \beta_j'(\omega)\partial_t\right] W_j^{(l)}(\omega,t,Z) = \sum_{n\neq j} \Gamma_{jn}(\omega)\left[W_n^{(l)}(\omega,t,Z) - W_j^{(l)}(\omega,t,Z)\right]$$

for $Z > 0$ with initial condition

$$W_j^{(l)}(\omega,t,Z=0) = \delta(t)\delta_{jl}.$$

• The source range $z_{\mathcal{A}} = Z/\varepsilon^2$ is encoded in the t peak of $\mathcal{W}_j^{(l)}(\omega,t,Z)$, i.e. in the cross-correlations $\int d\omega\, \mathcal{F}(\omega,t,r,r')$.

• We must estimate the transport speed. It differs from $\beta_j'(\omega)$.

Range estimation

- Given $p(t, \vec{x}_r)$ at the receivers, compute the cross-correlations

$$\int_{|\omega-\omega_c|\leq B} d\omega \mathcal{F}(\omega, t, r, r') = \int_{|\omega-\omega_c|\leq B} d\omega \int \frac{dh}{2\pi} \widehat{p}(\omega, \vec{x}_r) \overline{\widehat{p}(\omega - \varepsilon^2 h, \vec{x}'_r)} e^{-iht}$$

- Now project on the modes and backpropagate approximately

$$\mathcal{R}(\zeta, j) = \int_{\mathcal{A}} dr \phi_j(r) \int_{\mathcal{A}} dr' \phi_j(r') \int_{|\omega-\omega_c|\leq B} d\omega \, \mathcal{F}(\omega, t = \beta'_j(\omega_c)\zeta, r, r')$$

- This peaks at $\zeta = \zeta_j \neq Z$!

- We estimate the range Z by comparing $\mathcal{R}(\zeta, j)$ with its theoretical model $\mathcal{R}^M(\zeta, j; Z^s)$, at source search range Z^s.

Range estimation

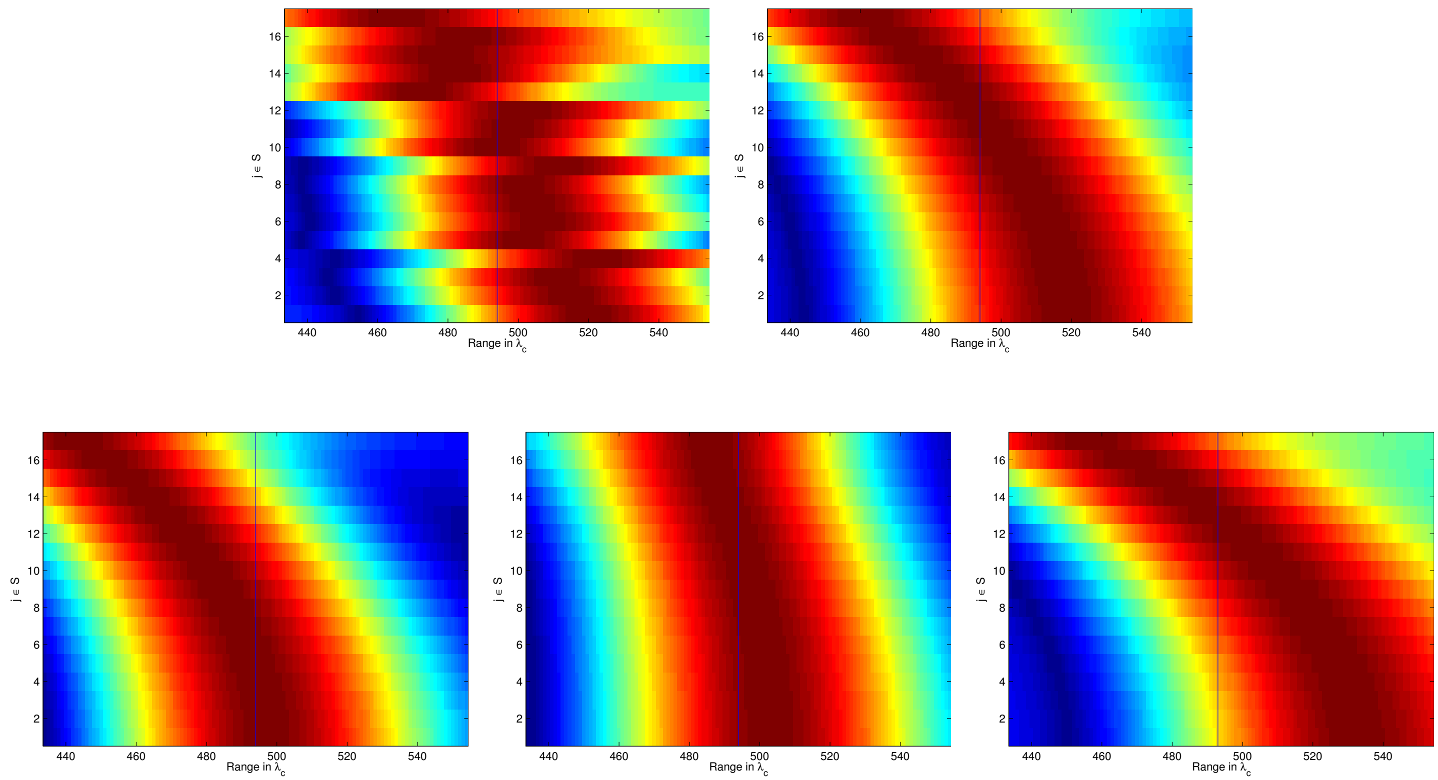
- Estimate Z by minimizing over Z^s

$$\mathbb{O}(Z^s) = \sum_{j\in\mathcal{S}} \int d\zeta \left| \frac{\mathcal{R}(\zeta,j)}{\max_{\zeta'} \mathcal{R}(\zeta',j)} - \frac{\mathcal{R}^M(\zeta,j;Z^s)}{\max_{\zeta'} \mathcal{R}^M(\zeta',j;Z^s)} \right|^2 .$$

- Computing $\mathcal{R}^M(\zeta',j;Z^s)$ requires correlation function of the fluctuations. If we don't know it $\rightsquigarrow$ estimate it using a model

\- We have used $E\left\{\nu(\vec{x})\nu(\vec{x}')\right\} = \sigma^s \mathcal{R}\left(\frac{\vec{x}-\vec{x}'}{\ell^s}\right)$.

\- We found that the range estimation is surprisingly robust with respect to the uncertainty in the above model.

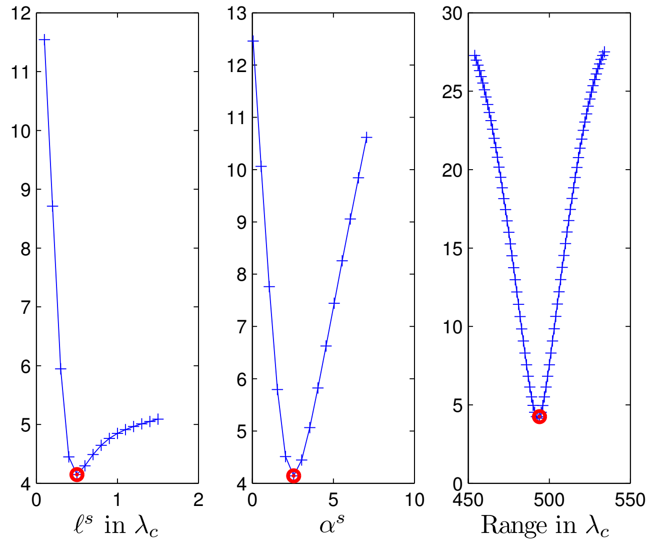


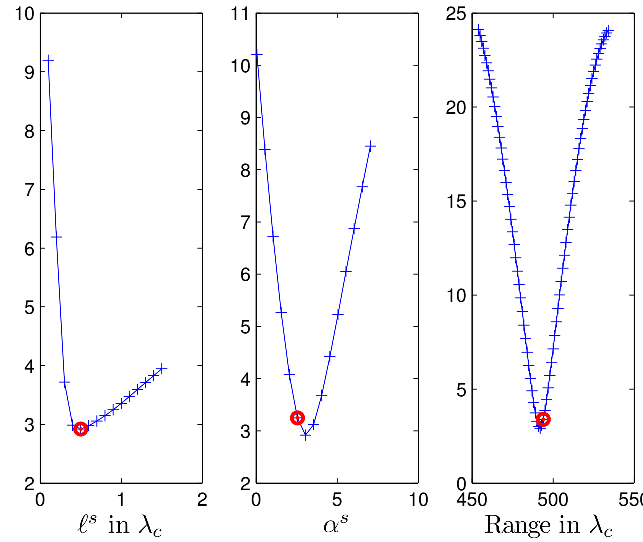
$\varepsilon = 3\%$, central frequency 2.09kHz and bandwidth 0.375kHz.

Top row: Left: $\mathcal{R}(\zeta, j)$. **Right:** $\mathcal{R}^M(\zeta, j; Z^\star, \sigma^\star, \ell^\star)$.

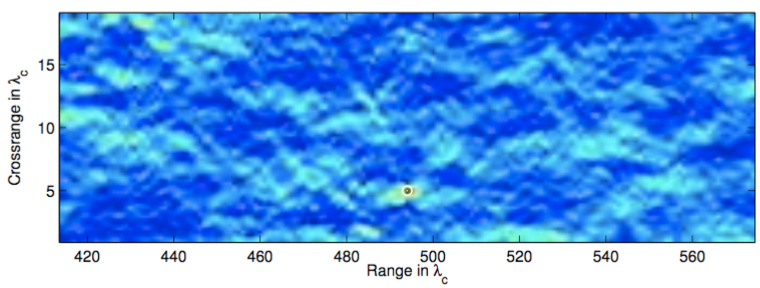
Bottom row: $\mathcal{R}^M(\zeta, j; Z^s, \sigma^s, \ell^s)$ for: **Left:** $\frac{Z^s - Z^\star}{\varepsilon^2} = -20\lambda_c$. **Middle:** $\ell^s = \ell^\star/2$. **Right:** $\sigma^s = 1.34\sigma^\star$.

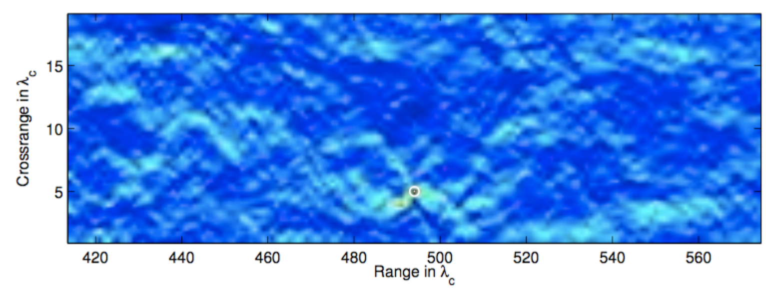
Estimation results. Full aperture, $\varepsilon = 2\%$, central frequency 2.69kHz and bandwidth 0.375kHz



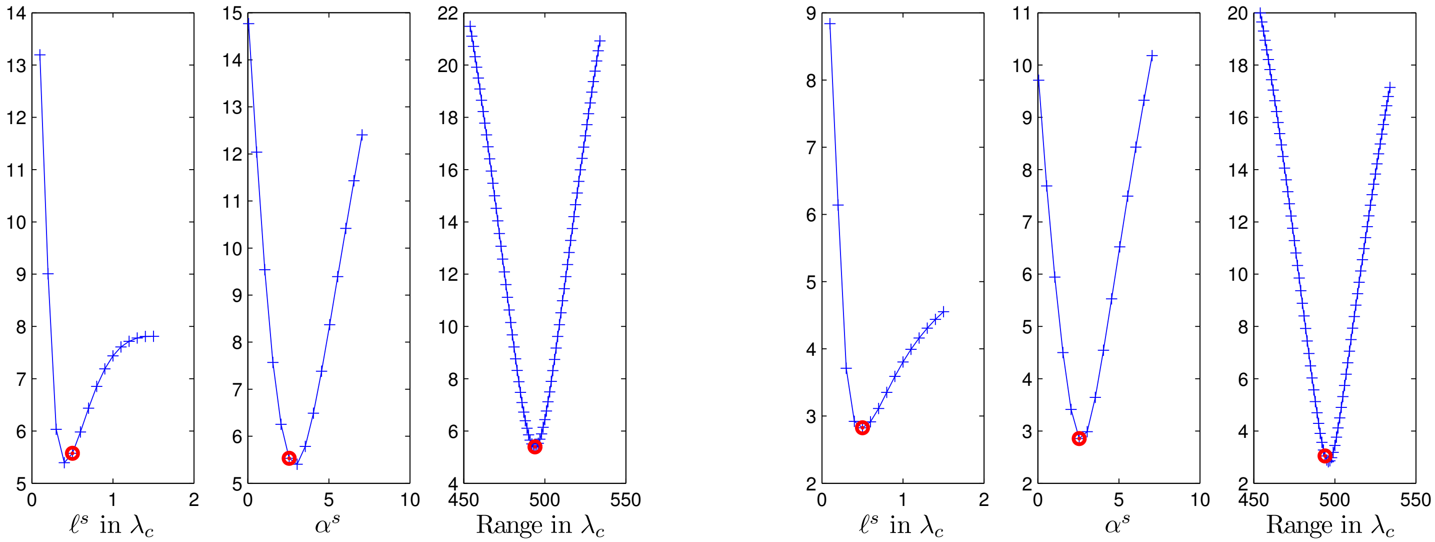


MF

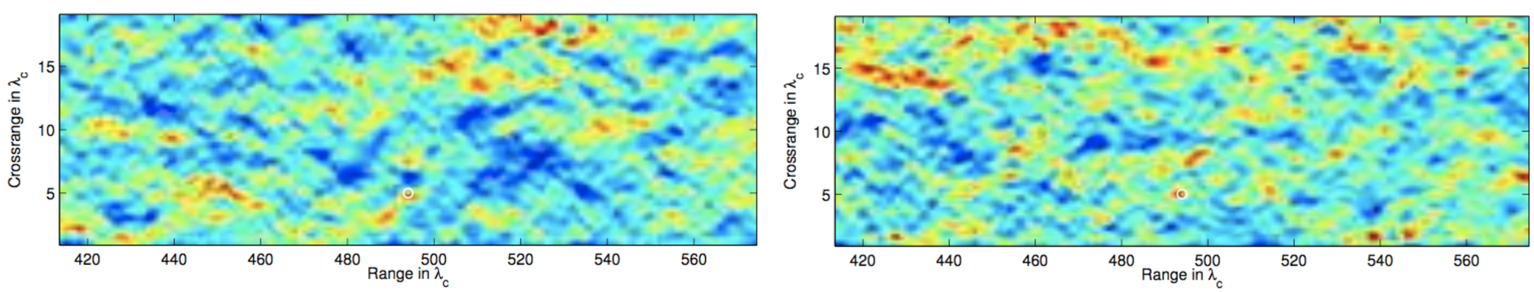




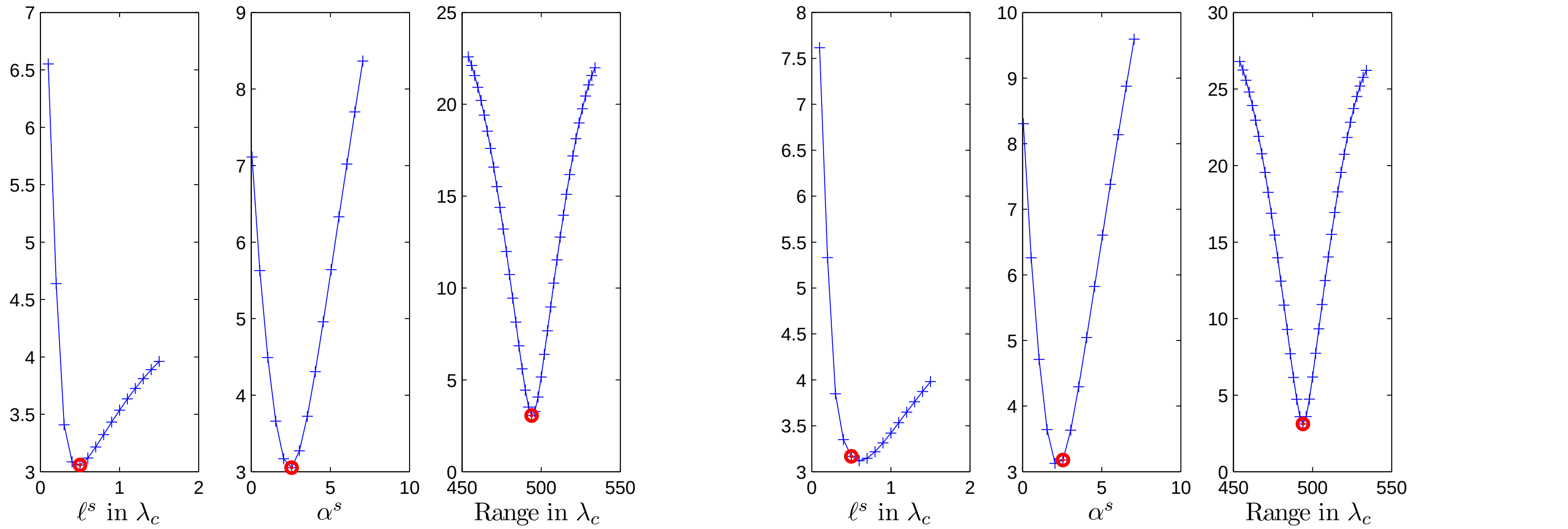
Estimation results. Full aperture, $\varepsilon = 3\%$, central frequency 2.09kHz and bandwidth 0.375kHz



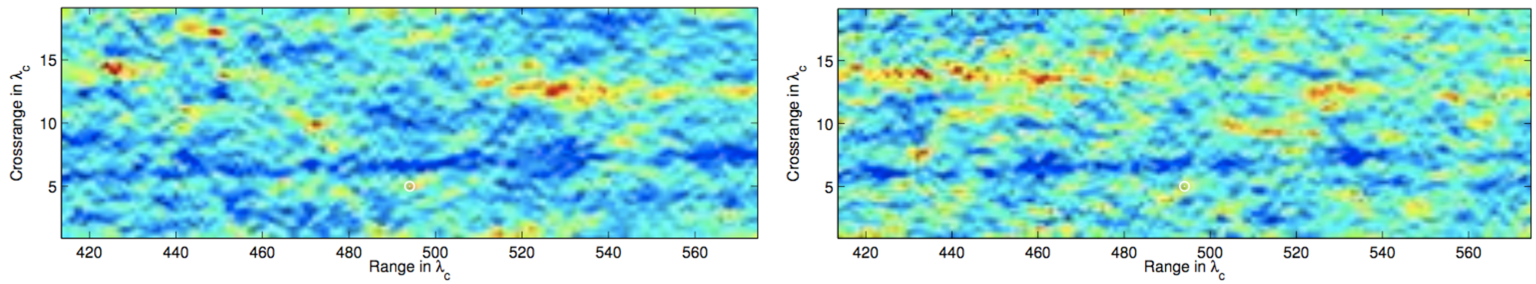
MF



Estimation results. 40% aperture, $\varepsilon = 2\%$, central frequency 2.09kHz and bandwidth 0.375kHz



MF



- We compare

$$\mathcal{X}(j) = \int \frac{d\omega}{2\pi} \widehat{P}_j(\omega, z_{\mathcal{A}}) \overline{\widehat{P}_j}(\omega, z_{\mathcal{A}}), \quad \widehat{P}_j(\omega, z_{\mathcal{A}}) = \int_{\mathcal{A}} dr\, \widehat{p}(\omega, r, z_{\mathcal{A}}) \phi_j(r)$$

with its model

$$\mathcal{X}^M(j; x^s) \sim \|f_B\|^2 \sum_{q,l=1}^{N(\omega_c)} \mathcal{M}_{jq}^2 \frac{\beta_l(\omega_c)}{\beta_q(\omega_c)} \phi_l^2(x^s) \left\{ e^{\Gamma^{(c)}(\omega_c) Z^\star} \right\}_{ql}$$

for a source at $(x^s, Z^\star)$.

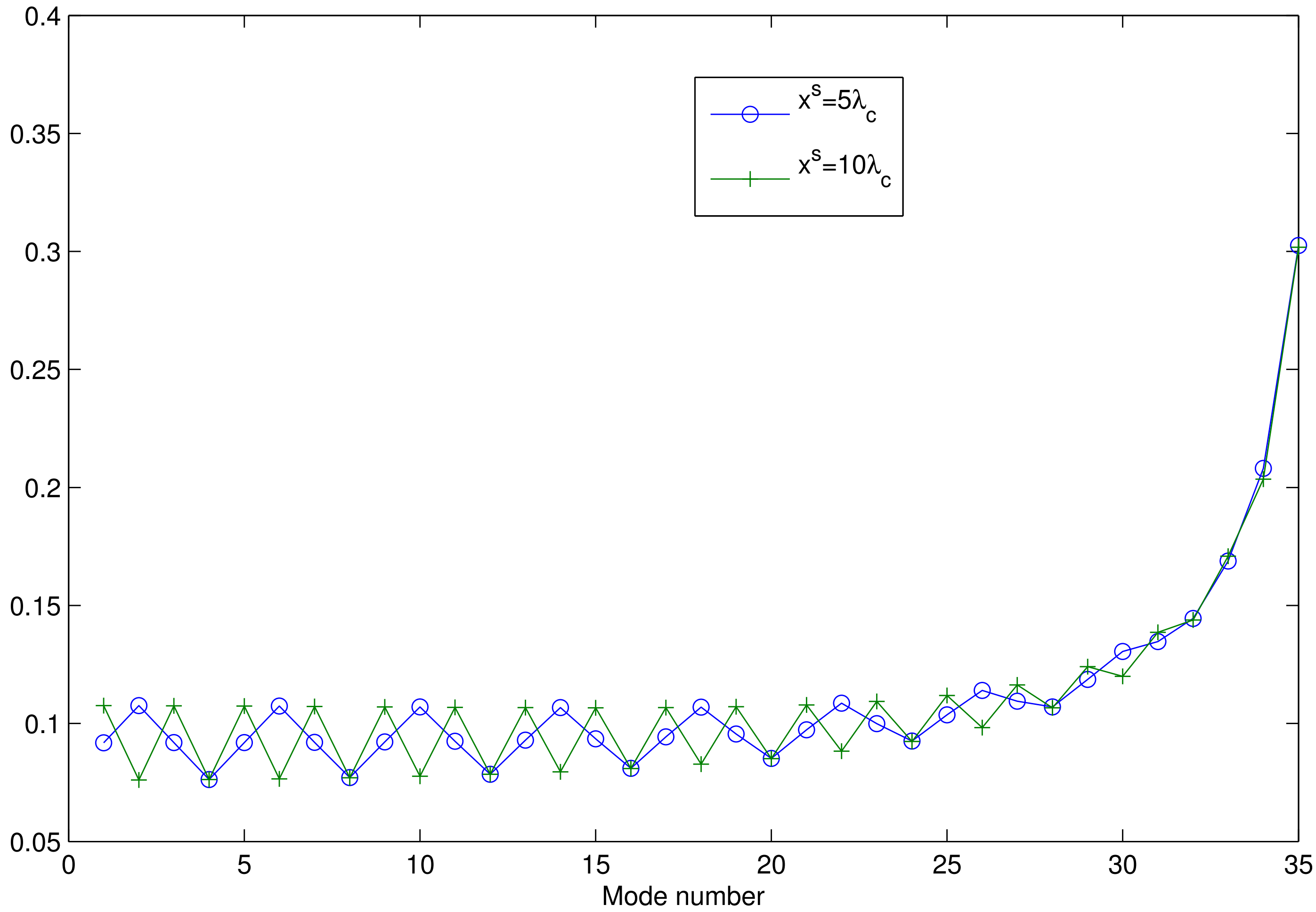
- We estimate the source cross-range by minimizing the misfit.

$$\mathbb{O}(x^s) = \sum_{j \in \mathcal{S}} \left| \frac{\mathcal{X}(j)}{< \mathcal{X}(\cdot) >} - \frac{\mathcal{X}^M(j; x^s)}{< \mathcal{X}^M(\cdot, x^s) >} \right|^2,$$

where

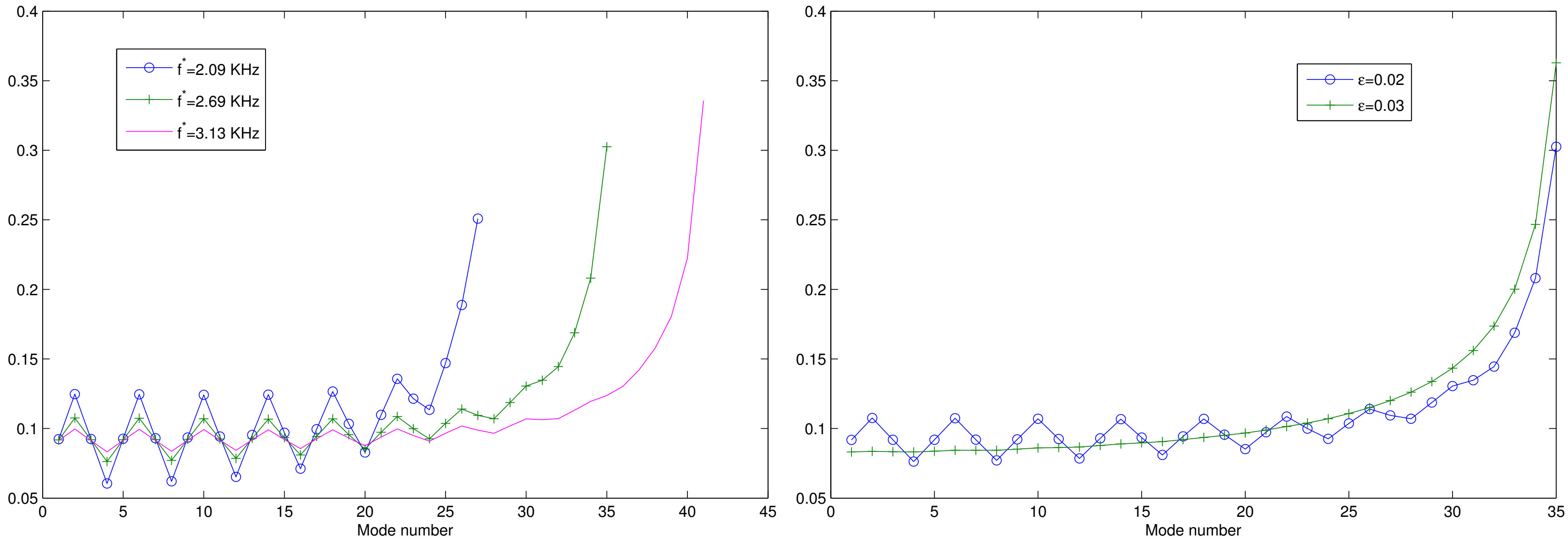
$$< \mathcal{X}(\cdot) >= \frac{1}{|\mathcal{S}|} \sum_{j \in \mathcal{S}} \mathcal{X}(j), \qquad < \mathcal{X}^M(\cdot; x^s) >= \frac{1}{|\mathcal{S}|} \sum_{j \in \mathcal{S}} \mathcal{X}^M(j; x^s)$$

Explanation



$\mathcal{X}^M(j; x^s)$ for $x^s = 5\lambda_c$ and $10\lambda_c$, for $\varepsilon = 2\%$, $\omega_c/(2\pi) = 2.69$kHz and 0.375kHz bandwidth.

Explanation

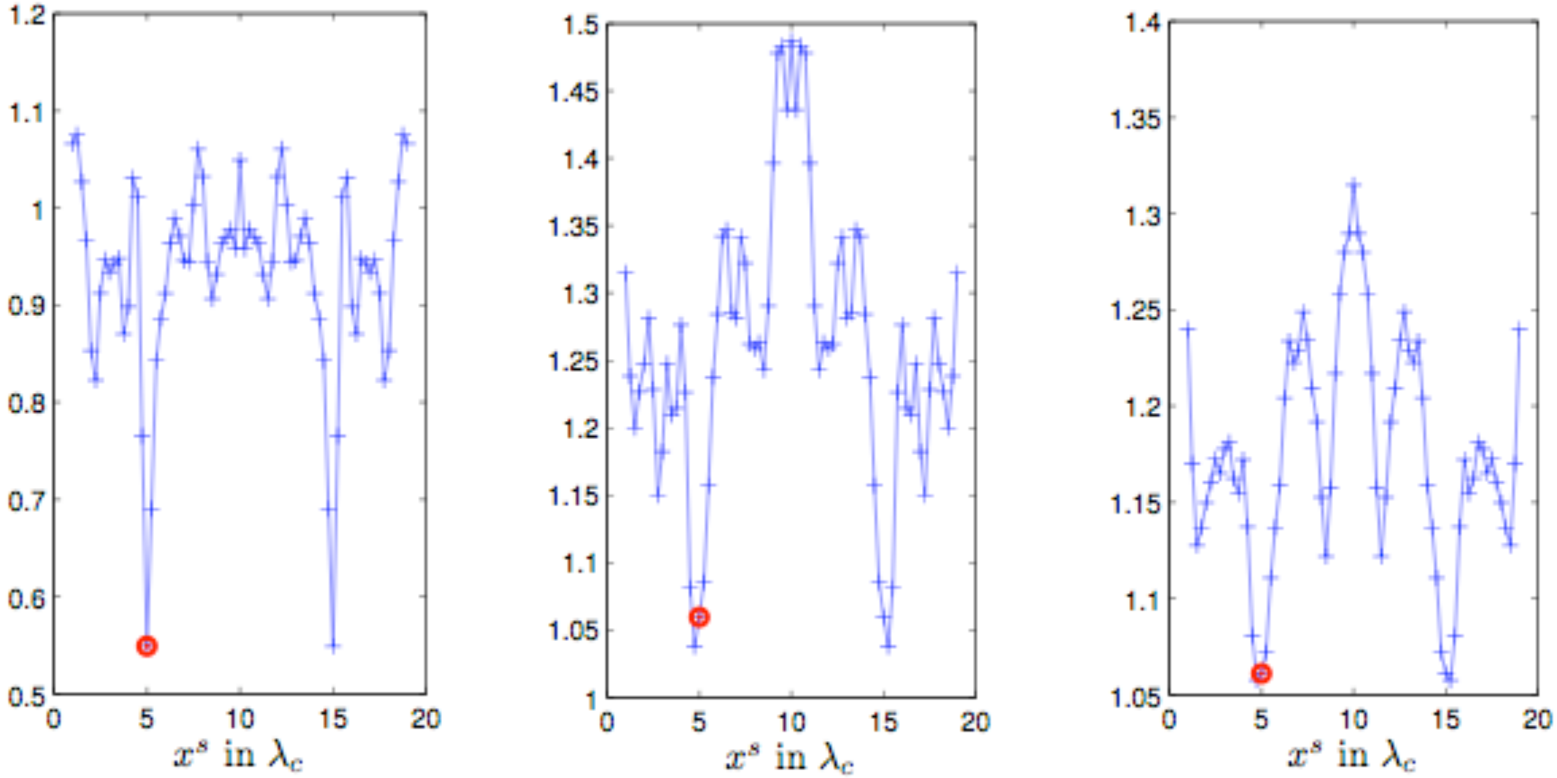


$\mathcal{X}^M(j; x^s)$ for $x^s = 5\lambda_c$ and full aperture.

Left: $\varepsilon = 2\%$, for $\omega_c/(2\pi) = 2.09$kHz, 2.69kHz and 3.13kHz, respectively.

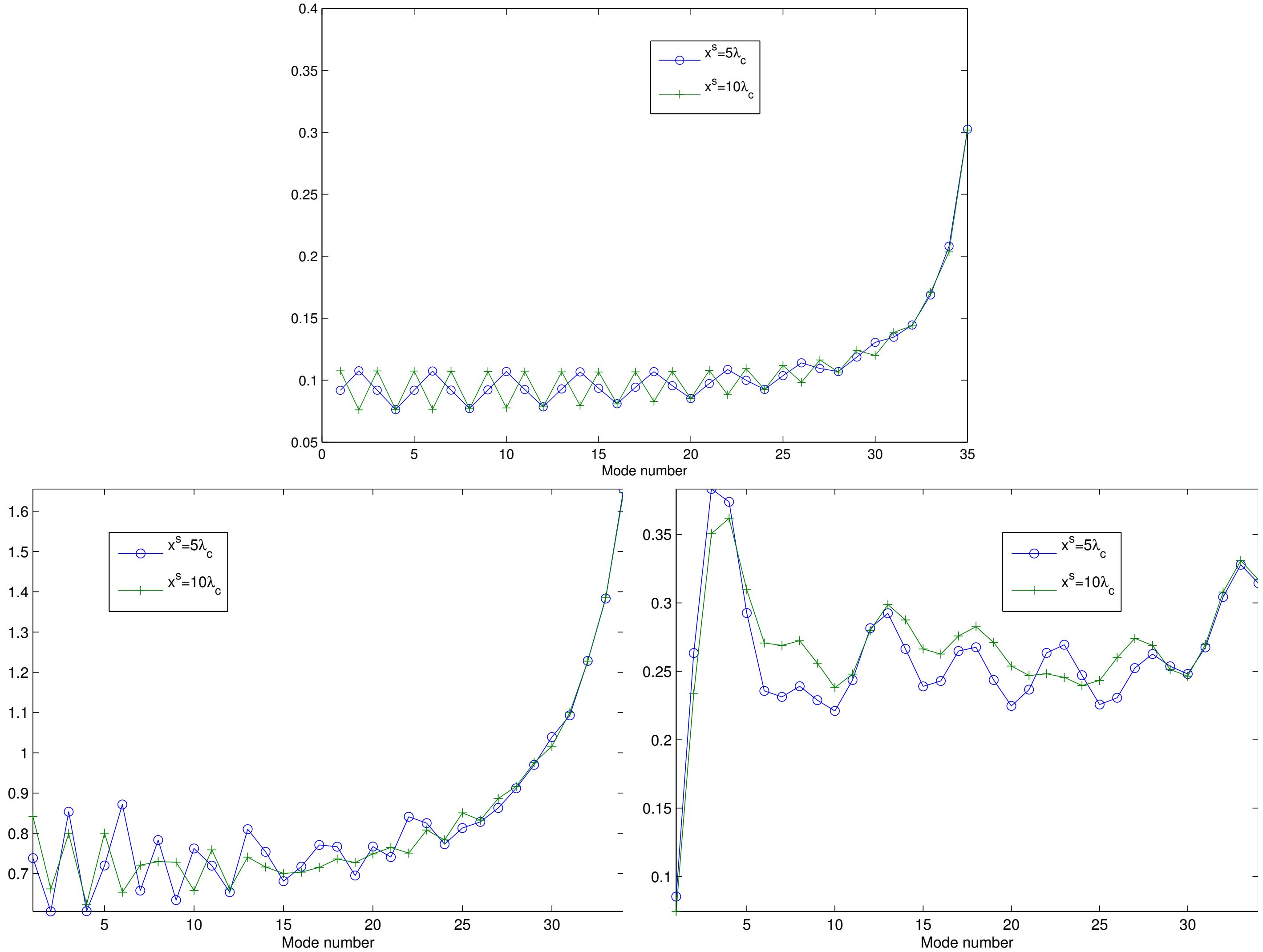
Right: $\omega_c/(2\pi) = 2.69$kHz and $\varepsilon = 2\%$ and 3%. The bandwidth is 0.375kHz. At 3% there is no cross-range information.

Numerical cross-range estimation at full aperture



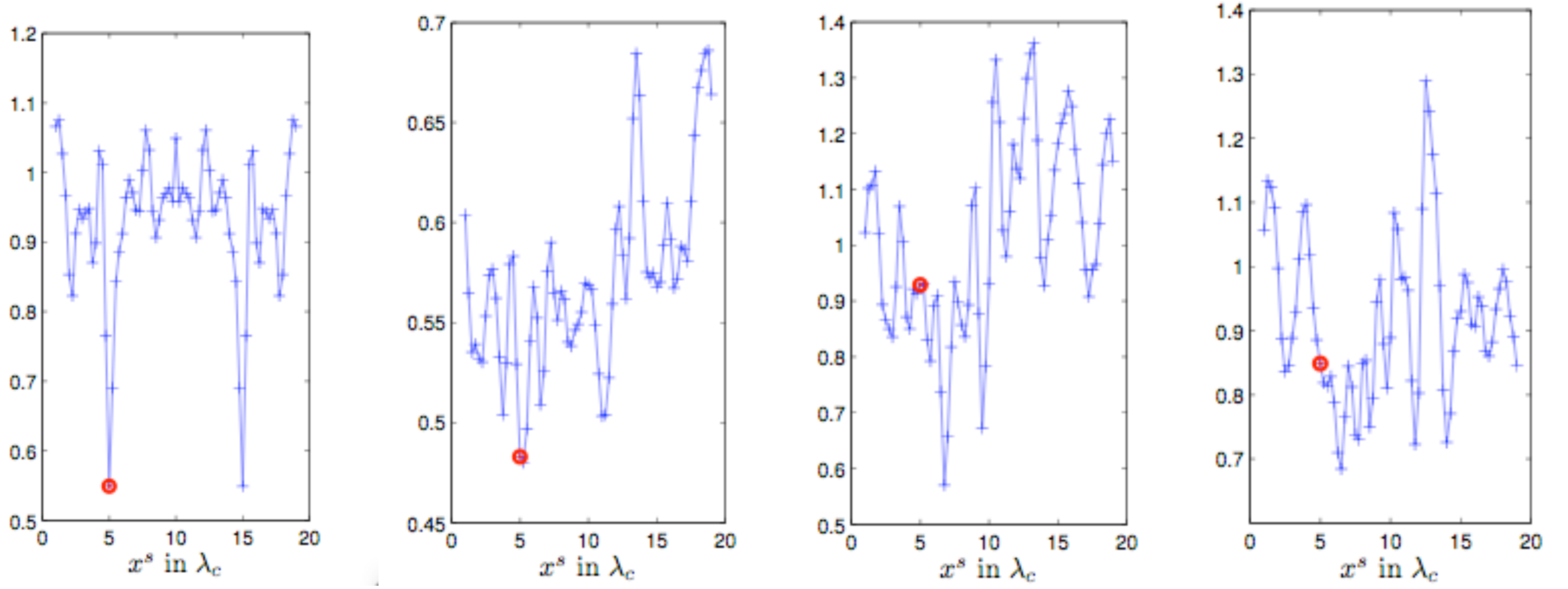
Full aperture cross-range estimation results at $\epsilon = 2\%$, and bandwidth 0.375kHz. **Left**: central frequency 2.69kHz, **middle**: 2.99kHz and **right**: 3.1.3kHz

Partial aperture effects



$\mathcal{X}^M(j; x^s)$ for $x^s = 5\lambda_c$ and $10\lambda_c$, for $\varepsilon = 2\%$, $\omega_c/(2\pi) = 2.69$kHz and 0.375kHz bandwidth. **Top:** full aperture $\mathcal{A} = [0, 20\lambda_c]$. **Bottom:** $\mathcal{A} = [0, 12\lambda_c]$ and $\mathcal{A} = [0, 4\lambda_c]$.

Cross-range estimation results. Partial aperture



$\varepsilon = 2\%$, $\omega_c/(2\pi) = 2.69$kHz and bandwidth 0.375KHz.
From left: full aperture $\mathcal{A} = [0, 20\lambda_c]$, $\mathcal{A} = [0, 12\lambda_c]$, $\mathcal{A} = [0, 8\lambda_c]$, $\mathcal{A} = [0, 4\lambda_c]$.