Coupling network models with porous media equations

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in collaboration with

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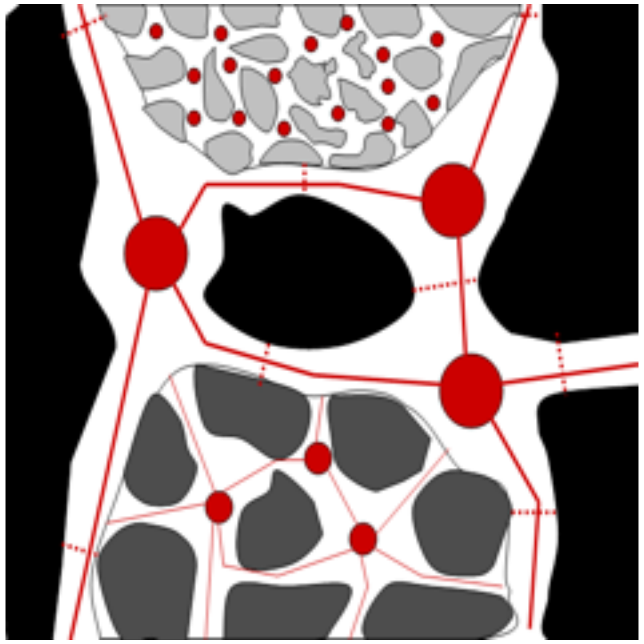
Objectives

- Effective solution of a conservation law

$$\nabla \cdot F(P, \nabla P, x) = h(x), \quad x_L < x < x_R$$
$$F(x_L) = P_L, F(x_R) = P_R$$

$$\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla g(S) = h_w$$

- Upscaling very large (nonlinear) network models



Conservation law at continuum

$$\nabla \cdot F(P, \nabla P, x) = h(x) \qquad F(P, 0, x) \equiv 0$$

E.g. linear flux: $F = -k(x)\nabla P$

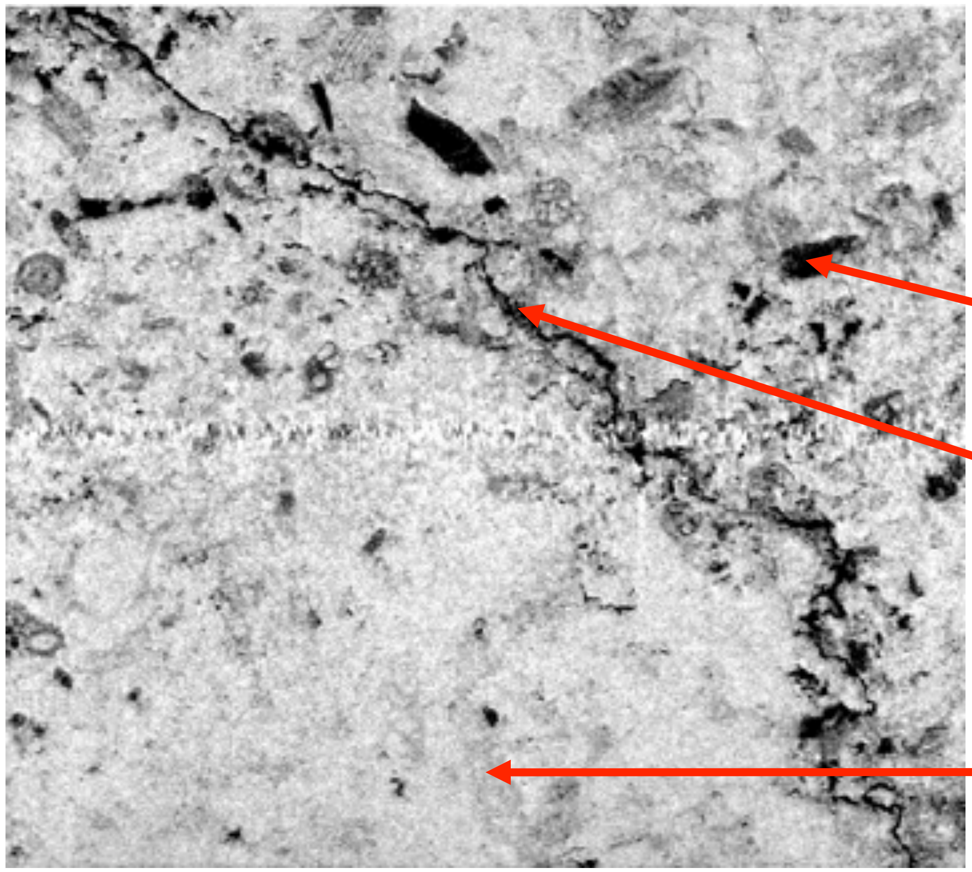
$$k(x, \frac{x}{\epsilon}; \omega) \qquad k_{\text{hom}}(x)$$ effective flux

numerical homogenization:

multiscale finite element methods, dual-porocity method

Effective flux of a pore scale network model.

Challenges from heterogeneous media

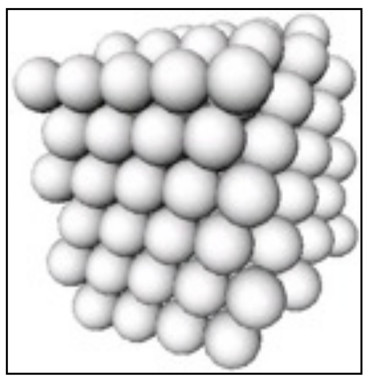


Many such media are of economic interest (oil and gas), but heterogeneity makes predictions difficult.

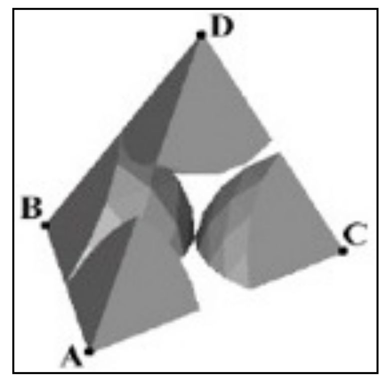
Naturally fractured carbonate, dx =3.1*μm*
Image courtesy of M. Knackstedt & R. Sok, Australian Nat'l Univ.

Obtaining network models

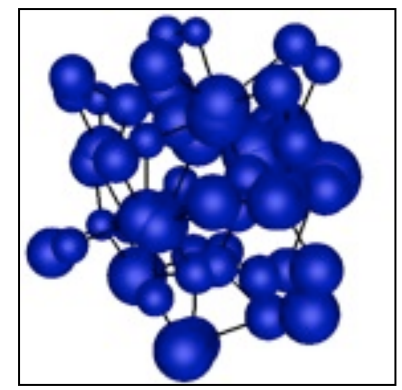
Model/granular media



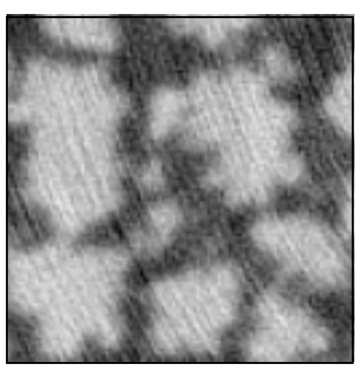
Delaunay tesselation



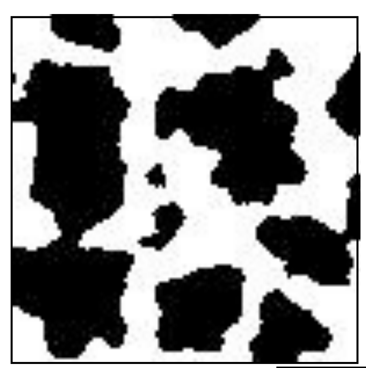
Merging?



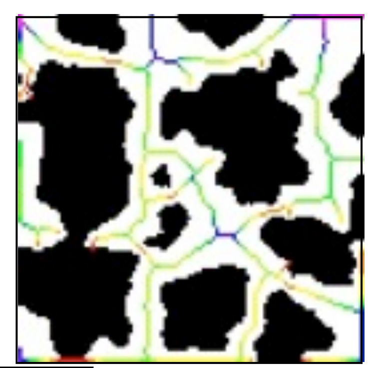
Imaged / real media



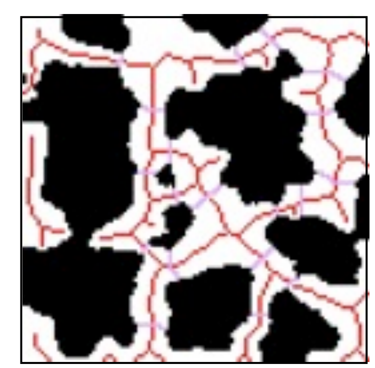
Segmentation

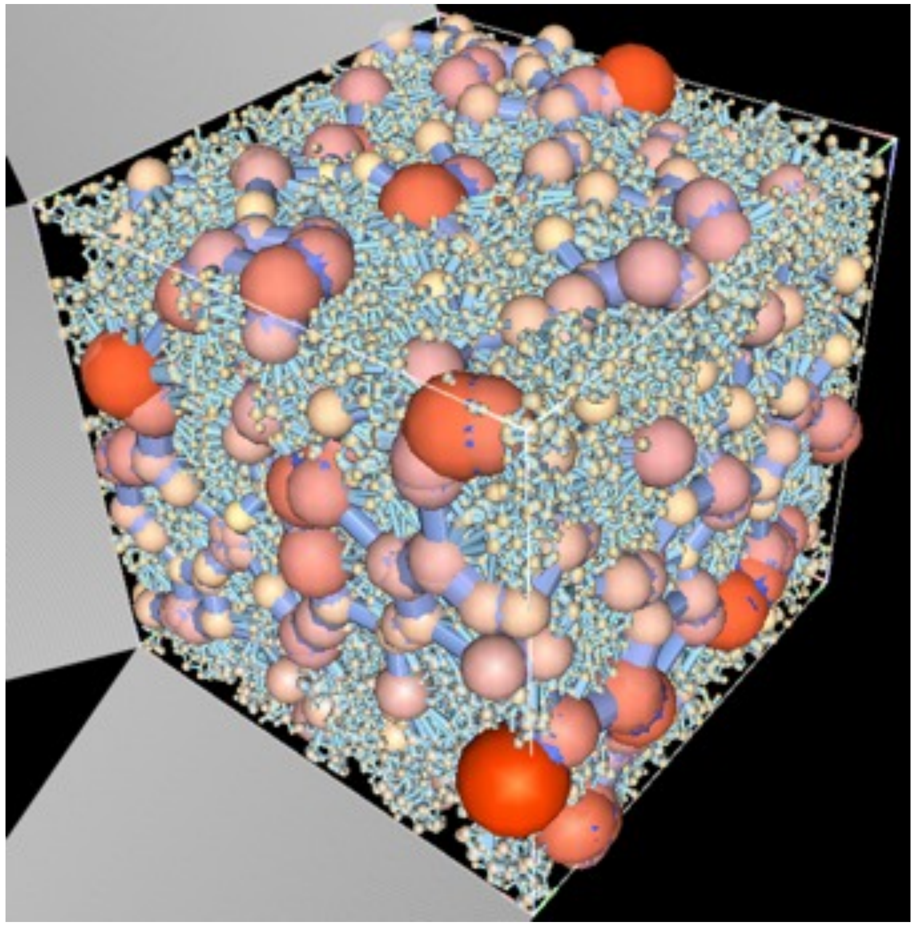


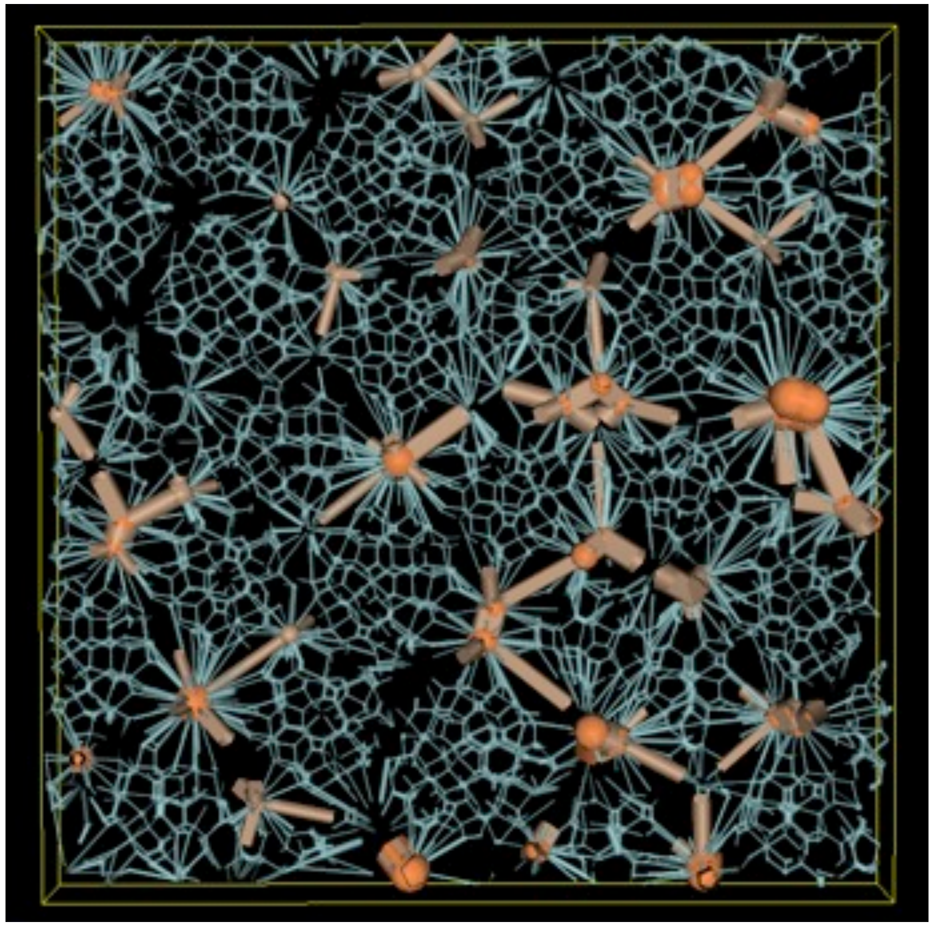
Medial axis extraction



Throat finding + pore partitioning







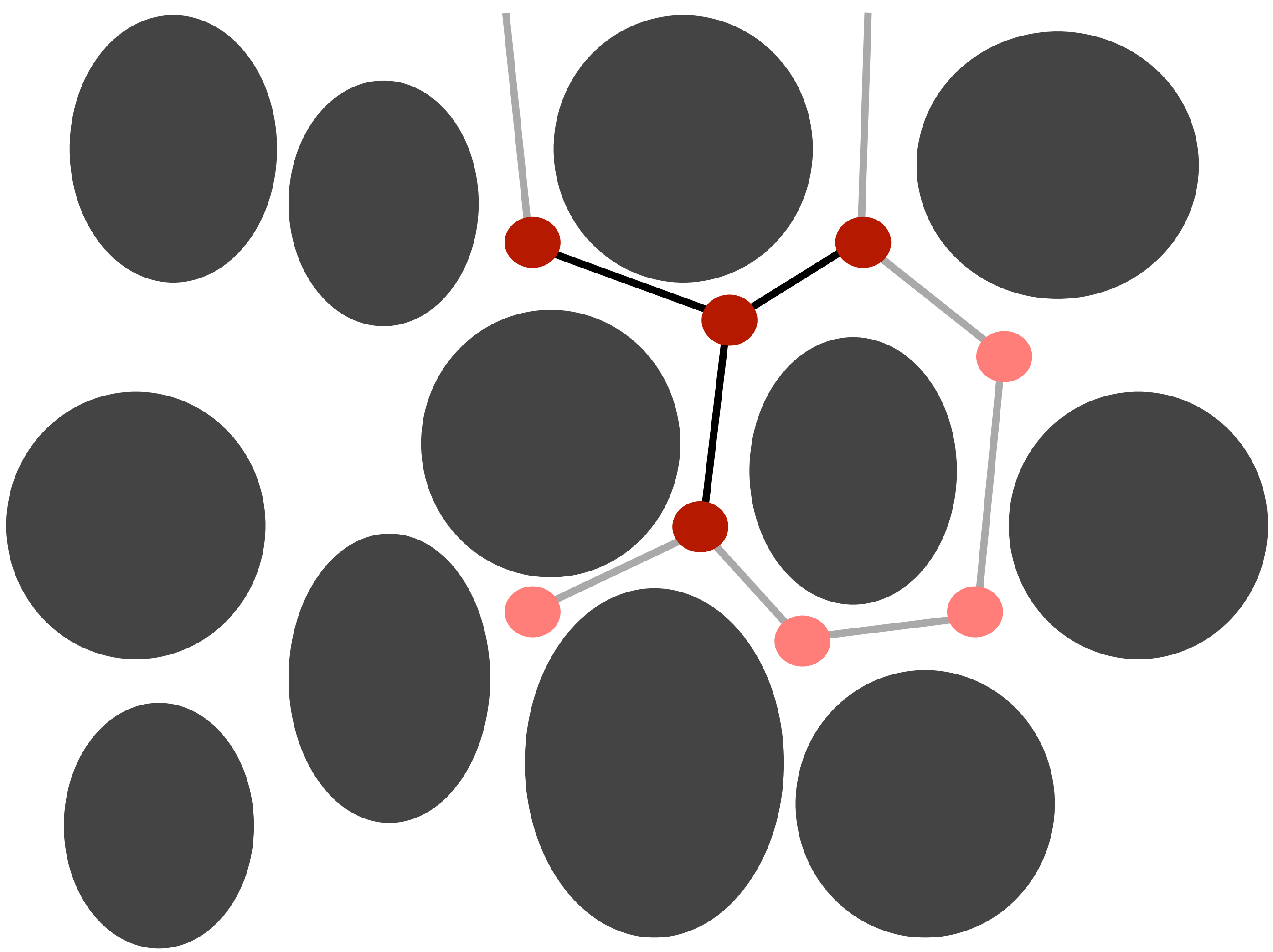
76673 pores and 166853 throats

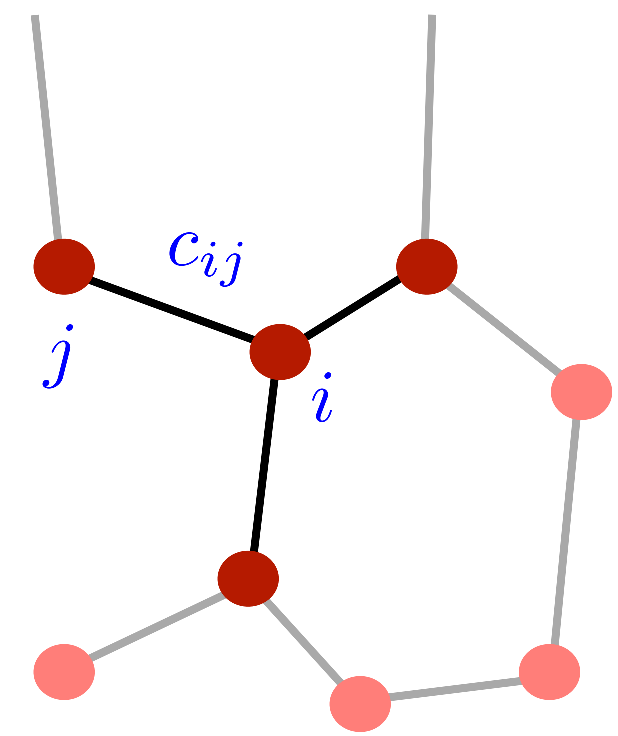
Top view of a slice through 3D network

Pore scale network model

conductance:

$$c_{ij} = c(x_i, p_i, p_j)$$





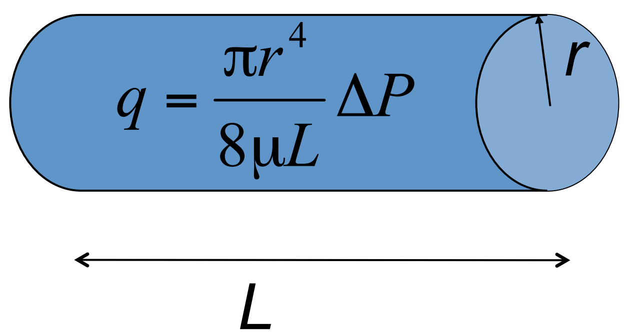
conservation of mass:

$$\sum_{j \in \mathrm{Nbr}(i)} c_{ij}(p_j - p_i) = 0$$

Conductance

- Conductance contains the physics and geometry
- Newtonian fluid of viscosity μ in a tube

$$Cp = b$$



Properties of linear networks:

- positive conductance
- invertibility
- maximum principle

q: discharge of fluid

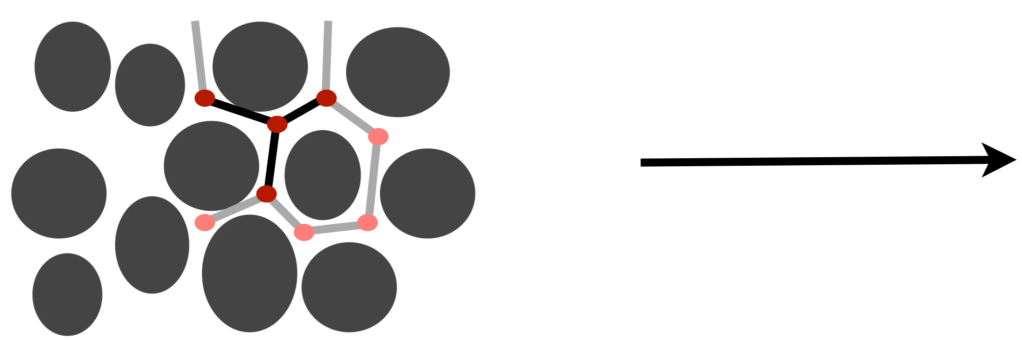
Coupling under HMM

- Finite volume discretization for the PDE

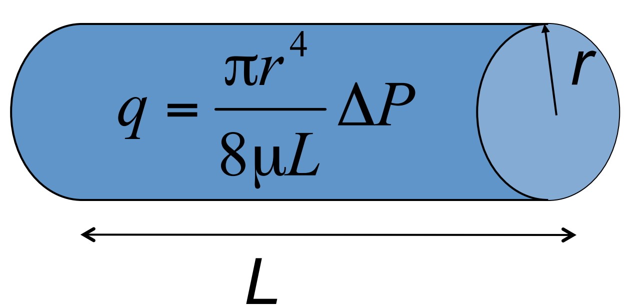
$$\nabla \cdot F(P, \nabla P, x) = h(x), \quad x_L < x < x_R$$
$$F(x_L) = P_L, F(x_R) = P_R$$

$$F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} = \Delta x \; h(x_j)$$

- Flux evaluated by small-size network simulations

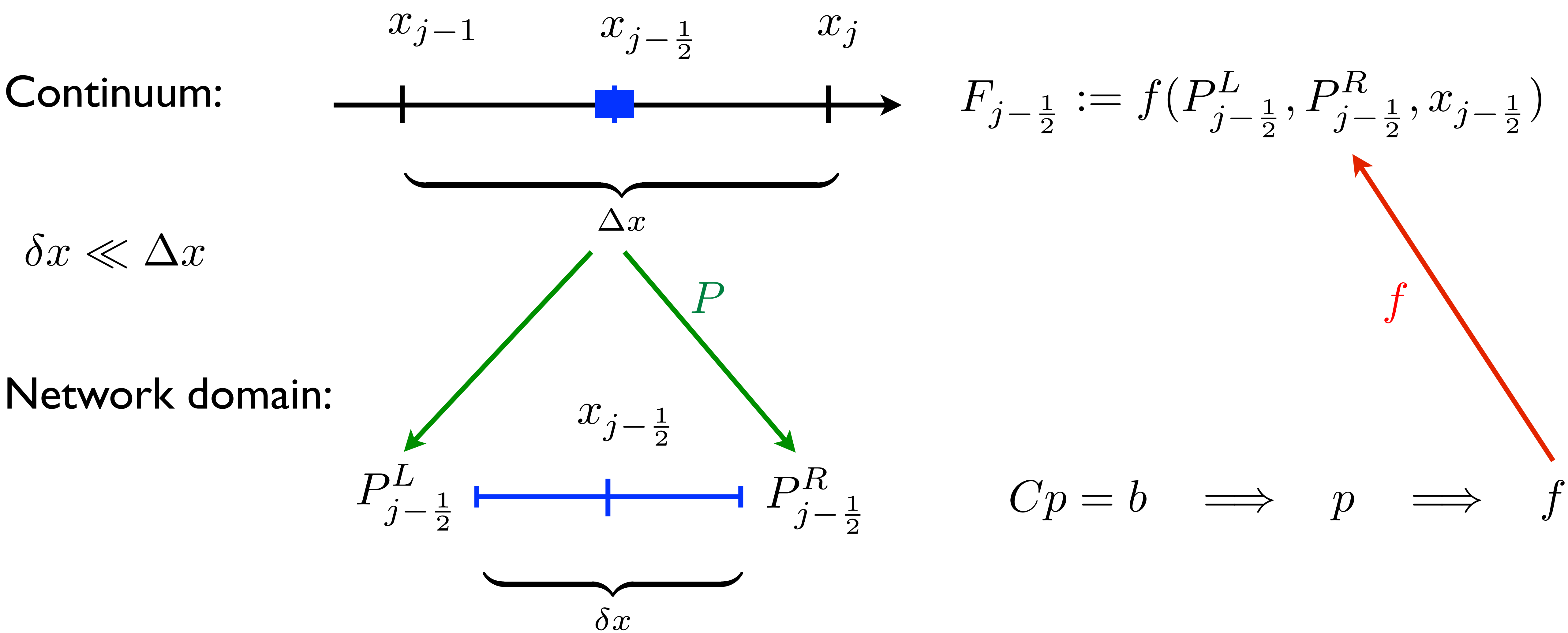


$F_{j+\frac{1}{2}}$



Cf: Balhoff et al. 2007
(domain decomp.)

Coupling

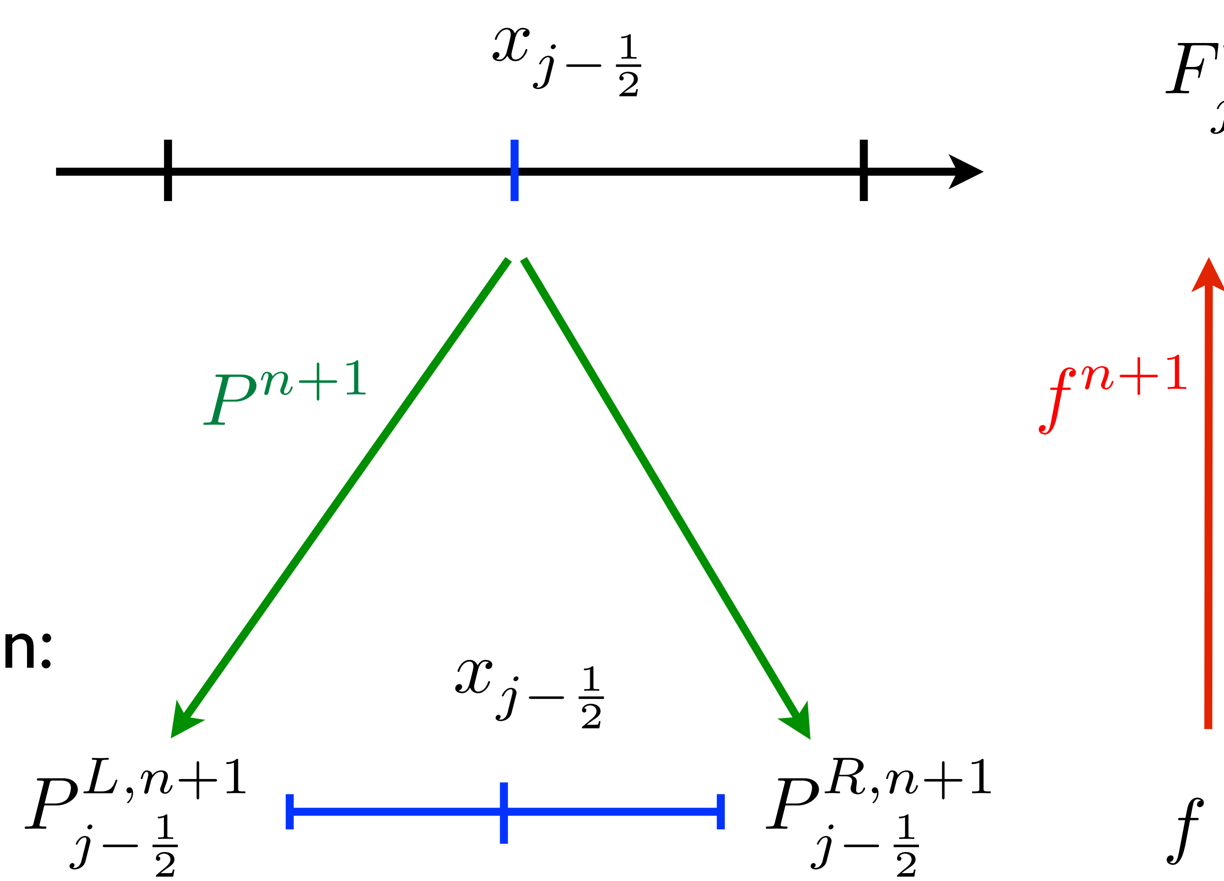


Need to recover the effective pressure field P.

Macro-micro iterations

Continuum:

Network domain:



$$F^{n+1}_{j-\frac{1}{2}} := f^{n+1} \simeq K^n \frac{P^{n+1}_j - P^{n+1}_{j-1}}{\Delta x}$$

$$K^n := \frac{F^n}{\Delta P^n} \Delta x$$

$$f = \sum \sum f_{ij}$$

$$C^{n+1} p^{n+1} = b^{n+1}$$

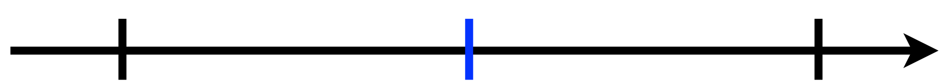
$$f_{ij} = -c_{ij}(p_i + p_j)(p_i - p_j)$$

Macro-micro iterations

$$\mathbf{P}^{n+1} := \mathbf{P}^n - \Delta x^2 \left(\mathbf{K}^n\right)^{-1} G(\mathbf{P}^n)$$

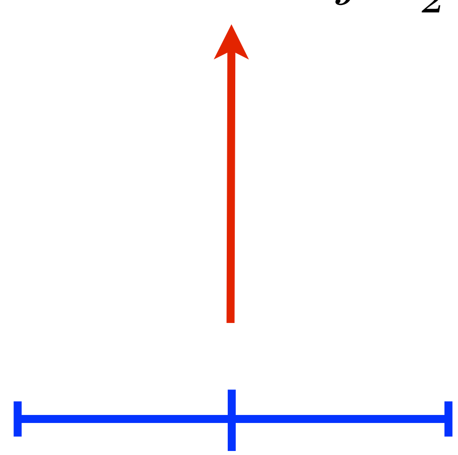
$$G(\mathbf{P}) := \left(D_0 F_j - h_j\right)^T$$

Continuum:



$$f = F^{n+1}_{j-\frac{1}{2}} \simeq K^n \, \frac{P^{n+1}_j - P^{n+1}_{j-1}}{\Delta x} \qquad K^n := \frac{F^n}{\Delta P^n} \Delta x$$

Network domain:

$P^{L,n+1}_{j-\frac{1}{2}}$  $P^{R,n+1}_{j-\frac{1}{2}}$ $\quad f^{n+1}$

Properties of the scheme

$$\mathbf{P}^{n+1} := \mathbf{P}^n - \Delta x^2 \left(\mathbf{K}^n\right)^{-1} G(\mathbf{P}^n)$$

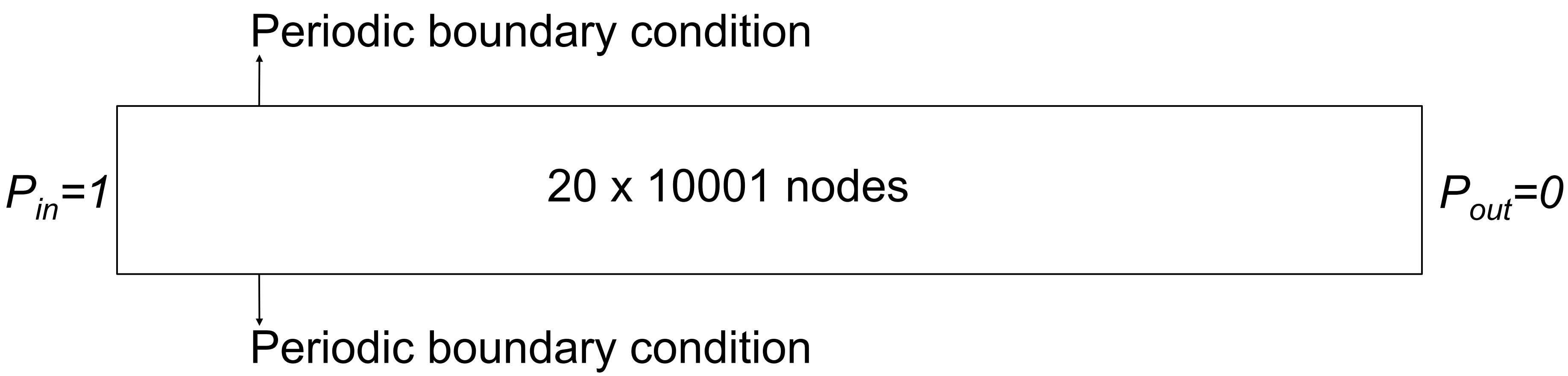
- Iterations converge under suitable conditions

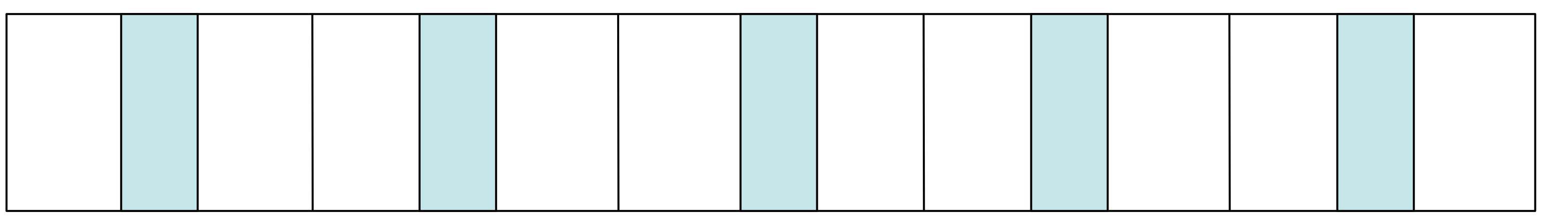
$$\frac{\partial G}{\partial \mathbf{P}} = \frac{1}{\Delta x^2}\left(\mathbf{K} + \mathbf{A}\right)$$

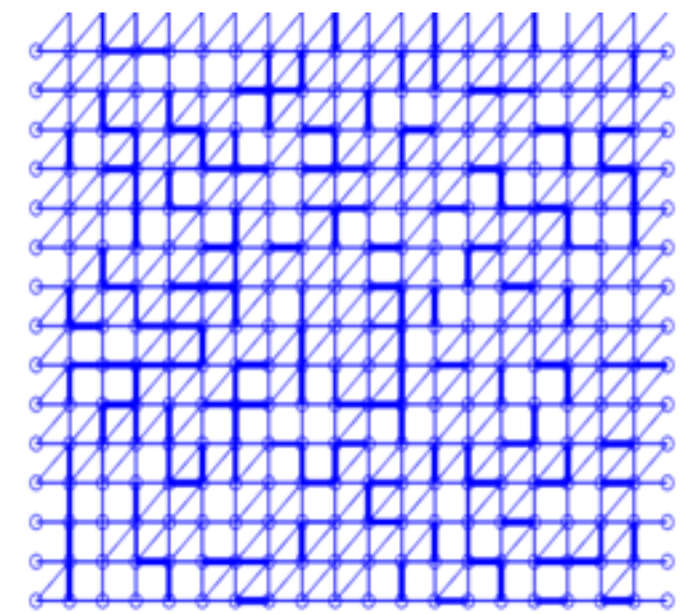
- Linear network: $\mathbf{A} := 0$
  Newton's method and converges in 1 step.

- Nonlinear network: $\mathbf{A} := \left((D^- P_{j-1}) \frac{\partial K_{j-1/2}}{\partial P_k} - (D^- P_j) \frac{\partial K_{j+1/2}}{\partial P_k} \right)$
  Quasi-Newton style iterations.
  Convergence under some conditions.

Simulation setup







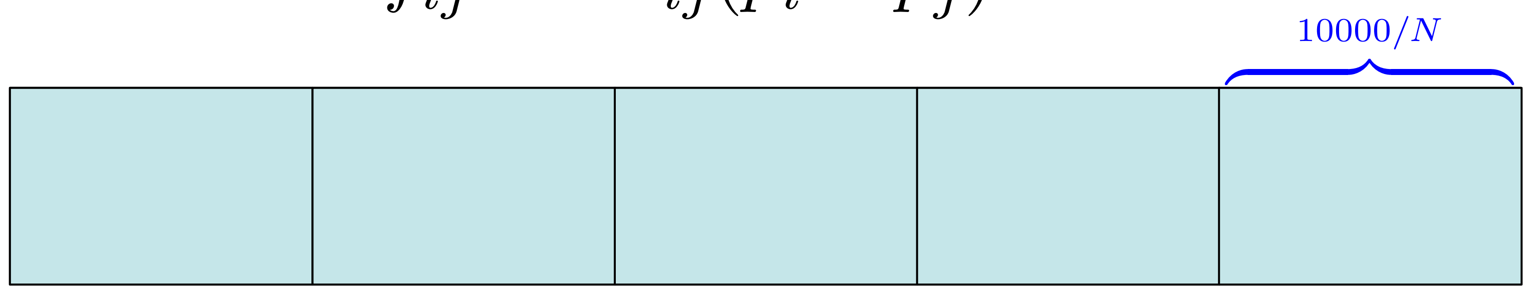
- The conductance c_{ij} is randomly distributed from (0, 1000) with uniform distribution.
- Compare the pressure and the flux from full sampling and partial sampling with results from the direct numerical simulation using the full system.

  Error averaged over 100 different random conductance.

Linear network model

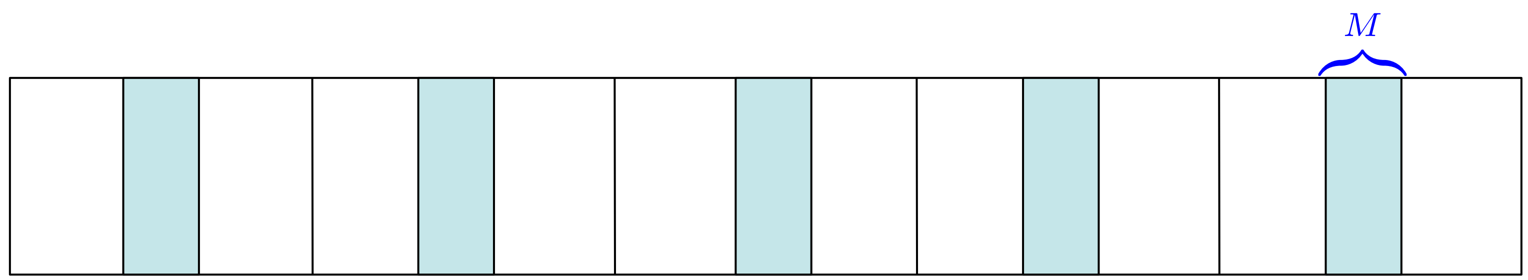
$$f_{ij} = -c_{ij}(p_i - p_j)$$

Full sampling



	$N=5$	$N=10$	$N=20$	$N=50$
Error in pressure	0.0008	0.0014	0.0022	0.0041
Error in flux	0.0048	0.0111	0.0235	0.0620

Partial sampling



Error in pressure

	$M=40$	$M=100$	$M=200$	$M=400$
$N=3$	0.0567	0.0273	0.0188	0.0117
$N=5$	0.0688	0.0370	0.0218	0.0122
$N=10$	0.1454	0.0442	0.0257	0.0158

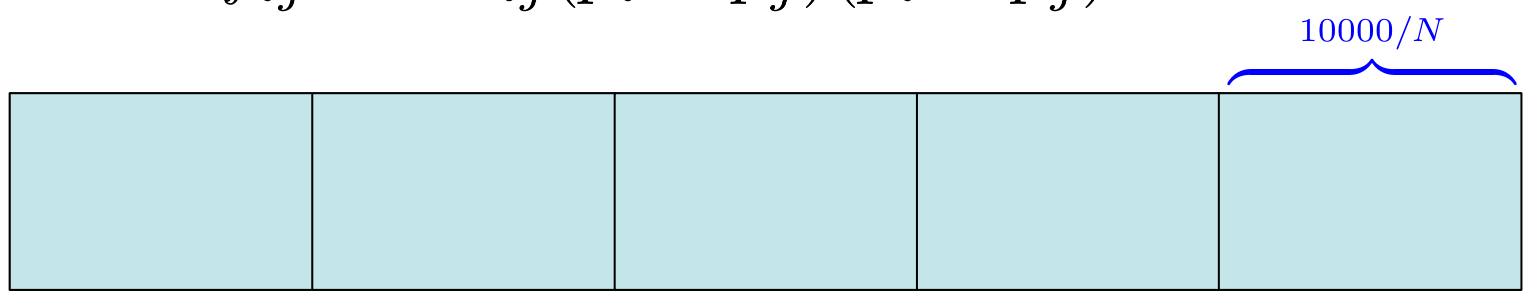
Error in flux

	$M=40$	$M=100$	$M=200$	$M=400$
$N=1$	0.0760	0.0514	0.0299	0.0170
$N=3$	0.1226	0.0513	0.0334	0.0192
$N=5$	0.2111	0.0685	0.0374	0.0214
$N=10$	0.6638	0.1347	0.0571	0.0277

Nonlinear network model I

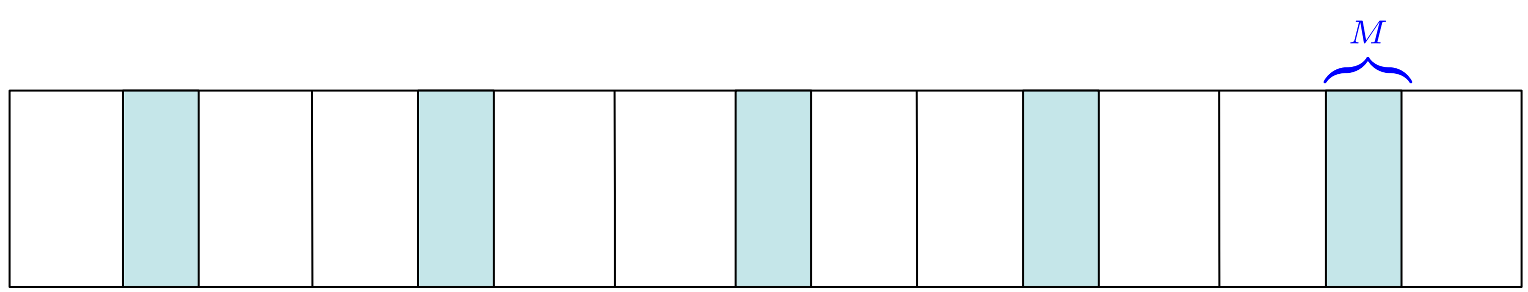
$$f_{ij} = -c_{ij}(p_i + p_j)(p_i - p_j)$$

Full sampling



	$N = 5$	$N = 10$	$N = 20$	$N = 50$
Error in pressure	0.0005	0.0009	0.0016	0.0032
Error in flux	0.0036	0.0082	0.0177	0.0473

Partial sampling



Error in pressure

	$M = 40$	$M = 100$	$M = 200$	$M = 400$
$N = 3$	0.0373	0.0184	0.0116	0.0072
$N = 5$	0.0532	0.0253	0.0176	0.0093
$N = 10$	0.1076	0.0365	0.0207	0.0118

Error in flux

	$M = 40$	$M = 100$	$M = 200$	$M = 400$
$N = 1$	0.0661	0.0370	0.0244	0.0173
$N = 3$	0.0987	0.0382	0.0258	0.0140
$N = 5$	0.1358	0.0560	0.0308	0.0168
$N = 10$	0.4330	0.1048	0.0470	0.0242

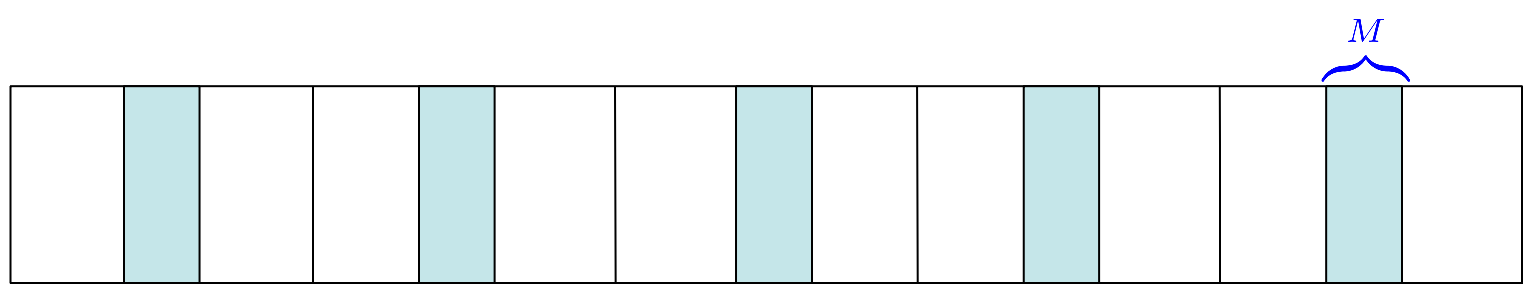
Nonlinear network model II

$$f_{ij} = -(c_{ij} + \beta c_{ij}^2 |p_i - p_j|)(p_i - p_j)$$ (The Forchheimer equation)

Full sampling

	$N=5$	$N=10$	$N=20$	$N=50$
Error in pressure	0.0007	0.0012	0.0021	0.0039
Error in flux	0.0047	0.0108	0.0235	0.0626

Partial sampling



Error in pressure

	$M=40$	$M=100$	$M=200$	$M=400$
$N=3$	0.0481	0.0274	0.0176	0.0101
$N=5$	0.0695	0.0332	0.0213	0.0118
$N=10$	0.1474	0.0442	0.0259	0.0150

Error in flux

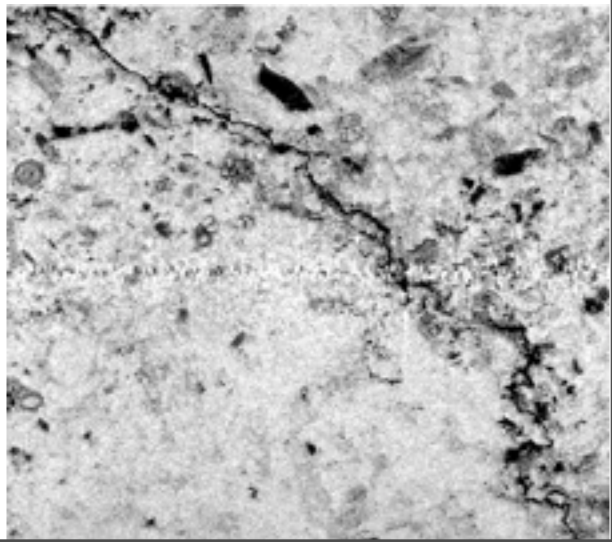
	$M=40$	$M=100$	$M=200$	$M=400$
$N=1$	0.0749	0.0474	0.0298	0.0184
$N=3$	0.1424	0.0602	0.0373	0.0209
$N=5$	0.2133	0.0795	0.0403	0.0203
$N=10$	0.7410	0.1524	0.0657	0.0296

Further macro-micro interaction

- Fluid pressure causes the formation of new crack/fracture.
- Formation of new fracture allows the fluid to enter and extend the crack further.
- Fracture is represented as throats with very high conductance.
- Iterations:

  micro: given network conductance and boundary pressure) solve network pressure --> update network conductance --> solve network pressure

  Macro: (update continuum model) & update pressure

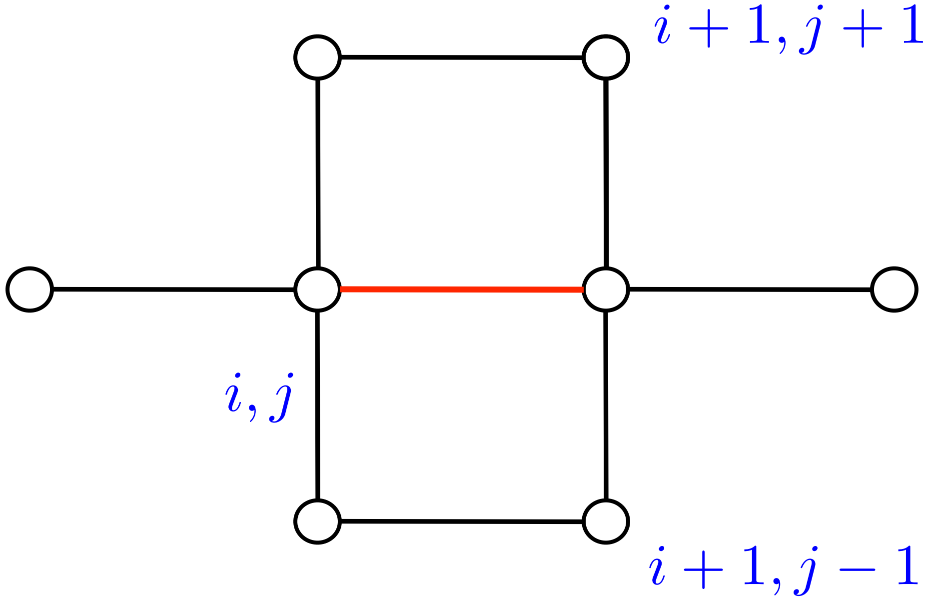


Conductance and stress

Conductance increases (i.e. crack propagates) if $G=K_I^2+K_{II}^2 \geqq G_C$.

$$K_I = C_I \sigma_N \qquad K_{II} = C_{II}(|\sigma_T| - \mu|\sigma_N|)$$

Estimate the normal and tangential stresses (σ_N and σ_T) by local pressure.



$$\sigma_N \approx p_{i+\frac{1}{2},j+1} - 2\, p_{i+\frac{1}{2},j} + p_{i+\frac{1}{2},j-1}$$

$$\sigma_T \approx |(p_{i+1,j+1} - p_{i,j+1}) - (p_{i+1,j-1} - p_{i,j-1})|$$

Compute stresses in similar fashion by suitable projections in non-cartesian network models.

Reference: T. Reuschle (1998),Yuan and Harrison (2006)

Simulation result

Memory effect: $c_{ij}^{n,m+1} = \max(c_{ij}^{n,m}, c_{frac})$

Simulation domain : 21 x 100 nodes.

Initial conductance is 1 or 1000 (red dot).

G_C (critical crack extension force) is randomly distributed: i.e. some throats open more easily

Summary

- New collaborative work in progress. A lot more work to be done.
- The proposed scheme (multiscale, domain decomposition, subsampling) produces reasonable approximations for linear and nonlinear fluxes defined by uniformly distributed random conductance.
- Local stress computation may be used to capture hydraulic fracturing behavior.

Thank you for your attention.

1D periodic case

The network is 1 x 1001 linear model and the conductance c_i is given by

$$c_i = \frac{1.1 + \cos(x/\epsilon)}{1.1 + \sin(x/\epsilon)},$$

where $x = 2\pi i/1000$. Partially sampling is used: 1, 3, 5, 8 blocks with 10, 20, ..., 90, 100 nodes in each sampling domain.

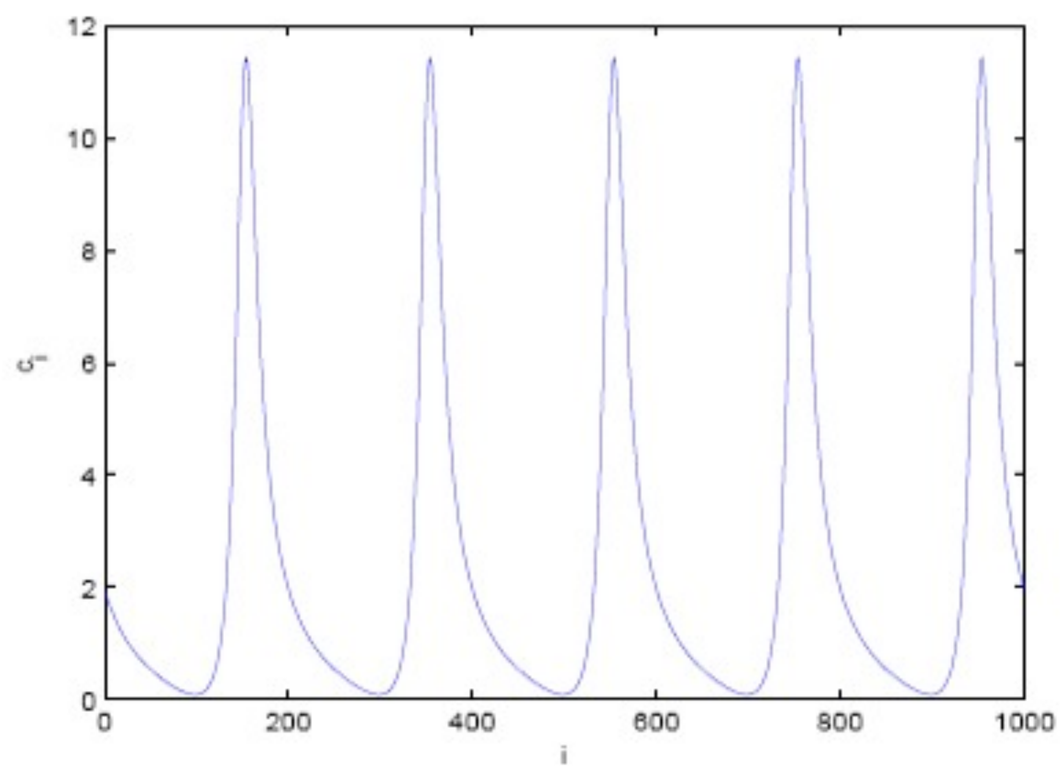
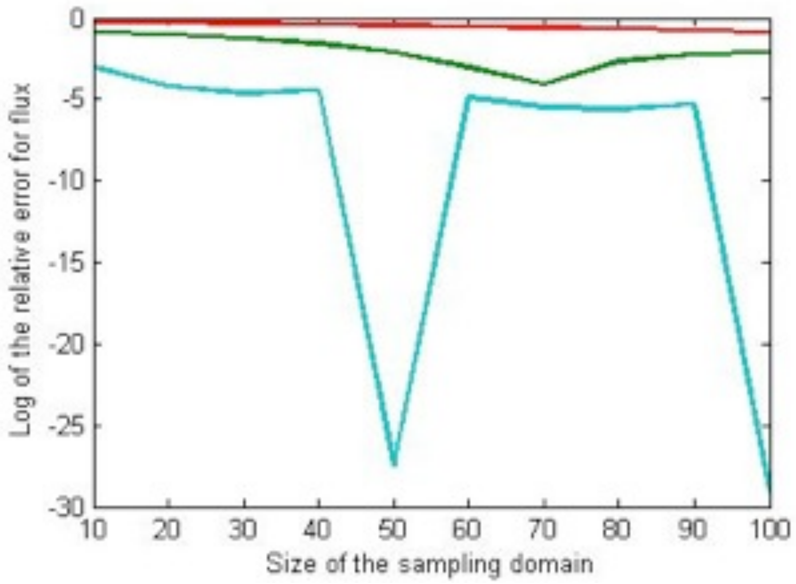
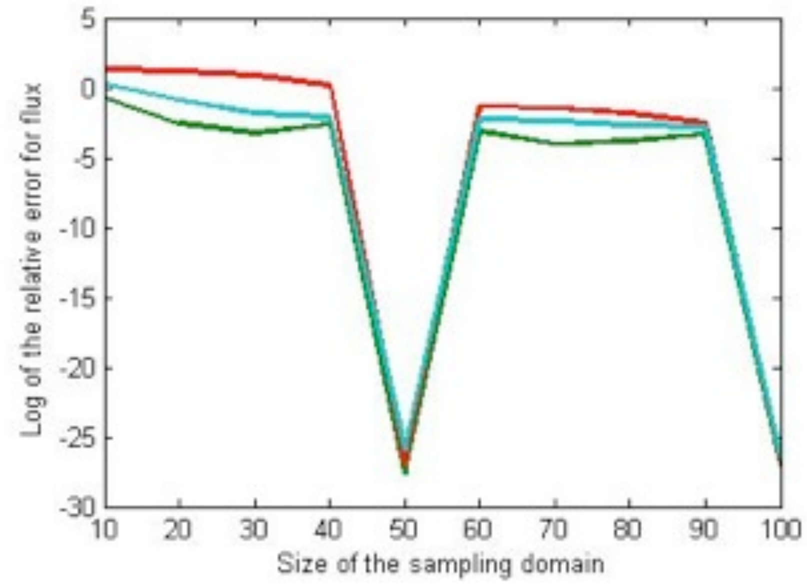


Figure 1: Illustration of c_i with $\epsilon = 1/5$.

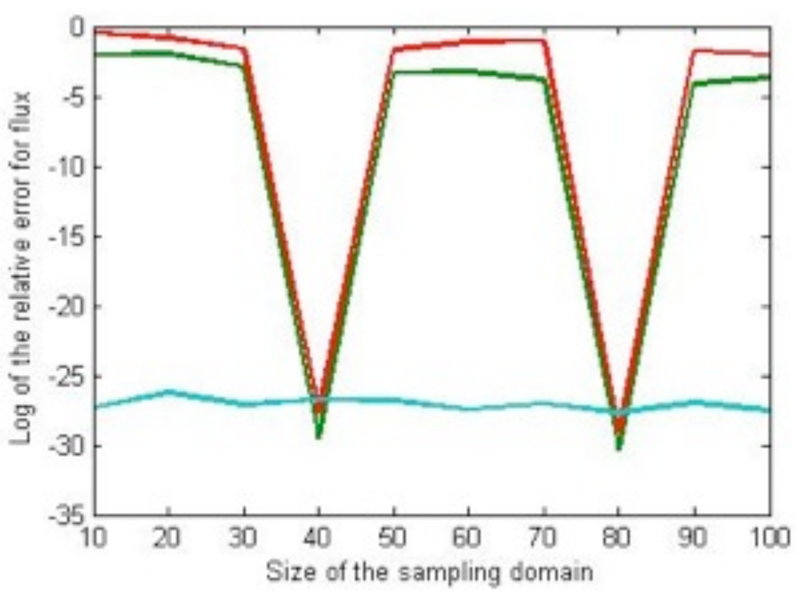
ε = 0.2



ε = 0.05



ε = 0.04



ε = 0.025