The Good Pants Homology and the Ehrenpreis Conjecture

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The aim of the talk

In this talk we will discuss

- A proof of the Ehrenpreis conjecture
- The Surface Subgroup Theorem
- Further results

This is joint work with Jeremy Kahn.

Locally nearly isometric surfaces

- We let S denote a Riemann surface and $\mathbf{M}^n$ a closed hyperbolic n-manifold. Let

  $$f : S \to \mathbf{M}^n$$

  be an immersion and choose $\delta > 0$. We say that such f is δ-*locally nearly isometric* if the induced map

  $$\widetilde{f} : \widetilde{S} \to \widetilde{\mathbf{M}^n}$$

  between the universal covers is locally δ close to an isometric embedding (in the C^0 sense).

Simple (but important) Observation

There exists a universal constant $\delta_0 > 0$ such that for every $\delta \leq \delta_0$ the induced map $f_* : \pi_1(S) \to \pi_1(\mathbf{M}^n)$ between fundamental groups is an injection.

Locally nearly isometric surfaces

- We always assume $\delta < \delta_0$, that is if a map $f : S \to \mathbf{M}^n$ is δ-*locally nearly isometric* then the induced map $f_* : \pi_1(S) \to \pi_1(\mathbf{M}^n)$ between fundamental groups is an injection.
- When $n = 2$, then a δ-locally nearly isometric map $f : S \to \mathbf{M}^2$ is a topological covering. Moreover, we can find a new Riemann surface S_1 and a $K(\delta)$-quasiconformal map $g : S_1 \to S$ (where $K(\delta) \to 1$ when $\delta \to 0$) such that the map $f \circ g : S_1 \to \mathbf{M}^2$ represents a regular holomorphic covering.
- When $n \geq 3$, then such $f : S \to \mathbf{M}^n$ yields the surface $f(S)$ that is incompressible (or essential) inside of $\mathbf{M}^n$.

The Perfect pants and the Model Orbifold

- A hyperbolic structure on a pair of pants Π is uniquely determined by the lengths of the three cuffs from $\partial\Pi$.

A Perfect Pair of Pants

We say that a pair of pants Π *is* R*-perfect if all three cuffs of* Π *have the same hyperbolic length* $R > 0$.

- A marked complex structure on a closed surface with a given pants decomposition is uniquely determined by its Fenchel-Nielsen coordinates.

The Model Orbifold

Let $\mathbf{S}(R)$ *denote a genus* 2 *Riemann surface that is obtained by gluing two* R*-perfect pairs of pants along their cuffs with the twist of* $+1$*. By* $\mathrm{Orb}(R)$ *we denote the induced orbifold.*

The Main Theorem

- Our central result claims the existence of locally nearly isometric surfaces in $\mathbf{M}^n$.

Nearly isometric surfaces in $\mathbf{M}^n$

Fix a closed hyperbolic n-manifold $\mathbf{M}^n$ *and let* $\delta > 0$. *There exists* $R_0 = R_0(\delta, \mathbf{M}^n) > 0$ *such that for every* $R > R_0$ *there exists a Riemann surface* $S = S(R)$ *that is a regular holomorphic cover of the Model Orbifold* $\mathrm{Orb}(R)$, *and a* δ*-locally nearly isometric map* $f : S \to \mathbf{M}^n$.

- Moreover, the surface S admits a decomposition into R-perfect pants that are glued to each other with the twist of $+1$, that is if we let $\iota : S \to \mathrm{Orb}(R)$ denote the corresponding covering map, then ι lifts the perfect pants decomposition of $\mathrm{Orb}(R)$ to a perfect pants decomposition of S.

Applications to 3-manifolds

- The first application of the above theorem is to show that every closed hyperbolic 3-manifold contains an incompressible surface.

Surface Subgroup Theorem

Let $\mathbf{M}^3$ denote a closed hyperbolic 3-manifold. There exists a closed surface S_g of genus $g \geq 2$, and a continuous map

$$f : S_g \to \mathbf{M}^3,$$

such that the surface $f(S_g)$ is incompressible, that is the induced homomorphism between fundamental groups

$$f_* : \pi_1(S_g) \to \pi_1(\mathbf{M}^3),$$

is injective.

- Moreover, given any $\delta > 0$ we can find a Riemann surface S_g so that the map $f : S_g \to \mathbf{M}^3$ is δ-locally nearly isometric. Then the group $f^*(\pi_1(S_g))$ is a quasifuchsian subgroup of $\pi_1(\mathbf{M}^3)$ that is δ-close to being Fuchsian.

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Applications to 3-manifolds

- Incompressible closed surfaces in $\mathbf{M}^3$ that we construct in the previous theorem are by construction all homologically trivial (which is to be expected since there are hyperbolic 3-manifolds $\mathbf{M}^3$ with trivial homology).
- In the case when $\mathbf{M}^3$ has non-trivial homology we can still realise every homology class in $H_2(\mathbf{M}^3, \mathbb{Z})$ as a nearly geodesic surface (that is, every homology class has a "nice" representative in this sense).

Homology and incompressible surfaces

Let $\mathbf{M}^3$ denote a closed hyperbolic 3-manifold and suppose that S_g, $g \geq 2$, is a closed surface and $f : S_g \to \mathbf{M}^3$, is a continuous map such that the surface $f(S_g)$ is incompressible in $\mathbf{M}^3$. Then there exists a closed surface S_h, $h \geq 2$, and a continuous map $f_1 : S_h \to \mathbf{M}^3$, such that $f_1(S_h)$ is a nearly geodesic surface in $\mathbf{M}^3$ (and thus incompressible) and $f_1(S_h)$ and $f(S_g)$ are homologous in $\mathbf{M}^3$.

- The surface S_h will have large genus (much larger than S_g).

Applications to Riemann surfaces

- The following statement follows from the main theorem.

Every closed Riemann surface has a good cover

Let S denote a closed Riemann surfaces and let $\epsilon > 0$. Then for every large enough R, there exists a finite degree cover $S_1 \to S$ such that S_1 has a decomposition into pants with cuff lengths ϵ close to R and that are glued to each other with a twist $\frac{\epsilon}{R}$-close to $+1$.

- Let M_1 be the Riemann surface that is obtained from S_1 by turning the good pants into the perfect pants that are glued by the twist $+1$. Then M_1 is a finite cover of the Model Orbifold $\mathrm{Orb}(R)$. It turns out that there exists a K-quasiconformal map $f : S_1 \to M_1$, such that $K = K(\epsilon)$ and $K(\epsilon) \to 1$ when $\epsilon \to 0$.

Theorem: The Ehrenpreis Conjecture

Let S and M denote two closed Riemann surfaces. For any $K > 1$, one can find finite degree covers S_1 and M_1 of S and M respectively, such that there exists a K-quasiconformal map $f : S_1 \to M_1$.

Counting the number of incompressible surfaces in $\mathbf{M}^3$

- Let $s(\mathbf{M}^3, g)$ denote the number (up to homotopy) of genus g incompressible surfaces of $\mathbf{M}^3$. We have the following counting result:

Counting surfaces in $\mathbf{M}^3$

Let $\mathbf{M}^3$ be a closed hyperbolic 3-manifold. There exist constants $0 < c_1 \leq c_2$, such that the inequality

$$(c_1 g)^{2g} \leq s(\mathbf{M}^3, g) \leq (c_2 g)^{2g},$$

holds for every large g.

- A difficult (but perhaps deep) conjecture is to prove that for some constant $c = c(\mathbf{M}^3) > 0$ we have

$$\lim_{g \to \infty} \frac{\sqrt[2g]{s(\mathbf{M}^3, g)}}{g} = c.$$

Applications to 3-manifolds

- More generally, we can prove that every closed homologically trivial curve in a closed hyperbolic n-manifold $\mathbf{M}^n$ rationally bounds an incompressible surface in $\mathbf{M}^n$.

Homologically trivial curve rationally bounds an incompressible surface

Let Γ denote the set of closed curves on $\mathbf{M}^n$ (up to homotopy). Suppose that γ is a formal sum $\gamma \in \mathbb{Q}\Gamma$ that is equal to zero in the standard homology $H^1(\mathbf{M}^n, \mathbb{Q})$. Then there exists an incompressible surface with boundary $S \subset \mathbf{M}^n$ (meaning that the inclusion $S \subset \mathbf{M}^n$ is π_1 injective) such that

$$\frac{1}{m}\partial S = \gamma,$$

for some $m \in \mathbb{N}$.

- Once again, we can arrange that the Riemann surface S is locally nearly isometric in $\mathbf{M}^n$ and that the boundary of S is geodesic.

The Good Pants

The Good Pants

Let $\delta, R > 0$. Let Π_R denote the R-perfect pants and suppose that $f : \Pi_R \to \mathbf{M}^n$ is a δ-locally nearly isometric immersion. We call the homotopy class $[f] = \Pi$ a (δ, R)-good pants, or just the good pants if δ and R are understood. The set of all good pants is denoted by $\mathbf{\Pi} = \mathbf{\Pi}_{\delta,R}$. In particular, the length of a cuff of a good pair of pants is ϵ close to R, where $\epsilon = \epsilon(\delta)$ and $\epsilon(\delta) \to 0$ when $\delta \to 0$.

- In order to find the map $f : S_R \to \mathbf{M}^n$ (recall that S_R denotes a finite cover of $\mathrm{Orb}(R)$) we are guided by the following principles:
  1. We do not start by trying to specify the surface S_R.
  2. Instead, we consider the good pants $\mathbf{\Pi}$ as the building blocks and we eventually construct $f(S_R)$ by appropriately assembling together a large collection of good pants from $\mathbf{\Pi}$.
- For example, consider any finite formal sum $W \in \mathbb{N}\mathbf{\Pi}$. Then the sum $2W$ yields a surface S_R and a map $g : S_R \to \mathbf{M}^n$, but the map g is not necessarily locally nearly isometric.

The two principles

- It turns out that in order to assemble the good pants and construct a locally nearly isometric map $f : S_R \to \mathbf{M}^n$, it suffices to prove the following two properties:
  1. The Equidistribution Principle: The set of good pants $\mathbf{\Pi}$ is exponentially well distributed in $\mathbf{M}^n$.
  2. The Good Pants Homology of $\mathbf{M}^n$ agrees with the standard first homology $H_1(\mathbf{M}^n, \mathbb{Q})$ with rational coefficients (we in fact need a quantitative version of this statement).
- In order to prove the main theorem when $n \geq 3$, we only need the Equidistribution Principle. This is why proving the Ehrenpreis conjecture is more demanding than proving the Surface Subgroup Theorem (however, in order to prove that every homologically trivial closed curve in $\mathbf{M}^n$ bounds an incompressible surface we need the both above statements).

The Equidistribution Principle

- Let γ be a good closed geodesic in $\mathbf{M}^n$ ("good" means that the length of γ is ϵ close to R). By $\mathbf{\Pi}(\gamma)$ we denote the good pants that contain γ as a cuff.
- The Equidistribution Principle is the statement that the good pants from $\mathbf{\Pi}(\gamma)$ are well distributed along γ.
- More precisely, to each good pants $\Pi \in \mathbf{\Pi}(\gamma)$ we associate a point $\text{foot}_\gamma(\Pi)$ in the unit normal bundle $N^1(\gamma)$. Then the projection $\text{foot}_\gamma : \mathbf{\Pi}(\gamma) \to N^1(\gamma)$ yields a measure μ on $N^1(\gamma)$.
- We say that the good pants from $\mathbf{\Pi}(\gamma)$ are δ- well distributed along γ if the measure μ is δ-close to the measure $K\lambda$, where $K > 0$ and λ denotes the standard Lebesgue measure on the torus $N^1(\gamma)$.

The Equidistribution Principle

The Equidistribution Principle

Let $\epsilon, R > 0$ *and fix a manifold* $\mathbf{M}^n$*. There exists* $q > 0$ *that depends only on the first eigenvalue of the Laplacian on* $\mathbf{M}^n$*, such that for every* R *large enough the set* $\mathbf{\Pi}(\gamma)$ *is* e^{-qR}*-well distributed along* γ*.*

- The above Equidistribution Principle follows from the fact that the frame flow on the frame bundle of $\mathbf{M}^n$ has excellent statistical properties.
- In fact the frame flow is exponentially mixing.

The frame flow is mixing

- There is a natural identification $\mathcal{F}(\mathbf{M}^3) = \Gamma \backslash \mathbf{PSL}(2,\mathbb{C})$, where $\Gamma < \mathbf{PSL}(2,\mathbb{C})$ is a lattice isomorphic to $\pi_1(\mathbf{M}^3)$. Importantly, the Lie group $\mathbf{PSL}(2,\mathbb{C})$ acts on the left on the frame bundle (we write $\mathbf{PSL}(2,\mathbb{C}) / \Gamma \backslash \mathbf{PSL}(2,\mathbb{C})$ to denote this action), and the frame flow is the restriction of this right action to a certain 1-parameter subgroup of $\mathbf{PSL}(2,\mathbb{C})$.
- The induced representation of $\mathbf{PSL}(2,\mathbb{C})$ on $L^2(\mathcal{F}(\mathbf{M}^3))$ is unitary, and we can decompose $L^2(\mathcal{F}(\mathbf{M}^3))$ into $\mathbf{PSL}(2,\mathbb{C})$ irreducible invariant subspaces. The irreducible unitary representations of $\mathbf{PSL}(2,\mathbb{C})$ are known and one then computes the rate of mixing of the frame flow.
- The proof of the Surface Subgroup Theorem and the Proof of the Ehrenpreis Conjecture are essentially powered by the analytical and geometric properties of the frame flow on hyperbolic manifolds and Gelfand's theory enables us to set up the calculus to compute the distribution of good geodesics and good pants inside $\mathbf{M}^3$ (or $\mathbf{M}^n$ in general).

The Pants Homology

- Consider the following problem. Let S be a topological surface and let G denote the set of closed curves on S up to homotopy. We say that a formal sum $\alpha \in \mathbb{Q}G$ is zero in the Rational Pants Homology if there exists a formal sum (with rational coefficients) W of immersed pairs of pants in S, such that

$$\partial W = \alpha.$$

- This defines a new homology which we call the pants homology. It turns out that the Pants Homology agrees with the standard homology $H_1(S, \mathbb{Q})$ with rational coefficients.
- However, proving this using purely topological methods (that is without using hyperbolic geometry) is perhaps no so easy.

The Good Pants Homology

- Let $\Gamma = \Gamma_{\epsilon,R}$ denote the set of (ϵ, R)- good geodesics in $\mathbf{M}^n$. We write
$$\Gamma = \{\gamma_1, \cdots, \gamma_n\},$$
and
$$\mathbf{\Pi} = \{\Pi_1, \cdots, \Pi_m\},$$
- Then every element $\gamma \in \mathbb{Q}\Gamma$ is uniquely written as $\gamma = \sum a_i \gamma_i$, for some $a_i \in \mathbb{Q}$, and every element $W \in \mathbb{Q}\mathbf{\Pi}$ is uniquely written as $W = \sum b_j \Pi_j$, for some $b_j \in \mathbb{Q}$.
- We define the supremum norms as follow. For $\gamma \in \mathbb{Q}\Gamma$ we let
$$||\gamma||_\infty = \max\{|a_i|\},$$
and for $W \in \mathbb{Q}\mathbf{\Pi}$ we let
$$||W||_\infty = \max\{|b_j|\},$$

The Good Pants Homology

Definition: The Good Pants Homology

We say that a formal sum $\alpha \in \mathbb{Q}\Gamma$ is zero in the Rational Good Pants Homology if there exists a formal sum of good pants $W \in \mathbb{Q}\mathbf{\Pi}$ such that

$$\partial W = \alpha.$$

- The main result in this direction is the following.

The Good Pants Homology is standard

Let $\mathbf{M}^n$ denote a hyperbolic manifold. If $\alpha \in \mathbb{Q}\Gamma$ is zero in the standard homology $H_1(\mathbf{M}^n, \mathbb{Q})$ then α is zero in the Rational Good Pants Homology.

The Good Pants Homology

- We actually prove a quantitative statement, which says that if $\alpha \in \Gamma$ is zero in the standard homology then we do not need to use too many pants to rationally close up γ.
- In the case of a Riemann surface $\mathbf{M}^2$, we have the following quantitative statement. Let $\alpha \in \mathbb{Q}\Gamma$ be zero in the standard homology. Then we can find a formal sum $W \in \mathbb{Q}\mathbf{\Pi}$ such that $\partial W = \alpha$, and

$$||W||_{\infty} \leq P(R)e^{-R}||\gamma||_{\infty},$$

where $P(R)$ denotes a fixed polynomial.