Deriving dynamical models from palaeoclimatic records using nonlinear Kalman filtering: estimation of parameters and noise levels

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Outline

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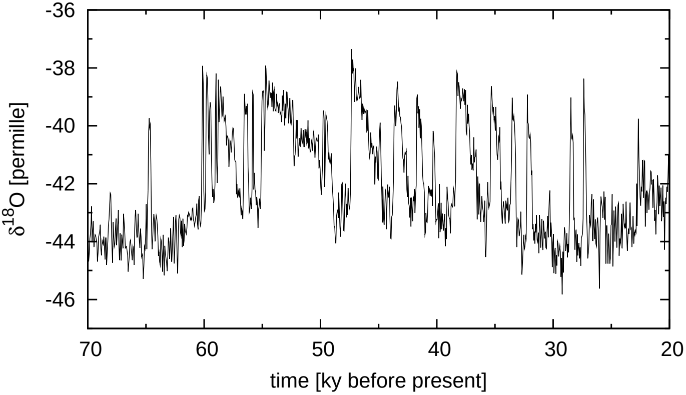
North Greenland ice-core record of $\delta^{18}O$



Proposed conceptual and low-order models for glacial millennial-scale climate transitions

- shifts between distinctly different states in a stochastically driven nonlinear system (*Alley et al. 2001; Ganopolski and Rahmstorf 2002; Ditlevsen et al. 2005*)

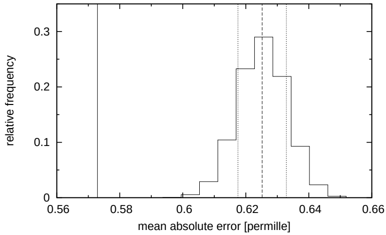
  simplest model: Brownian motion in a one-dimensional double-well potential

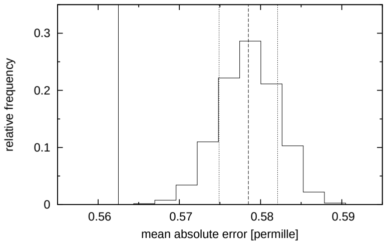
  stochastic resonance ?
- relaxation oscillator or system of nonlinearly coupled relaxation oscillators (*Schulz et al. 2002, 2004*)
- nonlinear thermal oscillator where the timing of the deterministic external forcing is crucial for generating DO-like oscillations (*Rial 2004*); nonlinear threshold model with external forcing (*Braun et al. 2007*)

Goal:

- deriving simple nonlinear deterministic and stochastic dynamical models from data
- model selection
- integrating theories and models with palaeoclimatic data

Nonlinearity versus stochasticity in the ice-core data





Surrogate time series (*Schreiber and Schmitz 1996*)

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Model: Stochastically driven motion in a double-well potential

$$\dot{z} = -U'(z) + \sigma\eta$$

$$U(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z$$

free parameters $\{a_i\}_{i=1}^4$ and noise level σ to be determined from data

Nonlinear state space model

dynamical (or state) equation

$$\begin{aligned} \mathbf{z}_t &= \mathbf{f}(\mathbf{z}_{t-1}, \boldsymbol{\lambda}) + \boldsymbol{\eta}_t \\ \mathbf{y}_t &= \mathbf{g}(\mathbf{z}_t) + \varepsilon_t \end{aligned}$$

observation (or measurement) equation

$\boldsymbol{\eta}_t \sim \mathcal{N}(0, \mathbf{Q})$

$\varepsilon_t \sim \mathcal{N}(0, \mathbf{R})$

Unscented Kalman Filter

- flexible tool for recursive estimation of unobserved states and parameters in nonlinear systems from incomplete, indirect and noisy observations
- nonlinear extension of the conventional Kalman filter
- Unlike the extended Kalman filter, it keeps the full nonlinear system dynamics rather than linearising it.
- truncates the filter probability density to a Gaussian by only propagating first and second moments
- applicable to deterministic as well as stochastic models
- Algorithm is deterministic even for a stochastic model.
- alternative to particle filters and MCMC

Estimation of noise levels

- not straightforward in Kalman filtering
- might be determined from model summary statistics (pdf, variance or time scale)
- more systematic approach: consistency of uncertainty estimates
- $y_t \sim \mathcal{N}(z_{t|t-1}, S_t)$
- Log-likelihood function:

  $$l(Q, R) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_t \left[\log(S_t) + \frac{\left(y_t - z_{t|t-1}\right)^2}{S_t}\right]$$

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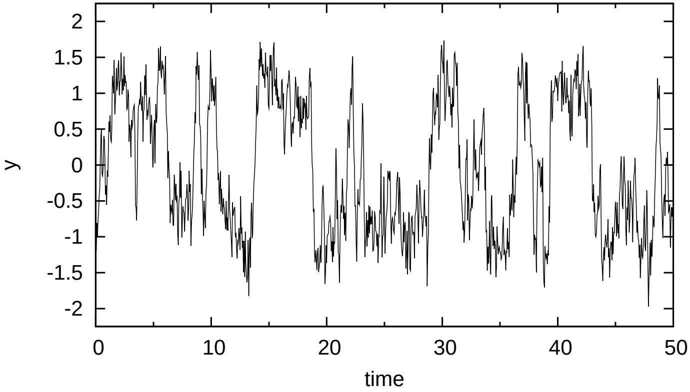
Model system

$$\dot{z} = -U'(z) + \sigma\eta$$

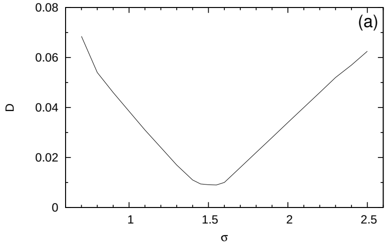
$$U(z) = z^4 - 2z^2$$

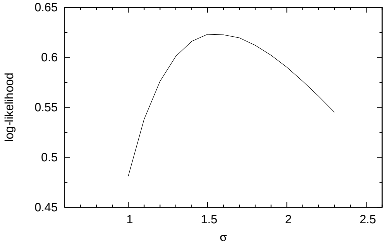
$$\sigma = 1.5, R = 0.01$$

Sample trajectory of symmetric double-well potential



Estimation of dynamical noise level ($R = 0.01$)

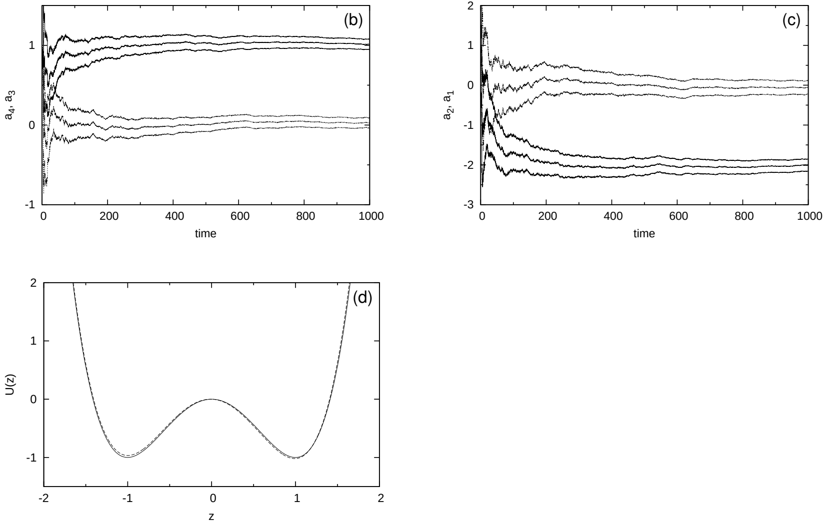




Estimation of both noise levels with likelihood function

	0.00	0.05	0.10	0.15
1.0	0.209	0.318	0.481	0.568
1.1	0.361	0.429	0.538	0.594
1.2	0.463	0.506	0.576	0.610
1.3	0.530	0.558	0.601	0.618
1.4	0.574	0.591	0.616	0.620
1.5	0.601	0.610	**0.623**	0.618
1.6	0.615	0.620	**0.623**	0.612
1.7	0.621	0.622	0.620	0.604
1.8	0.620	0.619	0.612	0.593
1.9	0.615	0.612	0.602	0.580
2.0	0.605	0.602	0.590	0.566
2.1	0.594	0.589	0.576	0.551

Parameter estimation



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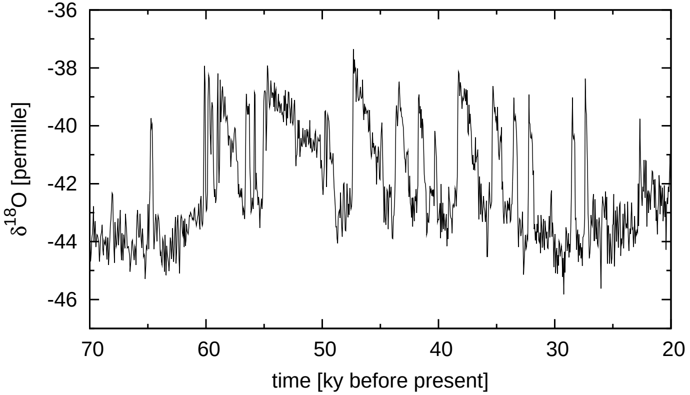
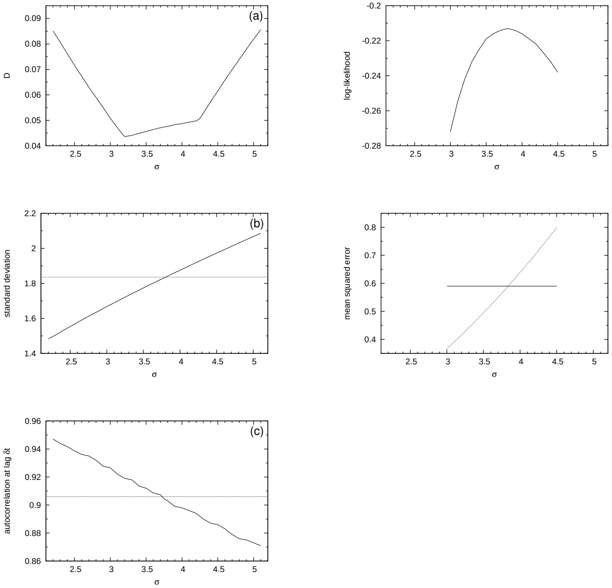
Ice-core record for last glacial period



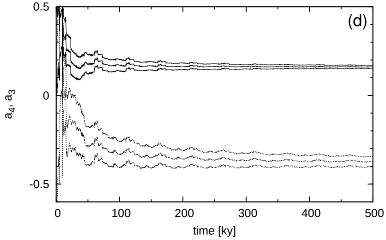
Ice-core data / Estimation of dynamical noise level ($R = 0$)

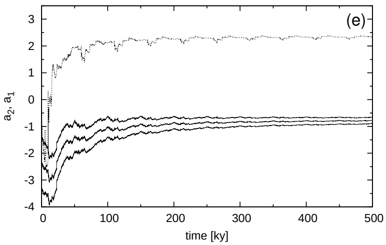


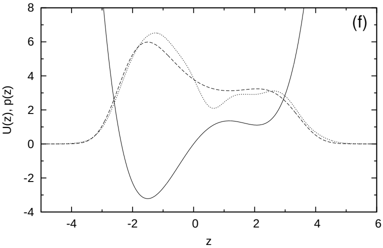
Estimation of both noise levels with likelihood function

	0.0	0.1	0.2
3.0	-0.272	-0.260	-0.241
3.1	-0.255	-0.247	-0.233
3.2	-0.242	-0.236	-0.227
3.3	-0.232	-0.228	-0.223
3.4	-0.225	-0.222	-0.220
3.5	-0.219	-0.218	-0.219
3.6	-0.216	-0.215	-0.219
3.7	-0.214	-0.214	-0.220
3.8	**-0.213**	-0.215	-0.222
3.9	-0.214	-0.216	-0.225
4.0	-0.216	-0.219	-0.228
4.1	-0.219	-0.222	-0.232
4.2	-0.222	-0.226	-0.237
4.3	-0.227	-0.231	-0.243
4.4	-0.232	-0.236	-0.248

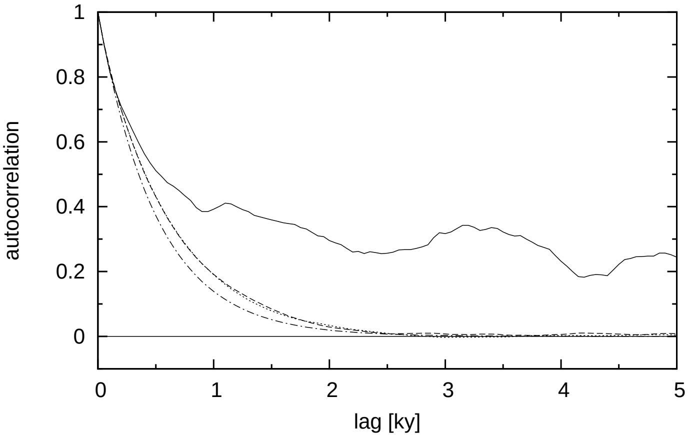
Ice-core data / Parameter estimation



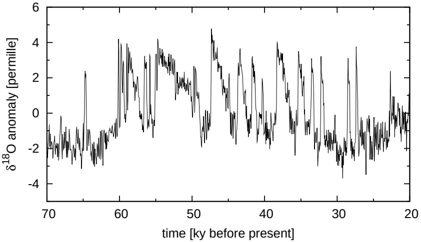


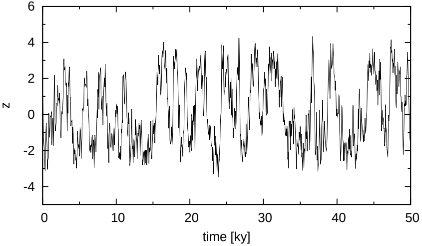


Autocorrelation function



Ice-core record and sample trajectory of the model





Stochastically driven oscillator in a double-well potential

$$\ddot{q} = -\gamma \dot{q} - V'(q) + \sigma \eta$$

$$\begin{aligned} \dot{q} &= p + \sigma_1 \eta_1 \\ \dot{p} &= -\gamma p - V'(q) + \sigma_2 \eta_2 \end{aligned}$$

$$U(q) = a_4 q^4 + a_3 q^3 + a_2 q^2 + a_1 q$$

free parameters $\{a_i\}_{i=1}^4$, γ and noise levels σ_1 and σ_2 to be determined from data

Van der Pol-type relaxation oscillator

$$\ddot{q} + \gamma(q, \dot{q})\dot{q} + V'(q) = \sigma\eta$$

References

- Kwasniok F., Lohmann G. (2009): Deriving dynamical models from paleoclimatic records: Application to glacial millennial-scale climate variability, Physical Review E 80, 066104.
- Kwasniok F., Lohmann G. (2011): A nonlinear stochastic oscillator model for glacial millennial-scale climate transitions derived from ice-core data, submitted.
- Kwasniok F. (2011): Modelling and analysis of Dansgaard-Oeschger events using a mixture of linear stochastic models, submitted.