Locality and Unitarity from Positivity

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Introduction

Motivation

New methods in QFT - test case: planar N=4 SYM

Object of interest: on-shell scattering amplitudes at weak coupling

At tree-level we have a well defined rational function.

Traditional method at loop level:

- Find the integrand - unique rational function in any planar theory.
- Integrate over the real contour.

Importance of the integrand

- Well defined finite object: rational function for any amplitude at all loop orders.
- A lot of structure is lost after integration over the real contour.
- There are also other interesting contours (leading singularities).

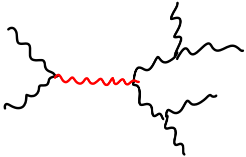
Motivation

Definition: Amplitude = tree-level + integrand of loop level.

How to calculate the amplitudes?

Feynman diagrams:

- Locality and unitarity manifest.



- Not all symmetries manifest, extremely inefficient.

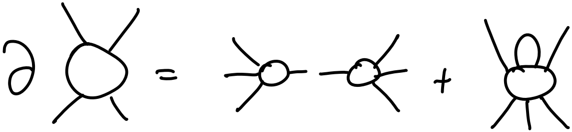
BCFW recursion relations:

- Locality not manifest - spurious poles.
- All symmetries manifest and very efficient.

Motivation

The amplitude can be defined using Locality and Unitarity

- It is a unique function that has local poles and factorization properties



- Feynman diagrams is a way how to make these properties manifest.
- BCFW is another way to satisfy the same equation.

This is still not a complete understanding.

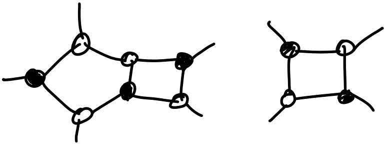
New mathematical structures underlying amplitudes: on-shell diagrams and positive Grassmannian.

Positive Grassmannian

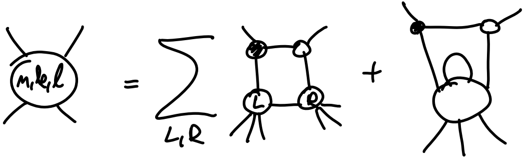
Remarkable relation between two different objects.

On-shell diagrams: physical quantities obtained by gluing together three-point amplitudes.

- They are cuts of higher loop amplitudes



- Any amplitude can be written as a sum of these objects via BCFW



Positive Grassmannian

Positive Grassmannian $G_+(k,n)$: basic object in algebraic geometry

- Grassmannian $G(k,n)$ describes a k-plane in n dimensional space,

$$C = \begin{pmatrix} * & * & * & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * \end{pmatrix} \qquad \text{with GL(k) redundancy}$$

- Positive Grassmannian = all minors are positive, $G_+(2,4)$,

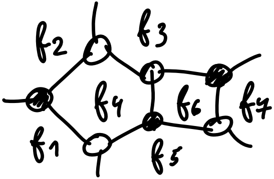
$$C = \begin{pmatrix} 1 & a & 0 & -c \\ 0 & b & 1 & d \end{pmatrix} \qquad \text{where } a,b,c,d > 0$$

- There is an incredible mathematical structure related to the Positive Grassmannian ranging from algebraic geometry to combinatorics: permutations, stratification, configuration of vectors.

Positive Grassmannian

Connection between these two objects: an on-shell diagram determines a point in the positive Grassmannian $G_+(k,n)$

The function that represents the on-shell diagram can be calculated using the integral over the Grassmannian



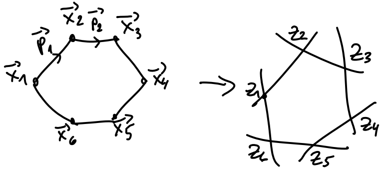
$$\rightarrow \qquad \int \frac{df_1}{f_1}\dots\frac{df_d}{f_d}\;\delta^{4|4}(C\cdot Z).$$

where C is the positive Grassmannian parametrized by face variables f_i. There are simple rules how to obtain C from the on-shell diagram.

There is still one unsatisfactory feature: the amplitude does not play a fundamental role in the story, construction via BCFW.

Momentum twistors

New variables for planar theories: momentum twistors Z_i^α,



Manifest dual conformal symmetry for planar $\mathcal{N}=4$ SYM.

External particles: Z_i, η_i, loop momenta $Z_A Z_B$.

Translation between p and Z:

$$(x_i - x_j)^2 = \frac{\langle i\, i{+}1\, j\, j{+}1\rangle}{\langle i\, i{+}1\rangle\langle j\, j{+}1\rangle}, \qquad (x-x_1)^2 = \frac{\langle AB12\rangle}{\langle AB\rangle\langle 12\rangle}$$

where $\langle a\, b\, c\, d\rangle = \epsilon_{\alpha\beta\gamma\delta} Z_a^\alpha Z_b^\beta Z_c^\gamma Z_d^\delta$.

Momentum twistors

Amplitudes in planar $\mathcal{N}=4$ SYM

$$\mathcal{A}_{n,k} = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\cdot A_{n,k-2}(Z,\eta)$$

$A_{n,k}$ is a function of $\langle Z_aZ_bZ_cZ_d\rangle$ and η's. E.g. 5pt NMHV amplitude:

$$A_{5,1} = \frac{(\langle 2345\rangle\eta_1 + \langle 3451\rangle\eta_2 + \langle 4512\rangle\eta_3 + \langle 5123\rangle\eta_4 + \langle 1234\rangle\eta_5)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

One-loop MHV amplitude:

$$A_{4,0}^{1-loop} = \frac{\langle AB\,d^2A\rangle\langle AB\,d^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

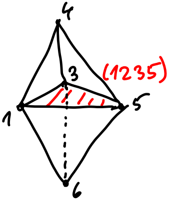
which corresponds to the 0-mass box.

NMHV polytopes

Hodges: the 6pt NMHV split helicity amplitude $1^-2^-3^-4^+5^+6^+$:

$$A_6 = \frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 2351\rangle} + \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 2361\rangle}$$

can be interpreted as a volume of polytope in $\mathbb{P}^3$.



Further developed for all NMHV amplitudes: polytopes in $\mathbb{P}^4$.

NMHV polytopes

Idea:

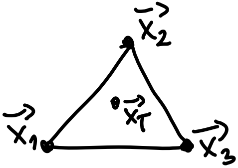
Amplitudes are "some volumes" of "some polytopes" in "some space".

We now know how to do this.

The New Positive Region

Inside of the simplex

Problem from classical mechanics: center-of-mass of three points



Imagine masses c_1, c_2, c_3 in the corners.

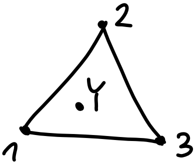
$$\vec{x_T} = \frac{c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}}{c_1 + c_2 + c_3}$$

Interior of the triangle: ranging over all positive c_1, c_2, c_3.

Triangle in projective space $\mathbb{P}^2$

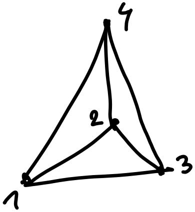
- Projective variables $Z_i = \begin{pmatrix} 1 \\ \vec{x_i} \end{pmatrix}$
- Point Y inside the triangle (mod GL(1))

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$



Inside of the simplex

Generalization to higher dimensions is straightforward.



Point Y inside tetrahedon in $\mathbb{P}^3$:

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4$$

Ranging over all positive c_i spans the interior of the simplex.

In general point Y inside a simplex in $\mathbb{P}^{m-1}$:

$$Y^I = C_{1a} Z_a^I \qquad \text{where } I = 1, 2, \ldots, m$$

and C is $(1 \times m)$ matrix of positive numbers,

$$C = (c_1\, c_2\, \ldots\, c_m)/GL(1) \qquad \text{which is } G_+(1, m)$$

Into the Grassmannian

Generalization of this notion to Grassmannian

Let us imagine the same triangle and a line Y,

$$Y_1 = c_1^{(1)} Z_1 + c_2^{(1)} Z_2 + c_3^{(1)} Z_3$$
$$Y_2 = c_1^{(2)} Z_1 + c_2^{(2)} Z_2 + c_3^{(2)} Z_3$$

writing in the compact form

$$Y_\alpha^I = C_{\alpha a} Z_a^I \qquad \text{where } \alpha = 1, 2$$

The matrix C is a (2×3) matrix mod GL(2) - Grassmannian $G(2,3)$.

Positivity of coefficients? No, minors are positive!

$$C = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & b \end{pmatrix}$$

Into the Grassmannian

In the general case we define a "generalized triangle"

$$Y^I_\alpha = C_{\alpha\, a} Z^I_a$$

where $\alpha = 1, 2, \ldots, k$, ie. it is a k-plane in $(k{+}m)$ dimensions, $a, I = 1, 2, \ldots k{+}m$. Simplex has $\alpha = 1$, for triangle also $m = 2$.

The matrix C is a 'top cell' (no constraint imposed) of the positive Grassmannian $G_+(k, k{+}m)$, it is $k \cdot m$ dimensional.

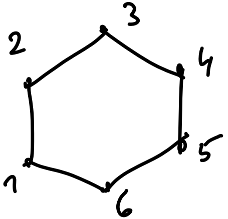
We know exactly what these matrices are!

Beyond triangles

External points Z_i did not play role, we could always choose the coordinate system such that Z is identity matrix, then $Y \sim C$.

For more vertices than the dimensionality of the space external Z's are crucial.

Let us consider the interior of the polygon in $\mathbb{P}^2$.



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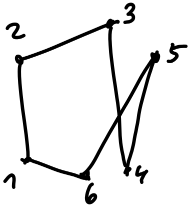


We need a convex polygon!

Key New Idea: Positivity of External Data

Beyond triangles

Convexity = positivity of external Z's. They form a $(3 \times n)$ matrix with all ordered minors being positive,

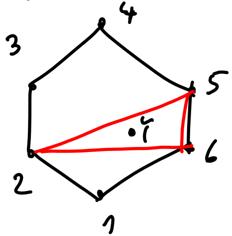
$$\langle Z_i Z_j Z_k \rangle > 0 \qquad \text{for all } i < j < k$$

The point Y inside this polygon is

$$Y = c_1 Z_1 + \cdots + c_n Z_n = C_{1a} Z_a$$

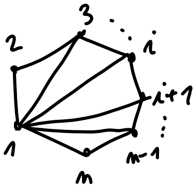
where $C \in G_+(1,n)$ and $Z \in G_+(3,n)$.

Correct but redundant description: Point Y is also inside some triangle



Beyond triangles

Triangulation: set of non-intersecting triangles that cover the region.



$$P_n = \sum_{i=2}^{n} [1\, i\, i{+}1]$$

The generic point Y is inside one of the triangles. The matrix C is

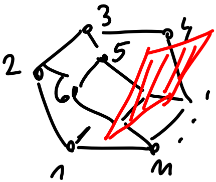
$$C = \begin{pmatrix} 1 & 0 & \ldots & 0 & c_i & c_{i+1} & 0 & \ldots & 0 \end{pmatrix}$$

Two descriptions:

- "Top cell" $(n{-}1)$-dimensional of $G_+(1,n)$ - redundant.
- Collection of 2-dimensional cells of $G_+(1,n)$ - triangulation.

Into the Grassmannian

In general case:



- A k-plane Y moving in the $(k+m)$ space.
- Positive region given by n external points Z_i.
- The definition of the space:

$$Y_\alpha^I = C_{\alpha\, a} Z_a^I$$

It is a map that defines a positive region $P_{n,k,m}$,

$$G_+(k,n) \times G_+(k+m,n) \to G(k,k+m)$$

The physical case is $m = 4$.

Conjecture: The positive region $P_{n,k,4}$ represents the n-pt N^{k}MHV tree-level amplitude.

Emergent Locality and Unitarity

How the locality and unitarity do emerge from positivity?

Locality: We show it for NMHV tree-level amplitudes

- Space is $\mathbb{P}^4$, vertices of the region are Z_i, boundaries are 3-planes $(Z_iZ_jZ_kZ_\ell)$. For what indices i,j,k,ℓ we get a boundary?
- Look at $\langle Y\,i\,j\,k\,\ell\rangle$: zero on the boundary and positive inside.

$$\langle Y\,i\,j\,k\,\ell\rangle = \sum_{a=1}^{n} c_a\langle a\,i\,j\,k\,\ell\rangle$$

- Always positive: $\langle Y\,i\,i{+}1\,j\,j{+}1\rangle > 0$ for Y inside the positive region.
- Reminder: $(x_i - x_j)^2 \sim \langle i\,i{+}1\,j\,j{+}1\rangle$, boundaries of the positive region correspond to local poles!
- Same proof holds for higher k.

Emergent Locality and Unitarity

Unitarity: Show on the example of N^{2}MHV amplitudes.

- The space is defined by the equation (Y is a line, $Y = Y_1Y_2$)

$$Y^I_\alpha = C_{\alpha\, a} Z^I_a$$

  where C is a top-cell of $G_+(2,n)$, all (2×2) minors are positive.

- On the factorization channel

$$\langle Y_1 Y_2\, 1\, 2\, j\, j{+}1\rangle = 0 \quad\rightarrow\quad Y_1 = a_1 Z_1 + a_2 Z_2 + a_j Z_j + a_{j+1} Z_{j+1}.$$

  Therefore,

$$C = \begin{pmatrix} a_1 & a_2 & 0 & \dots & 0 & a_j & a_{j+1} & 0 & \dots & 0 \\ b_1 & b_2 & b_3 & \dots & b_{j-1} & b_j & b_{j+1} & b_{j+2} & \dots & b_n \end{pmatrix}$$

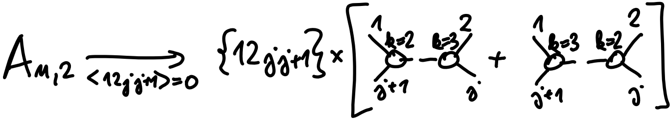
- Positivity: $b_3 = \cdots = b_{j-1} = 0$ or $b_{j+2} = \cdots = b_n = 0$.

Emergent Locality and Unitarity

- There are two options how to satisfy the positivity conditions:

$$\begin{pmatrix} * & * & 0 & \dots & 0 & * & * & 0 & \dots & 0 \\ * & * & 0 & \dots & 0 & * & * & * & \dots & * \end{pmatrix}\begin{pmatrix} * & * & 0 & \dots & 0 & * & * & 0 & \dots & 0 \\ * & * & * & \dots & * & * & * & 0 & \dots & 0 \end{pmatrix}$$

- Factorization of N^2MHV amplitude to MHV and NMHV,



- Same argument for all trees. Positivity forces C to split to C_L, C_R.
- Positivity of external data Z forces also positivity of Z_L, Z_R.
- We are not moving with external data to probe the factorization channel, Y is localized to more special position!

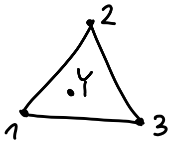
Canonical forms and amplitudes

Canonical form

How to get the actual formula from the positive region?

We define a canonical form Ω_P which has **logarithmic** singularities on the boundaries of P.

Example of triangle in $\mathbb{P}^2$:



$$\Omega_P = \frac{\langle Y\, dY\, dY\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

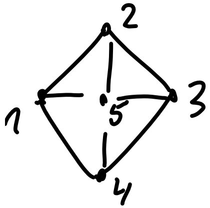
We parametrize $Y = Z_1 + c_2 Z_2 + c_3 Z_3$ and get

$$\Omega_P = \frac{dc_2}{c_2}\frac{dc_3}{c_3} = \mathrm{dlog}\,\mathrm{c}_2\ \mathrm{dlog}\,\mathrm{c}_3$$

Logarithmic singularities when moving with Y on a line (12) for $c_3 = 0$ or a line (13) for $c_2 = 0$.

Canonical form

Simplex in $\mathbb{P}^4$ - this is relevant for physics.



$$\Omega_P = \frac{\langle Y\, dY\, dY\, dY\, dY\rangle \langle 12345\rangle^2}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

For $Y = Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4 + c_5 Z_5$ we get

$$\Omega_P = \mathrm{dlog\, c_2\ dlog\, c_3\ dlog\, c_4\ dlog\, c_5}$$

"Generalized triangle" given by $Y^I_\alpha = C_{\alpha\, a} Z^I_a$ with $C_{\alpha\, a} \in G_+(k, k+m)$.

- C is parametrized by $k \cdot m$ parameters - it is a $k \cdot m$ dimensional "top" cell of $G_+(k, k+m)$.
- We know all the matrices C as functions of km positive variables c_j.
- The form associated with this region is

$$\Omega_P = \mathrm{dlog}\, c_1\ \mathrm{dlog}\, c_2\ \dots\ \mathrm{dlog}\, c_{km}$$

Canonical form

For general positive region P we have the same definition of Ω_P: canonical form with logarithmic singularities on the boundaries of P.

$$\Omega_P = \frac{\text{Measure of Y } \times \text{ Numerator}(Y, Z_i)}{\prod \langle Y\,\text{boundary}\rangle}$$

such that the form has logarithmic singularities on the boundaries.

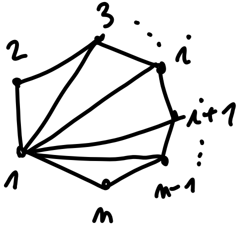
There is a natural strategy how to find the form:

- Triangulate the space, ie. find the set of non-overlapping "generalized triangles" that cover the space.
- Write the form for each triangle: dlogs of all variables $c_1, \ldots, c_{km}$.
- Solve for variables c_j in terms of Y, Z_i for each "triangle", plug into the form and sum all "triangles".

The non-trivial operation: Triangulation of the positive region!

Canonical form

Example: Polygon



$$\Omega_P = \sum_{i=2}^{n} \frac{\langle Y\, dY\, dY\rangle\langle 1\, i\, i{+}1\rangle^2}{\langle Y\, 1\, i\rangle\langle Y\, 1\, i{+}1\rangle\langle Y\, i\, i{+}1\rangle}$$

Spurious poles $\langle Y\, 1\, i\rangle$ cancel in the sum.

We know how to do the triangulation for some cases, e.g. for all $m = 2$ but not in general. The positive region is not known to mathematicians (only the "triangles"which are positive Grassmannians $G_+(k, n)$).

Canonical form

The case of physical relevance is $m = 4$.

BCFW provides for us a triangulation of the space, different representations are different triangulations.

Spurious poles are internal boundaries that are absent once we put all pieces together.

Using BCFW we did many checks that the the picture is indeed correct!

We have also examples of triangulations that are not BCFW or anything else coming from physics.

From canonical forms to amplitudes

How to extract the amplitude from Ω_P?

Look at the example of simplex in $\mathbb{P}^4$.

$$\Omega_P = \frac{\langle Y\, dY\, dY\, dY\, dY\rangle\langle 12345\rangle^4}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

Note that the data are five-dimensional, it is purely bosonic and it is a form rather than function.

Let us rewrite Z_i as four-dimensional part and its complement

$$Z_i = \begin{pmatrix} z_i \\ \delta z_i \end{pmatrix} \qquad \text{where} \quad \delta z_i = (\eta_i \cdot \phi)$$

We define a reference point Y^* which is in the complement of 4d data z_i,

$$Y^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

From canonical forms to amplitudes

We integrate the form, using $\langle Y^* 1234\rangle = \langle 1234\rangle$, etc. we get

$$\int d^4\phi \int \delta(Y - Y^*)\,\Omega_P = \frac{(\langle 1234\rangle\eta_5 + \langle 2345\rangle\eta_1 + \dots + \langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

For higher k we have $(k{+}4)$ dimensional external Z_i,

$$Z_i = \begin{pmatrix} z_i \\ (\eta_i\cdot\phi_1) \\ \vdots \\ (\eta_i\cdot\phi_k) \end{pmatrix} \qquad\qquad Y^* = \begin{pmatrix} \vec{0} & \vec{0} & \dots & \vec{0} \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Reference k-plane Y^* orthogonal to external z_i. We consider integral

$$A_{n,k} = \int d^4\phi_1 \dots d^4\phi_k \int \delta(Y - Y^*)\,\Omega_{P_{n,k}}$$

Loop amplitudes

MHV amplitudes

Let us start with MHV amplitudes where there is no dependence on η. External data are just original $Z_i = z_i$.

The loop variable is represented by a line $Z_A Z_B$, at one-loop we have just one line parametrized as

$$A_\alpha^I = C_{\alpha\, a} Z_a^I, \qquad \text{where } \alpha = 1, 2$$

where $A_\alpha = (A, B)$. We demand the matrix of coefficients to be positive, ie. $C \in G_+(2, n)$ and $Z \in G_+(4, n)$.

Boundaries of this region are $\langle AB\, i\, j\rangle$:

$$\langle AB\, i\, j\rangle = \sum_{a<b} (a\, b)\langle a\, b\, i\, j\rangle$$

Only $\langle AB\, i\, i{+}1\rangle$ are always positive - boundaries of the space.

MHV amplitudes

"Triangles" are just 4-dimensional cells of $G_+(2,n)$: "kermits"

Natural triangulation

$$P_n = \sum_{i<j} [1, i, i{+}1; 1, j, j{+}1]$$

where

$$C_{1,i,i+1;1,j,j+1} = \begin{pmatrix} 1 & 0 & \dots & 0 & c_i & c_{i+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & c_j & c_{j+1} & 0 & \dots & 0 \end{pmatrix}$$

Each kermit has a simple form $\Omega_P = \mathrm{dlog}\, c_i \; \mathrm{dlog}\, c_{i+1} \; \mathrm{dlog}\, c_j \; \mathrm{dlog}\, c_{j+1}$, the full MHV one-loop amplitude is then

$$\Omega_P = \sum_{i<j} \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle AB(i{-}1\,i\,i{+}1)\bigcap(j{-}1\,j\,j{+}1)\rangle^2}{\langle AB\,1\,i\rangle\langle AB\,1\,i{+}1\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,1\,j\rangle\langle AB\,1\,j{+}1\rangle\langle AB\,j\,j{+}1\rangle}$$

MHV amplitudes

At two-loop we have two lines $Z_A Z_B$, $Z_C Z_D$,

$$A^{(1)\ I}_{\alpha} = C^{(1)}_{\alpha\, a} Z^I_a$$
$$A^{(2)\ I}_{\alpha} = C^{(2)}_{\alpha\, a} Z^I_a$$

We combine matrices into

$$C = \begin{pmatrix} C^{(1)} \\ C^{(2)} \end{pmatrix}$$

We demand $C^{(1)}, C^{(2)}$ to be both $G_+(2,n)$. This is a "square" of one-loop problem: $(A_n^{1-loop})^2$.

Additional constraint: All (4×4) minors of C are positive! This gives MHV two-loop amplitude.

We did many numerical checks that this picture is correct.

MHV amplitudes

New feature: "triangles" are not known to mathematicians, it is a generalization of the positive Grassmannian, the form for each "triangle" is again the dlog of all positive variables.

One way to triangulate: BCFW loop recursion - we checked it triangulates the space. But geometrically it is not very natural.

New geometric triangulation for 4pt 2-loop: new formula not derivable from any physical approach.

Local expansion: not positive term by term! It is not a triangulation (perhaps some external triangulation).

MHV amplitudes

At L-loop we have L lines A_α^I.

$$A_\alpha^{(1)\ I} = C_{\alpha\, a}^{(1)} Z_a^I$$

$$\vdots$$

$$A_\alpha^{(L)\ I} = C_{\alpha\, a}^{(L)} Z_a^I$$

$$C = \begin{pmatrix} C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}$$

Positivity constraints:

- External data Z are positive.
- All minors of $C^{(1)}$ are positive.
- All (4×4) minors made of $C^{(i)}$, $C^{(j)}$ are positive, all (6×6) minors of $C^{(i)}$, $C^{(j)}$, $C^{(k)}$, etc. are also positive.

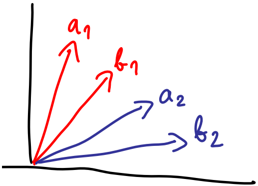
This conjecture passes many checks: locality, unitarity but also planarity are consequences of positivity.

MHV amplitudes

The geometry problem for 4pt is incredible simple and it should be tractable to triangulate this space to all loop orders.

We have 2d vectors a_i, b_i for $i = 1, \ldots, L$ and we demand

- They all live in the first quadrant.
- For any pair $(a_i - b_i) \cdot (a_j - b_j) < 0$.
- Triangulation: Find all possible configurations of vectors!



We did it manually up to 3-loops.

Cuts from positivity

Let us take 4pt ℓ-loop MHV amplitude and consider its cuts.

$$C_1 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix} \qquad \dots \qquad C_\ell = \begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Unitarity cut: $\langle AB12\rangle = \langle AB34\rangle = 0$.

$$C_1 = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 0 & y & 1 \end{pmatrix}$$

Positivity: $x, y > 0$ and also

$$(23)_i + xy(14)_i - x(13)_i - y(24)_i > 0$$

for all other matrices $i = 2, \dots, \ell$.

Cuts from positivity

For each loop we have:

$$(12),(13),(14),(23),(24),(34)>0, \qquad (23)+xy(14)-x(13)-y(24)>0$$

splits the problem into two regions:

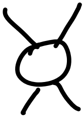
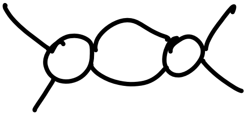
$$C^{(1)}=C(1,2-x1,3-y4,4), \qquad C^{(2)}=C\left(1-\frac{1}{x}2,2,3,4-\frac{1}{y}3\right)$$

We can do it for all $\ell-1$ loops, all internal positivities magically work out and the result is

Cuts from positivity

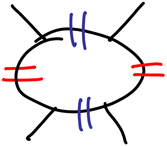
$$
\begin{aligned}
&\mathrm{Cut}\, A_\ell(1,2,3,4) \\
&\quad = \sum_{\ell_1+\ell_2=\ell-1} A_{\ell_1}\left(1, 2-x1, 3-y4, 4\right) A_{\ell_2}\left(1-\frac{1}{x}2, 2, 3, 4-\frac{1}{y}3\right)
\end{aligned}
$$

which is the classical example of unitarity cut,

Cut  = 

Cuts from positivity

Another example is the non-planar cut,



corresponding to $\langle AB12\rangle = \langle AB34\rangle = \langle CD23\rangle = \langle CD41\rangle = 0$.
In this case first two C-matrices are localized to

$$C_1 = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 0 & y & 1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 0 & 1 & a & 0 \\ -1 & 0 & 0 & b \end{pmatrix}$$

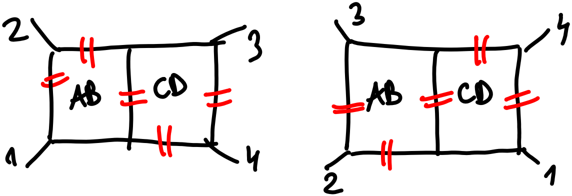
But the (4×4) minor of C_1C_2 is then

$$|C_1C_2| = -yb - xa < 0$$

and therefore the cut must be vanishing.

Cuts from positivity

There are examples that are not images of any known properties of amplitude: $\langle AB12\rangle = \langle AB23\rangle = \langle CD34\rangle = \langle CD41\rangle = \langle ABCD\rangle = 0$.



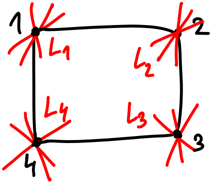
This appears non-trivially even at two-loops!

Solution to this cut is not compatible with positivity of all minors and therefore it is forbidden.

Indeed this 5-cut vanishes at all loop orders for any n, invisible in local expansion.

Cuts from positivity

Let us consider 2L cut at any loop order:



Parametrization of C matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_i & 1 & y_i^{-1} \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -u_j^{-1} & 0 & v_j & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & -w_k^{-1} & 0 & z_k \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \\ -s_l & -1 & -t_l^{-1} & 0 \end{pmatrix}$$

Cuts from positivity

Define

$$\Omega(x,y) = \sum_\sigma \frac{dy_1\dots dy_\ell}{y_{\sigma_1}(y_{\sigma_2}-y_{\sigma_1})\dots(y_{\sigma_n}-y_{\sigma_{n-1}})}\prod_i \frac{dx_i}{(x_i-y_{\sigma_n})}$$

The residue is then

$$\Omega = \Omega(v,y)\Omega(x,t)\Omega(z,u)\Omega(s,w)$$

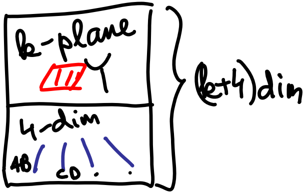
All-loop information, impossible to get using any standard method.

General case

In the general case of n-pt L-loop N^{k}MHV amplitude we have

- Positive $k{+}4$-dimensional external data Z.
- k-plane Y in $k{+}4$ dimensions
- L lines in 4-dimensional complement to Y plane

$$\begin{aligned} Y^I_\sigma &= C_{\sigma\,a} Z^I_a \\ A^{(1)\;I}_\alpha &= C^{(1)}_{\alpha\,a} Z^I_a \\ &\vdots \\ A^{(L)\;I}_\alpha &= C^{(L)}_{\alpha\,a} Z^I_a \end{aligned} \qquad C = \begin{pmatrix} C \\ C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}$$



Positivity constraints:

- C is positive.
- C + any combination of $C^{(i)}$'s is positive.

Everything is Positive

Positivity of amplitude

The integrand itself is positive in the positive region

- Choose positive external data Z_i.
- Choose Y-plane and lines AB_j to be inside the positive region.
- The numerical value of the function is positive!

Checked for all available results available in the literature:

- Many tree amplitudes.
- MHV up to 3-loop, and up to 7-loop for 4pt.
- NMHV up to 2-loop.
- N^{k}MHV for 1-loop.

Even more surprising: the integrated IR finite expressions are positive for positive external data!

Positivity of amplitude

We checked the one-loop ratio functions for high n, k.

$$R_{n,k}^{1-loop} = A_{n,k}^{1-loop} - A_{n,k}^{tree} \cdot A_{n,0}^{1-loop}$$

For six-point NMHV:

$$R_{6,1}^{1-loop} = H_1 \cdot [(2) - (3) + (4)] + \text{cycl.}$$

where H_1 is the chiral hexagon integral with unit LS,

$$H_1 = \frac{1}{2}\left[\mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) + \log(w)\log(u) - 2\zeta_2\right]$$

In the positive region the sum is positive. Very non-trivial: combines rational functions $(1), (2), \ldots (6)$ with dilogs.

Positivity of amplitude

Positivity of MHV remainder function:

$$R_6^{2-loop} = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \frac{\pi^2 J^2}{12} + \frac{\pi^4}{72}$$

is positive in the positive region. Here the positivity implies

$$u,v,w>0, \qquad 1-u-v-w>0, \qquad \Delta = (1-u-v-w)^2 - 4uvw > 0$$

Lance checked the 3-loop remainder to be uniformly negative in this positive region (looking forward to 4-loops!)

Relation to cluster coordinates (see Mark tomorrow).

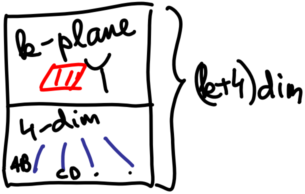
Conclusions

Definition of the object

The n-pt L-loop N^{k}MHV amplitude:

- Positive $k{+}4$-dimensional external data Z.
- k-plane Y in $k{+}4$ dimensions
- L lines in 4-dimensional complement to Y plane

$$\begin{aligned} Y^I_\sigma &= C_{\sigma\, a} Z^I_a \\ A^{(1)\ I}_\alpha &= C^{(1)}_{\alpha\, a} Z^I_a \\ &\ \vdots \\ A^{(L)\ I}_\alpha &= C^{(L)}_{\alpha\, a} Z^I_a \end{aligned} \qquad C = \begin{pmatrix} C \\ C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}$$



Positivity constraints:

- C is positive.
- C + any combination of $C^{(i)}$'s is positive.

- We defined the Positive region $P_{n,k,\ell}$ and canonical form $\Omega_{n,k,\ell}$ with logarithmic singularities on the boundaries of this region.
- The n-pt ℓ-loop N^{k}MHV amplitude then directly corresponds to this form.
- Calculating amplitude = triangulating the positive region $\Omega_{n,k,\ell}$.
- Positivity of the integrand and also integrated expressions!

It is remarkable that this mathematical structure, generalizing positivity beyond the usual positive grassmannian, gives a complete definition of scattering amplitudes in planar N=4 SYM

- No reference to usual field theory notions whatsoever: no Feynman diagrams, not even on-shell diagams or recursion relations.
- Locality and Unitarity emerge from positivity.
- This rich structure is also completely new to the mathematicians.

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THANK YOU!