# Spectral gaps for periodic discrete and metric graphs

Fernando Lledó

Department of Mathematics, University Carlos III, Madrid
and
ICMAT, Madrid

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## Overview:

1. Idea: Decoupling fundamental domains in periodic structures
2. Discrete and metric graphs and their Laplacians
3. Eigenvalue bracketing for metric graphs
4. Spectral localization and gaps. Examples
5. Summary and conclusions

Joint work with Olaf Post

# 1. Motivation: decoupling fundamental domains

- If $M$ compact "structure"

  $\rightsquigarrow$ spectrum of the corresponding Laplacian $\sigma(\Delta_M)$ ✓
- If $X$ is a non-compact structure

  $\rightsquigarrow \sigma(\Delta_X)$ ?

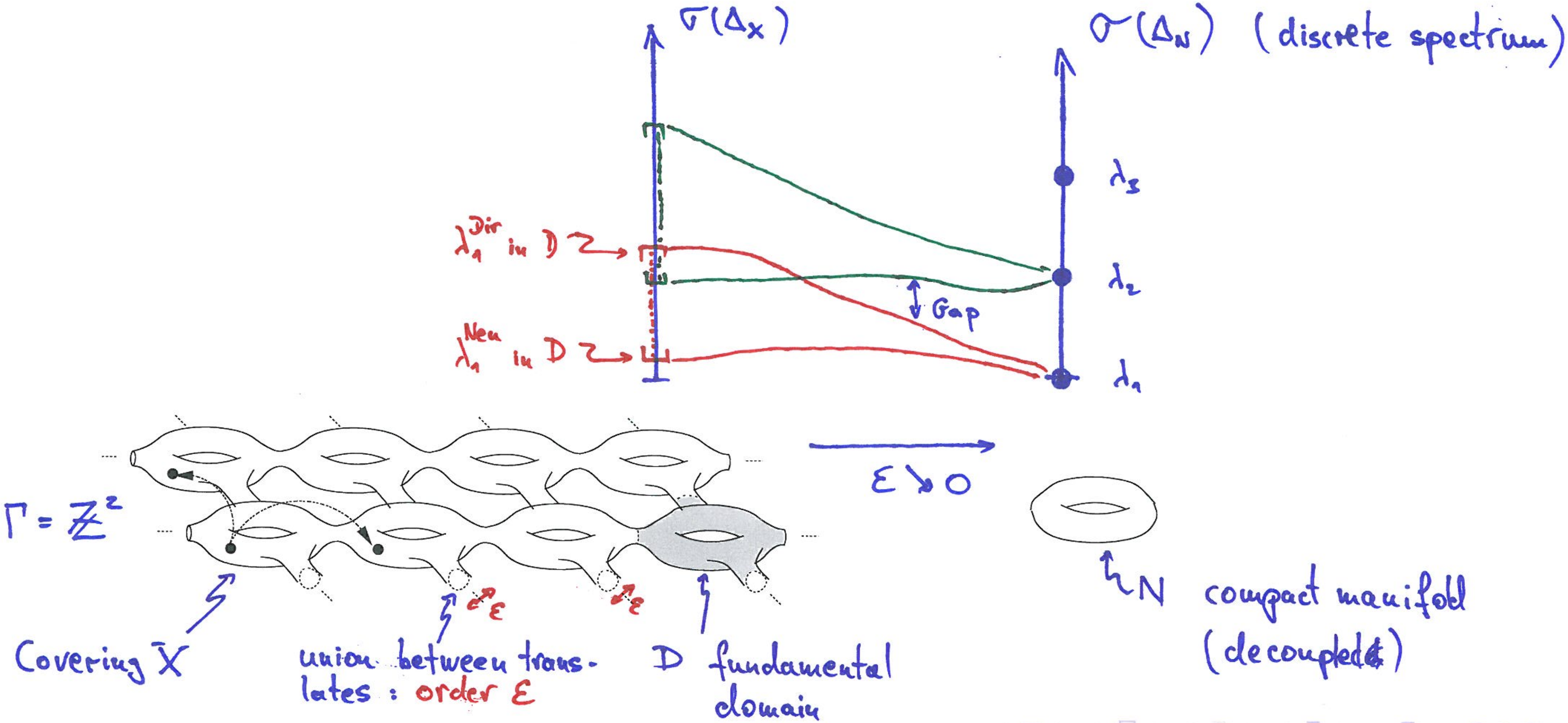
  (Think of $M$ as, e.g., cpt. Riemannian mfd. or finite graph and $X$ non-cpt. Riemannian mfd. or infinite graph)

In this talk we will focus on periodic structures:

- Covering $X \to M$ with fundamental domain $D$ and discrete group $\Gamma$ acting on it.
  - Analyze $\sigma(\Delta_X)$, in particular when gaps appear.
  - Physically, spectral gaps may be thought as energy values at which there is no transmission.

## Spectral gaps and Riemannian coverings:

**Theorem:** (Ll./Post, Cont. Math. '07, Rev.Math.Phys. '08)
Given a Riemannian covering $X \to M$ with residually finite group $\Gamma$ and given $n \in \mathbb{N}$, there is a deformed covering $X_\varepsilon \to M_\varepsilon$, $\varepsilon = \varepsilon(n)$, such that $\sigma(\Delta_{X_\varepsilon})$ has at least $n$ *gaps*.

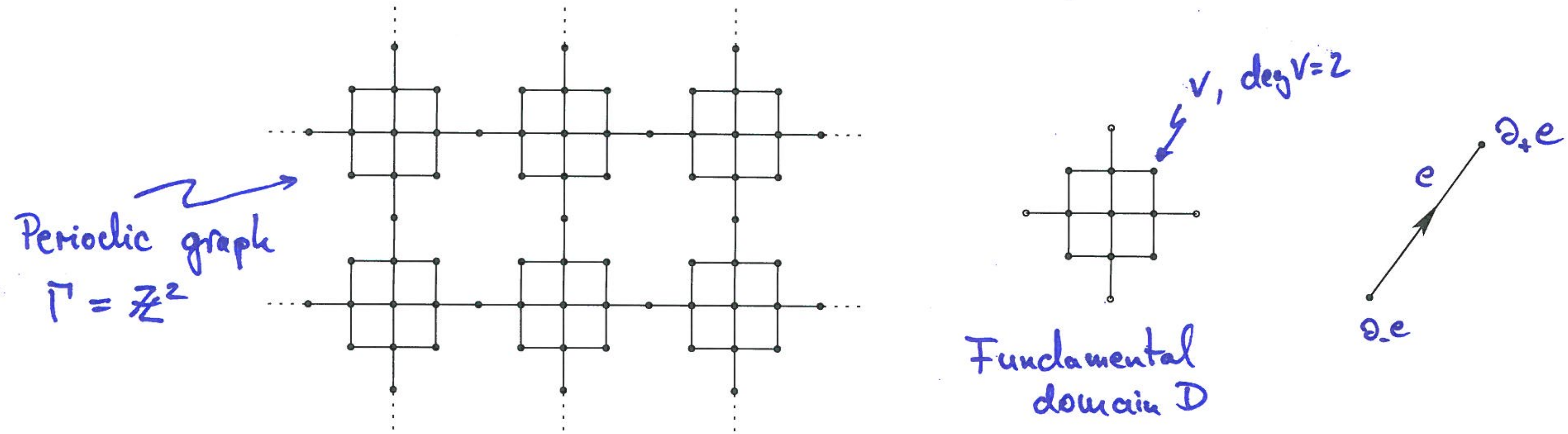


# 2. Laplacians on discrete and metric graphs

**Can we localize spectral gaps in graph Laplacians ?**

## 2.1 Discrete graphs and Laplacians

- $G := (V, E, \partial)$ be an oriented graph. $E_v$ edges at $v \in V$
- Orientation: $\partial : E \to V \times V$, $\partial e = (\partial_- e, \partial_+ e)$.




Exterior derivative: $d\colon \ell_2(V) \to \ell_2(E)$, $(dF)_e = F(\partial_+ e) - F(\partial_- e)$,

- discr. (normalized) Laplacian: $\check\Delta_G := d^* d\colon \ell_2(V) \to \ell_2(V)$

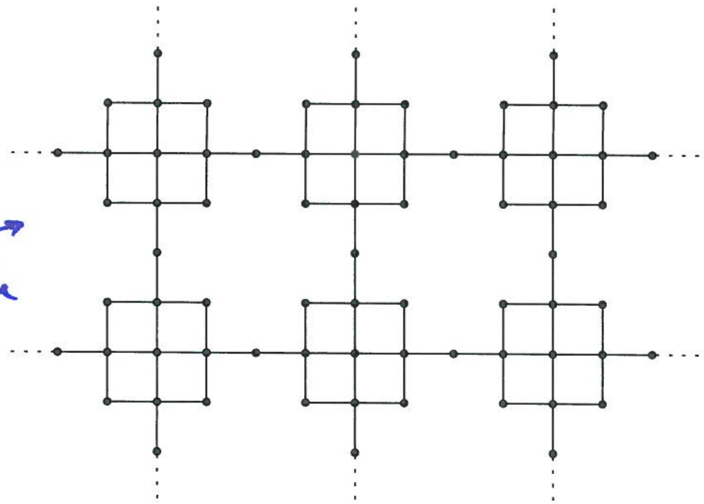
$$(\check\Delta_G F)(v) := F(v) - \frac{1}{\deg(v)} \sum_{e \in E_v} F(v_e), \text{ with } F \in \ell_2(V).$$

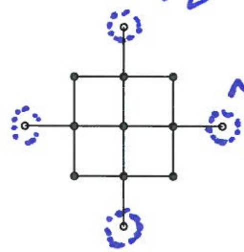
## Discrete Dirichlet Laplacians

**Boundary** of $G = (V, E, \partial)$: It is a choice of a set $\partial V \subset V$

- Define the Dirichlet Laplacian $\check{\Delta}_G^{\partial V}$ similarly as before, but on $F \in \ell_2(V)$ such that $F \restriction \partial V = 0$.
- If $G$ is finite (e.g., a fund. domain $D$ of a periodic graph) then
  - $\mu_k$, $\mu_k^{\partial V}$ eigenvalues of $\check{\Delta}_G$, $\check{\Delta}_G^{\partial V}$ respectively.

periodic graph $\Gamma = \mathbb{Z}^2$





Boundary $\partial V$ (typically those vertices that are "glued" together to produce the periodic graph)

Fundamental domain $D$

Remark:

- In discrete Laplacians the vertices $V$ play a primary role, while edges $E$ are secondary objects.

## 2.2 Metric graphs and Laplacians

Let $G := (V, E, \partial)$ be an oriented graph.
The **metric graph** associated with $G$ is constructed as follows:

- **Equilateral graph:** $E \ni e \mapsto I_e = (0, 1)$
- **Metric space:** $X := \dot{\bigcup}_{e \in E} \overline{I}_e / \sim$
- **Basic $L^2$-space:** $\mathscr{H} = L^2(X) \cong \bigoplus_{e \in E} L^2(I_e)$ (Hilbert space).

We define **Laplacians in terms of quadratic forms!**

Consider first on each edge/intervall: the following

unbounded positive quadratic $\longleftrightarrow$ operators

$$\mathfrak{h}(f) := \int_0^1 |f'(x)|^2 \, dx \overset{\text{reprsent.}}{\longrightarrow} -\langle f, \Delta f \rangle \text{ and } \Delta f = -f'' \;\&\; \text{b.c.}$$

Most important advantage of the quadratic forms language:

- $\mathrm{dom}\,\mathfrak{h} \supseteq \mathrm{dom}\,\Delta$. E.g., $H_0^1(0,1) \supset H_0^2(0,1)$.

## Laplacians on the metric graph: $G := (V, E, \partial) \rightsquigarrow X$

Kirchhoff Laplacian:

- Quadratic form: $\mathfrak{h}(f) := \|f'\|^2 = \sum_{e \in E} \int_0^1 |f_e'(x)|^2 \, dx$.
- $\operatorname{dom}\mathfrak{h} = \left(\bigoplus_e H^1(I_e)\right) \cap C(X)$.

**Remark:** Where does the name Kirchhoff come from?

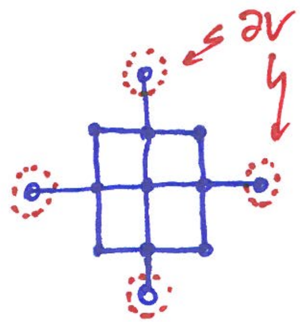
$$\Delta_X f_e = -f_e'' \text{ if } f = (f_e)_{e \in E} \in \left( \bigoplus_{e \in E} H^2(I_e) \right) \cap C(X)$$

Kirchhoff condition: $\sum_{e \in E_v} f_e'(v) = 0, \; v \in V.$

Dirichlet-Kirchhoff Laplacian: Let $X$ cpt. & $\partial V \subset V$ a boundary.

- Defined via the quadratic form $\mathfrak{h}^{\partial V}(f) := \|f'\|^2$ on

$$\operatorname{dom}\mathfrak{h}^{\partial V} = \{ f \in \operatorname{dom}\mathfrak{h} \mid f \restriction \partial V = 0 \}$$




- $\lambda_k$, $\lambda_k^{\partial V}$ eigenvalues of $\Delta_X$, $\Delta_X^{\partial V}$ respectively.

- blueSummary: For metric Laplacians the edges are primary objects and the vertices come in via boundary conditions.

## 2.3 Spectral relations (reminder): discrete vs. metric graphs

- $G$ finite graph and $X$ is the metric graph (compact and equilateral) associated to $G$.
- Discrete Laplacian $\check{\Delta}_G^{\partial V}$ vs. metric Laplacian $\Delta_X^{\partial V}$.Both Laplacians with Dirichlet data on the boundary $\partial V \subset V$.
- Dirichlet spectrum for $\Delta_X^{\partial V}$: $\Sigma^D = \{\lambda_n := n^2\pi^2 \mid n \in \mathbb{N}\}$.

**Proposition 1:** (Outside $\Sigma^D$) If $\lambda \notin \Sigma^D$, then

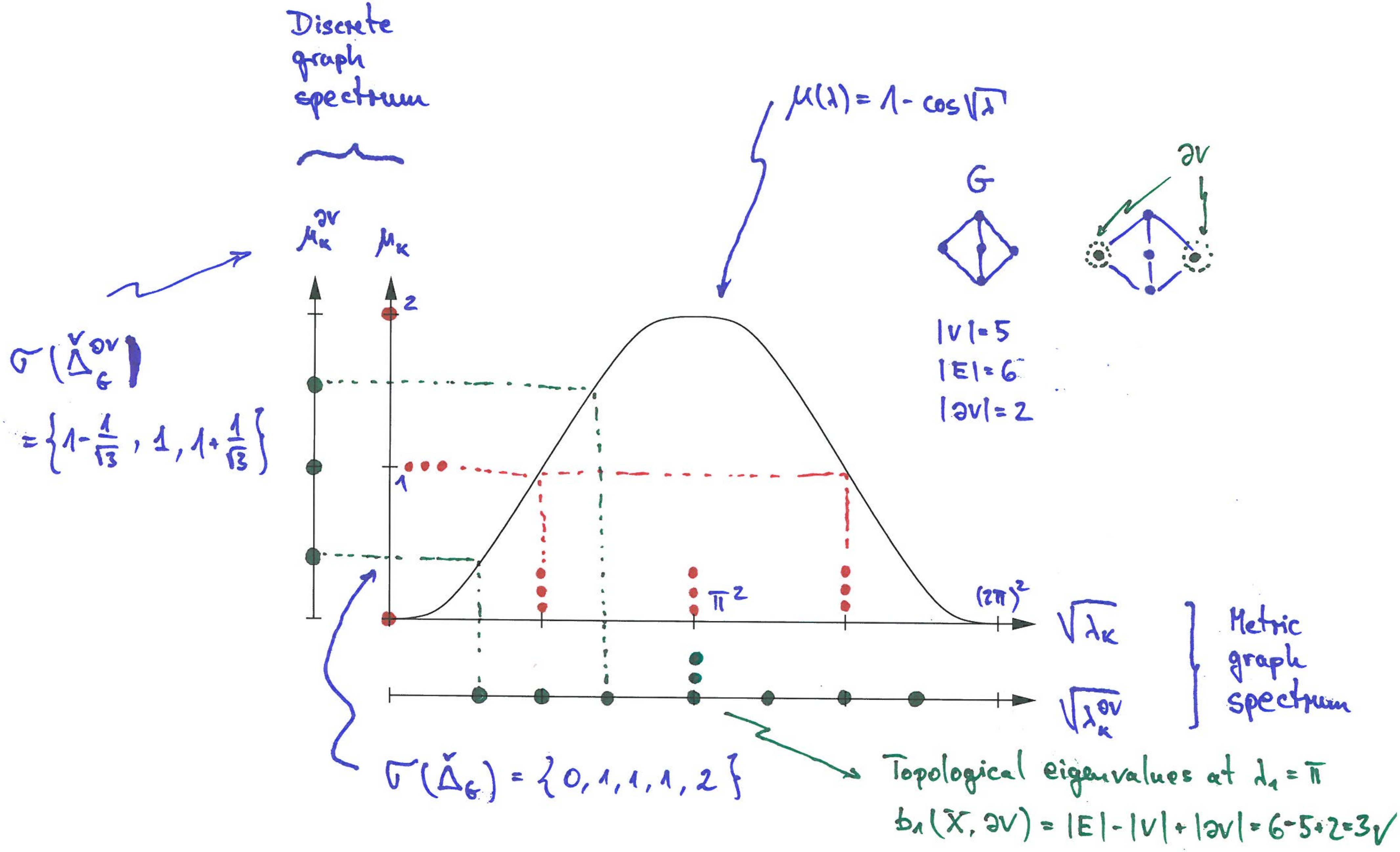
- $\lambda \in \sigma\left(\Delta_X^{\partial V}\right)$ iff $\mu(\lambda) := \left(1 - \cos\sqrt{\lambda}\right) \in \sigma\left(\check{\Delta}_G^{\partial V}\right) \subset [0,2]$

**Proposition 2:** (At $\Sigma^D$) If $\lambda = \lambda_n \in \Sigma^D$, then for $n = 2l$ even there is an isomorphism between the relative homology $H_1(X, \partial V)$ and the set $\left\{f \in \ker\left(\Delta_X^{\partial V} - \lambda_n 1\right) \mid f \restriction V = 0\right\}$. In particular,

$$\boxed{\dim\ker(\Delta_X^{\partial V} - \lambda\mathbb{1}) = b_1(X, \partial V) = |E| - |V| + |\partial V|.}$$

$\vdots$

# Spectral relations in a concrete example



# 3. Eigenvalue bracketing for the metric Laplacian

Quadratic forms are also useful to order eigenvalues of metric Laplacian! (In general this is not possible at the level of operators!)

1. Recall that the Dirichlet and Kirchhoff domains satisfy $\mathrm{dom}\mathfrak{h}^{\partial V} \subset \mathrm{dom}\mathfrak{h} \subset \mathscr{H}$ with $\mathfrak{h}(f) = \|f'\|^2$.
2. Extending the quadratic forms on the whole Hilbert $\mathscr{H}$ space by $\infty$ and using the variational characterization of eigenvalues in terms quadratic forms

$$\lambda_k = \inf_{L \subset \mathscr{H}} \sup_{f \in L} \frac{\mathfrak{h}(f)}{\|f\|^2},$$

where $L$ is a $k$-dimensional subspace. Similarly for $\mathfrak{h}^{\partial V}$.

$$\mathrm{dom}\mathfrak{h}^{\partial V} \subset \mathrm{dom}\mathfrak{h} \qquad \Rightarrow \qquad \lambda_k^{\partial V} \geq \lambda_k$$

3. Define the Dirichlet-Kirchoff intervals: $I_k := [\lambda_k, \lambda_k^{\partial V}]$.

# 4. Spectral localization and gaps

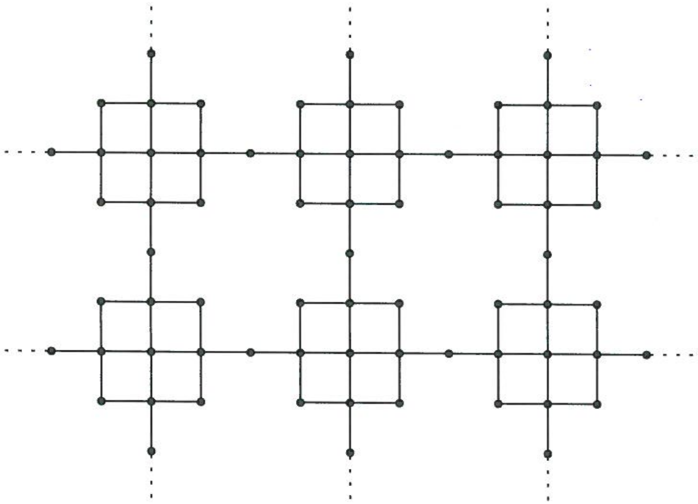
**Theorem 1** (Ll./Post '08) Let $X$ be a periodic metric graph (not necessarily equilateral) with residually finite group $\Gamma$ and fundamental domain $D$. If $\Delta_X$ is the Kirchhoff-Laplacian, then

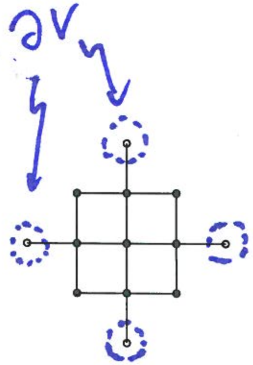
(i) $\sigma(\Delta_X) \subset \bigcup_{k\in\mathbb{N}} \left[\lambda_k, \lambda_k^{\partial V}\right]$.

(ii) If $\mathfrak{g} \subset [0,\infty)$ satisfies $\mathfrak{g} \cap \left(\bigcup_{k\in\mathbb{N}} \left[\lambda_k, \lambda_k^{\partial V}\right]\right) = \emptyset$, then $\mathfrak{g} \cap \sigma(\Delta_X) = \emptyset$ and $\Delta_X$ has a spectral gap.

Periodic graph

$\Gamma = \mathbb{Z}^2$






$\lambda_k$ and $\lambda_k^{\partial V}$ are associated to the Laplacian on $D$

Fundamental domain $D$

## Idea of the proof:

- Towers of coverings: The covering $X$ can be exhausted by finite coverings $X_i$ with finite group $\Gamma/\Gamma_i$

$$\begin{tikzcd} & X \arrow[dl, "\pi_i"', "\Gamma_i"] \arrow[dr, "\pi", "\Gamma"'] & \\ X_i \arrow[rr, "p_i", "\Gamma/\Gamma_i"'] & & X_0 \end{tikzcd}$$

- 1. step: $\sigma(\Delta_X) \subseteq \overline{\bigcup_{i\in\mathbb{N}} \sigma(\Delta_{X_i})}$

- 2. step: $\sigma(\Delta_{X_i}) \subseteq \bigcup_{k\in\mathbb{N}} \left[\lambda_k, \lambda_k^{\partial V}\right]$ using bracketing techniques.

**Theorem 2** (Ll./Post '08)
Let $G$ be a discrete periodic graph with residually finite group $\Gamma$ and fundamental domain $D$. Let $V(D)$ be the set of vertices of the fundamental domain and $\check{\Delta}_G$ the discrete Laplacian. Then

(i) $\sigma\left(\check{\Delta}_G\right) \subset \bigcup_{k=1}^{V(D)} \left[\mu_k, \mu_k^{\partial V}\right]$.

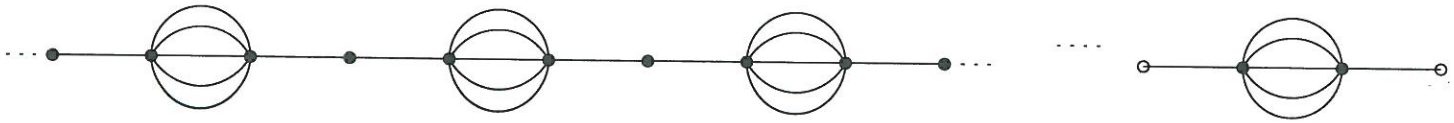
(ii) If $\mathfrak{g} \subset [0,2]$ satisfies $\mathfrak{g} \cap \left(\bigcup_{k=1}^{V(D)} \left[\lambda_k, \lambda_k^{\partial V}\right]\right) = \emptyset$, then $\mathfrak{g} \cap \sigma\left(\check{\Delta}_G\right) = \emptyset$ and the discrete Laplacian has a spectral gap.

Idea of the proof:

- Use first Theorem 1 for **equilateral** metric graph.
- Use the spectral relation between discrete and metric graphs:

$$\lambda \in \sigma\left(\Delta_X\right) \quad \text{iff} \quad \mu(\lambda) = \left(1 - \cos\sqrt{\lambda}\right) \in \sigma\left(\check{\Delta}_G\right).$$

**Example 1** (Periodic onion)



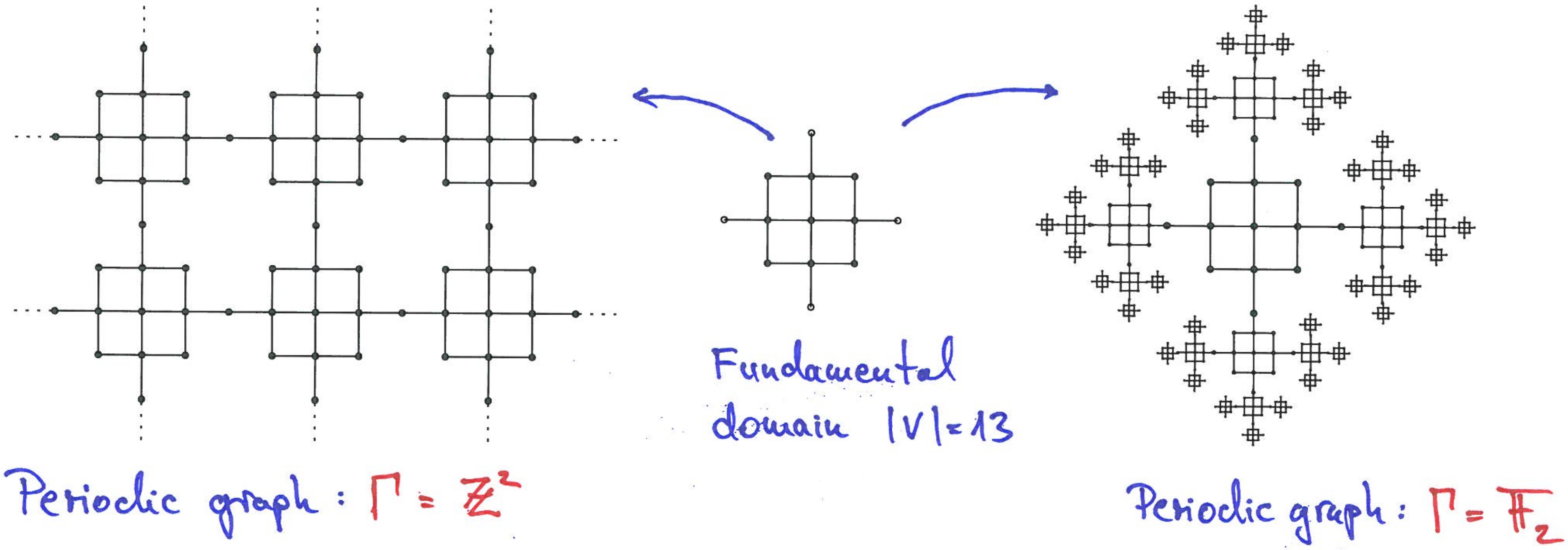
Metric and discrete Laplacians have a spectral gap if the number of edges $r \geq 2$: e.g.,

$$\sigma\left(\check{\Delta}_G\right) \subseteq \left[0, \frac{1}{1+r}\right] \cup \left[1 - \frac{1}{1+r}, 2 - \frac{1}{1+r}\right] \cup \left[1 + \frac{1}{1+r}, 2\right] .$$

**Motivational idea applied to graphs:**
Decouple fundamental domains by enlarging the fundamental domain. High contrast!

**Example 2** (One fundamental domain with two group actions)
Consider the periodic graphs with the **same fundamental domain D** and two radically different groups acting

- Abelian group $\Gamma = \mathbb{Z}^2$.
- Free group $\Gamma = \mathbb{F}_2$ with two generators.




- For both we can localize a spectral gap: e.g., discr. Laplacian

$$\sigma\left(\check{\Delta}_G\right) \cap \left(1 - \frac{\sqrt{3}}{2}, 1 - \frac{1}{\sqrt{2}}\right) = \emptyset\,.$$

# 5. Summary and conclusions

**Results:**

- Analysis of the spectrum of Laplacians for (infinite) periodic discrete and metric graphs.
- Sufficient conditions for the existence of spectral gaps in both cases. Examples.

**Techniques used:**

- Eigenvalue bracketing for metric graphs.
- Use the metric graphs to analyze discrete Laplacians.
- Work and estimate at the level of fundamental domains.

**What else?**

1. Localize spectral gaps directly for the discrete Laplacians.
2. Analyze tower of coverings with magnetic Laplacians.